Discussiones Mathematicae
General Algebra and Applications xx (xxxx) 1–10

ORDERED SEMIGROUPS IN WHICH PRIME IDEALS ARE MAXIMAL

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Abstract

In this paper, a class of ordered semigroups, namely semi-pseudo symmetric ordered semigroups, which includes the classes of commutative ordered semigroups, duo ordered semigroups, normal ordered semigroups and idempotent ordered semigroups is introduced. We obtain a characterization for semi-pseudo symmetric ordered semigroups with identity in which proper prime ideals are maximal and also characterize semi-pseudo symmetric ordered semigroups without identity in which proper prime ideals are maximal and the set of all globally idempotent principal ideals forms a chain under the set inclusion.

Keywords: ordered semigroup, semi-pseudo symmetric, duo, archimedean, primary ideal, prime ideal, maximal ideal.

2020 Mathematics Subject Classification: 06F05.

1. Introduction and preliminaries

Schwarz initiated the study of semigroups in which prime ideals are maximal in [11] and some interesting results regarding the classical radical in the ring theoretic sense were obtained. In [10], Satyanarayana characterized commutative semigroups in which prime ideals are maximal and idempotent forms a chain under natural ordering. A class of semigroups, namely semi-pseudo symmetric semigroups, which includes the classes of one-sided duo semigroups, one-sided pseudo commutative semigroups and band was introduced by Anjaneyulu. Moreover, in [1] Anjaneyulu obtained a characterization for semi-pseudo symmetric semigroups with identity in which proper prime ideals are maximal and also characterized semi-pseudo symmetric semigroups without identity in which proper prime ideals are maximal and the family of globally idempotent principal ideals forms a chain which are a generalization of the results in [10]. The findings presented in this paper extend the results obtained in [1]. Let us recall some certain definitions and results used throughout this paper. A semigroup $(S, \cdot)$ together with a partial order $\leq$ that is *compatible* with the semigroup operation, meaning that, for any x, y, z in S,

$$x \leq y \text{ implies } zx \leq zy \text{ and } xz \leq yz$$

is called a *partially ordered semigroup* (or simply an *ordered semigroup*)(see [6]). Under the trivial relation, $x \leq y$ if and only if $x = y$, it is observed that every semigroup is an ordered semigroup. Let $(S, \cdot, \leq)$ be an ordered semigroup. For two nonempty subsets A, B of S, we write AB for the set of all elements xy in S where $x \in A$ and $y \in B$, and write $(A]$ for the set of all elements x in S such that $x \leq a$ for some a in A, i.e.,

$$(A] = \{x \in S \mid x \leq a \text{ for some } a \in A\}.$$

In particular, we write Ax for $A\{x\}$, and xA for $\{x\}A$. It was shown in [5] that the followings hold:

(1) $A \subseteq (A]$ and $((A]] = (A]$;

(2) $A \subseteq B \Rightarrow (A] \subseteq (B]$;

(3) $((A](B]] = ((A]B] = (A(B]] = (AB]$;

(4) $(A](B] \subseteq (AB]$;

(5) $(A]B \subseteq (AB]$ and $A(B] \subseteq (AB]$;

(6) $(A \cup B] = (A] \cup (B]$.

The concepts of left, right and two-sided ideals of an ordered semigroup can be found in [6]. Let $(S, \cdot, \leq)$ be an ordered semigroup. A nonempty subset A of S is called a *left* (resp., *right*) *ideal* of S if it satisfies the following conditions:

(i) $SA \subseteq A$ (resp., $AS \subseteq A$);

(ii) $A = (A]$, that is, for any x in A and y in S, $y \leq x$ implies $y \in A$.

If A is both a left and a right ideal of S, then A is called a *two-sided ideal*, or simply an *ideal* of S. It is known that the union or intersection of two ideals of S is an ideal of S.

