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# DUALITY FOR STONEAN HILBERT ALGEBRAS

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**Abstract**

A Stonean Hilbert algebra is a bounded Hilbert algebra with supremum that satisfies the Stone identity. In this paper we characterize the subdirectly irreducible Stonean Hilbert algebras. We extend the duality of Hilbert algebras with supreumum to bounded Hilbert algebras with supremum and we identify among the dual spaces of bounded Hilbert algebras with supremum those that correspond to Stonean Hilbert algebras in general, and, in particular, those that corresponds to sub-directly irreducible Stonean Hilbert algebras. As an application we exhibit a special partial endomorphism of the dual space of a Stonean Hilbert algebra.

**Keywords:** Hilbert algebra, duality, monoid of endomorphisms.

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## 1. Introduction

Hilbert algebras (positive implication algebras in [22]) are the algebraic counterpart of the implicative fragment of Intuitionistic Propositional Logic. A Hilbert algebra is an algebra $\langle A, \to, 1\rangle$ of type $(2,0)$. Diego in [8] proves that the class of Hilbert algebras is a variety generated by the $\{\to, 1\}$-reduct of Heyting algebras. We recall that a Heyting algebra is an algebra $\langle H, \vee, \wedge, \to, 0, 1\rangle$ of type $(2,2,2,0,0)$. But there are examples of algebras $\langle L, \vee, \wedge, \to, 0, 1\rangle$ which are not Heyting algebras but their $\{\to, 1\}$-reduct is a Hilbert algebra. These examples encourage the study of Hilbert algebra with lattice operations $(\vee, \wedge)$. The class of Hilbert algebras is a subclass of the class of BCK-algebras (see [9]); indeed, Hilbert algebras are dual isomorphic positive implicative BCK-algebras (see [18]) and Hilbert algebras with lattice operations are a particular case of BCK-algebras with lattice operations; this class of BCK-algebras has been studied by Idziak in [16] and [17]. A Hilbert algebra with supremum is an algebra $\langle A, \vee, \to, 1\rangle$ such that its natural order form a join-semi-lattice, i.e., $a \to b = 1$ iff $a \vee b = b$.

In this paper we study bounded Hilbert algebras with supremum (Hilbert algebras with supremum which have a least element for their natural order) satisfying the Stone identity. They were introduced in [7] and in [20] where they are called Stonean Hilbert algebras. Our main motivation is to answer the question up to what extent the structure of such a kind of Hilbert algebra is determined by the monoid of its endomorphisms. We have addressed the same question for the case of finite Hilbert algebras (see [12]) and for the case of Hilbert algebras generated by finite chains (see [14]). With such a purpose, building on the duality for (bounded) Hilbert algebras of Celani, Cabrer and Montangie (see [4, 5, 7]), in Section 5 (Theorem 9) we characterize the dual space of a Stonean Hilbert algebra and we identify the dual space of a subdirectly irreducible Stonean Hilbert algebra; previously, in Section 3 we characterize the subdirectly irreducible Stonean Hilbert algebras (Proposition 7 and Corollary 8).

The class of bounded Hilbert algebras with morphisms the algebraic homomorphisms is a category dually equivalent to the category of dual spaces of bounded Hilbert algebras with morphisms a special kind of partial functions. In the last section (Section