An element a of an ordered semigroup $(S, \cdot, \leq)$, the *principal left* (*resp., right, two-sided*) *ideal* generated by a is of the form $L(a) = (a \cup Sa]$ (resp., $R(a) = (a \cup aS]$, $I(a) = (a \cup Sa \cup aS \cup SaS]$).

A nonempty subset B is called a *bi-ideal* of S if

(i) $BSB \subseteq B$;

(ii) $B = (B]$, that is, for any x in B and y in S, $y \leq x$ implies $y \in B$.

An element e of an ordered semigroup $(S, \cdot, \leq)$ is called an *identity element* of S if $ex = x = xe$ for any $x \in S$. The *zero element* of S, defined by Birkhoff, is an element 0 of S such that $0 \leq x$ and $0x = 0 = x0$ for all $x \in S$.

Let $(S, \cdot, \leq)$ be an ordered semigroup. A left ideal A of S is said to be *proper* if $A \subset S$. A proper right and two-sided ideals are defined similarly. S is said to be *left* (*resp., right*) *simple* if S does not contain proper left (resp., right) ideals. S is said to be *simple* if S does not contain proper ideals. S is said to be 0-*simple* if $S^2 \neq \{0\}$ and $\{0\}$ is the only proper ideal of S (see [3]). A proper ideal A of S is said to be *maximal* if for any ideal B of S such that $A \subset B \subseteq S$, then $B = S$.

Let $(S, \cdot, \leq)$ be an ordered semigroup. An ideal I of S is said to be *prime* if for any ideals A, B of S, $AB \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$. An ideal I of S is said to be *completely prime* if for any $a, b \in S$, $ab \in I$ implies $a \in I$ or $b \in I$. An ideal I of S is said to be *semiprime* if for any ideal A of S, $A^2 \subseteq I$ implies $A \subseteq I$. An ideal I of S is said to be *completely semiprime* if for any $a \in S$, $a^2 \in I$ implies $a \in I$. An ideal A of an ordered semigroup $(S, \cdot, \leq)$, the intersection of all prime ideals of S containing A, will be denoted by $Q^*(A)$ and we write

$$\overline{A} = \{x \in S \mid I(x)^n \subseteq A \text{ for some positive integer } n\}.$$

It is observed that $\overline{A} \subseteq Q^*(A)$. A subset A of an ordered semigroup $(S, \cdot, \leq)$, the radical of A, will be denoted by $\sqrt{A}$ defined by

$$\sqrt{A} = \{x \in S \mid x^n \in A \text{ for some positive integer } n\} \text{ (see [2]).}$$

Let $(S, \cdot, \leq)$ be an ordered semigroup. An ideal I of S is said to be *left*(*right*) *primary* if

(i) If A, B are ideals of S such that $AB \subseteq I$ and $B \nsubseteq I (A \nsubseteq I)$ implies $A \subseteq Q^*(I) (B \subseteq Q^*(I))$.

(ii) $Q^*(I)$ is a prime ideal (see [14]).

An ideal I of S satisfies condition (i) if and only if for every $x, y \in S$ such that $I(x)I(y) \subseteq I$ and $y \notin I (x \notin I)$, then $x \in Q^*(I) (y \in Q^*(I))$.

An ideal I of S is said to be *primary* if it is both the left and right primary ideal. An ideal I of S is said to be *semi-primary* if $Q^*(I)$ is a prime ideal. It is clear that every left(right) primary ideal is a semi-primary ideal. An ordered semigroup $(S, \cdot, \leq)$ is said to be (*left, right, semi-*) *primary* if every ideal of S is (left, right, semi-) primary.

An element a of an ordered semigroup $(S, \cdot, \leq)$ is called a *semisimple element* in S if $a \in (SaSaS]$. And S is said to be *semisimple* if every element of S is semisimple (see [12]).

An equivalence relation σ on S is called *congruence* if $(a,b) \in \sigma$ implies $(ac,bc) \in \sigma$ and $(ca,cb) \in \sigma$ for every $c \in S$. A congruence σ on S is called *semilattice congruence* if $(a^2,a) \in \sigma$ and $(ab,ba) \in \sigma$ for every $a,b \in S$. A semilattice congruence σ on S is called *complete* if $a \leq b$ implies $(a,ab) \in \sigma$. An ordered semigroup S is called a *semilattice of archimedean semigroups (resp., complete semilattice of archimedean semigroups)* if there exists a semilattice congruence (resp., complete semilattice congruence) σ on S such that the σ-class $(x)_\sigma$ of S containing x is a archimedean subsemigroup of S for every $x \in S$.

A subsemigroup F is called a *filter* of S if

(i) $a,b \in S$, $ab \in F$ implies $a \in F$ and $b \in F$;

(ii) if $a \in F$ and b in S, $a \leq b$, then $b \in F$ (see [7]).

For an element x of S, we denote by $N(x)$ the filter of S generated by x and $\mathcal{N}$ the equivalence relation on S defined by $\mathcal{N} := \{(x,y) \mid N(x) = N(y)\}$. The relation $\mathcal{N}$ is the least complete semilattice congruence on S.

An element a of an ordered semigroup $(S,\cdot,\leq)$ is called an *ordered idempotent* if $a \leq a^2$. We call an ordered semigroup S *idempotent ordered semigroup* if every element of S is an ordered idempotent (see [9]). The set of all ordered idempotents of an ordered semigroup S denoted by $E(S)$.

Let $(S,\cdot,\leq)$ be an ordered semigroup. A bi-ideal A of S is said to be *B-pure* if $A \cap (xS] = (xA]$ and $A \cap (Sx] = (Ax]$ for all $x \in S$. An ordered semigroup S is said to be *B^*-pure* if every bi-ideal of S is B-pure (see [13]).

An ordered semigroup $(S,\cdot,\leq)$ is called *archimedean* if for any a,b in S there exists a positive integer n such that $a^n \in (SbS]$ (see [12]). An ordered semigroup S is said to be *normal* if $(xS] = (Sx]$ for all $x \in S$ (see [13]). An ideal A of an ordered semigroup S is called *globally idempotent* if $A = (A^2]$ (see [4]). An ordered semigroup S is said to be *weakly commutative* if for any $a,b \in S$, then there exists positive integer n such that $(ab)^n \in (bSa]$ (see [7]).

An ordered semigroup $(S,\cdot,\leq)$ is said to be a *left(right) duo* if every left(right) ideal of S is a two-sided ideal of S. An ordered semigroup S is said to be a *duo* if it is both a left duo and a right duo.

Let $(S,\cdot,\leq_S)$, $(T,*,\leq_T)$ be an ordered semigroups, $f : S \to T$ a mapping from S into T. The mapping f is called *isotone* if $x,y \in S$, $x \leq_S y$ implies $f(x) \leq_T f(y)$ and *reverse isotone* if $x,y \in S$, $f(x) \leq_T f(y)$ implies $x \leq_S y$. The mapping f is called a *homomorphism* if it is isotone and satisfies $f(xy) = f(x) * f(y)$ for all $x,y \in S$. The mapping f is called a *isomorphism* if it is reverse isotone onto homomorphism. The ordered semigroups S and T are called *isomorphic*, in symbols $S \cong T$ if there exists an isomorphism between them.

An ordered semigroup V is called an *ideal extension*(or just an *extension*) of an ordered semigroup S by an ordered semigroup Q, if Q has a zero 0, $S \cap (Q \setminus \{0\}) = \emptyset$, and there exists an ideal K of V such that $K \cong S$ and $V/K \cong Q$

(see [8]).

Let $(S, \cdot, \leq)$ be an ordered semigroup and K an ideal of S. S/K is called the *Rees quotient ordered semigroup* of S, where 0 is an arbitrary element of K. It is observed that $K \cap [(S/K) \setminus \{0\}] = \emptyset$, $K \cong K$ and $S/K \cong S/K$ under the identity mapping and so S is an ideal extension of K by S/K.

2. MAIN RESULTS

First, we have the following definition.

Definition. Let $(S, \cdot, \leq)$ be an ordered semigroup. An ideal A of S is said to be *semi-pseudo symmetric* if for any $x \in S$ and for any positive integer n, $x^n \in A$ implies $I(x)^n \subseteq A$. An ordered semigroup S is said to be *semi-pseudo symmetric* if every ideal of S is a semi-pseudo symmetric ideal.

It is easy to see the following lemma:

Lemma 1. *An ordered semigroup $(S, \cdot, \leq)$ is a duo if and only if $(Sx] = (xS]$ for all $x \in S$.*

Corollary 2. *Let $(S, \cdot, \leq)$ be an ordered semigroup. Then S is a duo if and only if S is a normal.*

Proposition 3. *Every duo ordered semigroup is a semi-pseudo symmetric ordered semigroup.*

Proof. Let S be a duo ordered semigroup and A an ideal of S. For any $x \in S$ and for any positive integer n, $x^n \in A$. Let $b \in I(x)^n$. Then $b \leq s_1 x s_2 x \cdots x s_{n+1}$, where $s_i \in S$ or empty symbol. We have $b \leq sx^n$, where $s \in S$ or empty symbol by Lemma 1. This implies $b \in A$ and so $I(x)^n \subseteq A$. Thus S is a semi-pseudo symmetric. ■

Similarly, we prove the following.

Proposition 4. *Every idempotent ordered semigroup is a semi-pseudo symmetric ordered semigroup.*

Lemma 5 [14]**.** *Let $(S, \cdot, \leq)$ be an ordered semigroup and A an ideal of S. Then $Q^*(A) \subseteq \sqrt{A}$.*

Theorem 6. *Let $(S, \cdot, \leq)$ be an ordered semigroup and A an ideal of S. If A is a semi-pseudo symmetric, then $\overline{A} = Q^*(A) = \sqrt{A}$.*

Proof. As is easily seen, $\overline{A} \subseteq Q^*(A)$. We have $Q^*(A) \subseteq \sqrt{A}$ by Lemma 5. Since A is a semi-pseudo symmetric, $\sqrt{A} \subseteq \overline{A}$. Thus $\overline{A} = Q^*(A) = \sqrt{A}$. ■

Lemma 7 [14]**.** *Let $(S, \cdot, \leq)$ be an ordered semigroup with identity. If every (nonzero, assume this if S has 0) proper prime ideals are maximal, then S is a primary.*

Theorem 8. *Let $(S, \cdot, \leq)$ be a semi-pseudo symmetric ordered semigroup with identity. The following statements are equivalent:*

(1) *Proper prime ideals of S are maximal;*

(2) *S is either a simple and so an archimedean or S has a unique proper prime ideal P such that S is an ideal extension of the archimedean subsemigroup P by a 0-simple ordered semigroup S/P.*

In either case S is a primary ordered semigroup and S has at most one proper globally idempotent principal ideal.

Proof. As is easily seen, (2) implies (1).

(1)⇒(2). If S is a simple, then S is an archimedean. If S is not a simple, then S has a unique maximal ideal P and so P is a unique proper prime ideal. Since P is a maximal ideal, S/P is a 0-simple ordered semigroup by Lemma 1 in [8]. Let $a, b \in P$. Then $Q^*(I(a)) = P = Q^*(I(b))$. Since S is a semi-pseudo symmetric, we have $I(a)^n \subseteq I(b)$ for some positive integer n by Theorem 6. It follows that $a^{n+2} \in (PbP]$. Thus P is an archimedean subsemigroup of S.

We have S is a primary by Lemma 7. Let $I(a)$ and $I(b)$ be two proper globally idempotent principal ideals. Then $Q^*(I(a)) = P = Q^*(I(b))$. We have $(I(a)^n] \subseteq I(b)$ for some positive integer n. Since $I(a) = (I(a)^2]$, $I(a) \subseteq I(b)$. Similarly, we have $I(b) \subseteq I(a)$. Thus $I(a) = I(b)$. ■

Lemma 9 [14]**.** *Let $(S, \cdot, \leq)$ be an ordered semigroup. The following statements are equivalent:*

(1) *S is semisimple;*

(2) *$(A^2] = A$ for any ideal A of S;*

(3) *$A \cap B = (AB]$ for any ideals A, B of S;*

(4) *$I(a) \cap I(b) = (I(a)I(b)]$ for any $a, b \in S$;*

(5) *$(I(a)^2] = I(a)$ for any $a \in S$.*

Theorem 10. *Let $(S, \cdot, \leq)$ be a semi-pseudo symmetric ordered semigroup without identity. The following statements are equivalent:*

(1) *Proper prime ideals of S are maximal and the set of all globally idempotent principal ideals forms a chain under the set inclusion;*

(2) *S is an archimedean or there exists a unique proper prime ideal P such that S is an ideal extension of the archimedean subsemigroup P by a 0-simple ordered semigroup S/P;*

(3) *Proper prime ideals of S are maximal and S has at most two distinct globally idempotent principal ideals.*

If S has exactly two distinct globally idempotent principal ideals then one of their radicals is S.

Proof. The implication (3)⇒(1) is obvious.

(1)⇒(2). If S has no proper prime ideals. Then for any $a, b \in S$, $Q^*(I(a)) = S = Q^*(I(b))$. We have $I(a)^n \subseteq I(b)$ for some positive integer n. This implies $a^{n+2} \in (SbS]$. Thus S is an archimedean. If S has proper prime ideals. Let M and N be two proper prime ideals of S. Then M and N are maximal ideals of S. For any $x \in S \setminus M$, $I(x) \nsubseteq M$. Then $I(x)^2 \nsubseteq M$. Since S is a semi-pseudo symmetric, $x^2 \notin M$. We have $S = M \cup I(x) = M \cup I(x^2)$. This implies x is semisimple. Thus every element of $S \setminus M$ and $S \setminus N$ is semisimple. Let $a \in S \setminus M$ and $b \in S \setminus N$. Then $I(a)$ and $I(b)$ are globally idempotent by Lemma 9. We have $I(a) \subseteq I(b)$ or $I(b) \subseteq I(a)$. Suppose that $I(a) \subseteq I(b)$. If $b \in M$, then $a \in M$. This is a contradiction. Thus $b \in S \setminus M$. We have $I(a) = I(b)$. It follows that $M = N$. Thus S has a unique proper prime ideal, namely P. Since S is a semi-pseudo symmetric, S/P is a 0-simple ordered semigroup. By the same method given in Theorem 8 we have S is an ideal extension of the archimedean subsemigroup P by a 0-simple ordered semigroup S/P.

(2)⇒(3). If S is an archimedean. Let P be any prime ideal of S. Let $a \in P$ and $b \in S$. Then there exists positive integer n such that $b^n \in (SaS] \subseteq P$. Since S is a semi-pseudo symmetric, $I(b)^n \subseteq P$. This implies $b \in P$. It follows that $S = P$ and so S has no proper prime ideals. Thus proper prime ideals are maximal. Let $I(a)$ and $I(b)$ be two globally idempotent principal ideals. Then $Q^*(I(a)) = S = Q^*(I(b))$. Thus $I(a)^n \subseteq I(b)$ and $I(b)^m \subseteq I(a)$ for some positive integer n, m by Theorem 6. It follows that $I(a) \subseteq I(b)$ and $I(b) \subseteq I(a)$. Thus $I(a) = I(b)$. If S has a unique proper prime ideal P such that S is an ideal extension of the archimedean subsemigroup P by a 0-simple ordered semigroup S/P. Since S/P is 0-simple ordered semigroup, P is a maximal ideal. Then for any $a, b \in S \setminus P$, we have $I(a) = I(b)$ and $Q^*(I(a)) = S = Q^*(I(b))$. Let $I(a)$ and $I(b)$ be two proper globally idempotent principal ideals. Then $Q^*(I(a)) = P = Q^*(I(b))$ and so $I(a) = I(b)$. Thus S has at most two distinct globally idempotent principal ideals. Also if S has exactly two distinct globally idempotent principal ideals then one of their radicals is S. This completes the proof of the theorem. ■

Lemma 11 [14]**.** *Let $(S, \cdot, \leq)$ be an ordered semigroup. Then S is semi-primary if and only if the set of all prime ideals of S forms a chain under the set inclusion.*

Theorem 12. *Let $(S, \cdot, \leq)$ be a semi-pseudo symmetric ordered semigroup such that $S \neq (S^2]$. Then S is a primary in which proper prime ideals are maximal if and only if S is an archimedean.*

Proof. Assume that S is a primary in which proper prime ideals are maximal. Then S is a semi-primary. Thus the set of all prime ideals of S forms a chain under the set inclusion by Lemma 11. If S has proper prime ideals. Since proper prime ideals are maximal, S has a unique proper prime ideal P which is also a maximal ideal. Since every element of $S \setminus P$ is semisimple, we have $S \setminus P \subseteq (S^2]$. Let $a \in S \setminus P$ and $b \in P$. If $(I(a)I(b)] \neq I(b)$, then $b \notin (I(a)I(b)]$. Since S is a primary, we have $a \in Q^*((I(a)I(b)]) = P$. This is a contradiction. Thus $(I(a)I(b)] = I(b)$. This implies $P \subseteq (S^2]$ and so $S = (S^2]$. This is a contradiction. Thus S has no proper prime ideals. We have S is an archimedean follows from Theorem 10. Conversely, Assume that S is an archimedean. Then clearly S no proper prime ideals. Thus proper prime ideals are maximal. Let A be any ideal of S such that $I(x)I(y) \subseteq A$ and $y \notin A$. Since S is an archimedean, $x^n \in (SyS]$ for some positive integer n. Thus $x^{n+1} \in I(x)I(y) \subseteq A$. We have $x \in Q^*(A) = S$ by Theorem 6. Thus A is left primary. Similarly, A is right primary. Thus S is primary. ■

Corollary 13. *Let $(S, \cdot, \leq)$ be a normal ordered semigroup such that $S \neq (S^2]$. Then S is a primary in which proper prime ideals are maximal if and only if S is an archimedean.*

Theorem 14. *Let $(S, \cdot, \leq)$ be a weakly commutative ordered semigroup such that $(aS] = (a^2S]$ and $(Sa] = (Sa^2]$ for all a in S and $S \neq (S^2]$. Then S is a primary in which proper prime ideals are maximal if and only if S is an archimedean.*

Proof. It follows from Corollary 13 and Theorem 6 in [13]. ■

Theorem 15. *Let $(S, \cdot, \leq)$ be a B^*-pure ordered semigroup such that $S \neq (S^2]$. The following statements are equivalent:*

(1) *S is a primary in which proper prime ideals are maximal;*

(2) *S is an archimedean;*

(3) *$(SaS] = (SbS]$ for all $a, b \in S$;*

(4) *$(aS] = (bS]$ for all $a, b \in S$;*

(5) *$(aSa] = (bSb]$ for all $a, b \in S$;*

(6) *for any $e, f \in E(S)$, $(e, f) \in \mathcal{N}$;*

(7) *every bi-ideal of S is an archimedean subsemigroup.*

Proof. We have (1) and (2) are equivalent by Corollary 13 and Lemma 3 in [13] and (2) to (7) are equivalent by Theorem 12 in [13]. ■

Acknowledgements

The authors are deeply grateful to the referee for careful reading of the manuscript and helpful suggestions.

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Received 26 December 2023
Revised 17 June 2024
Accepted 17 June 2024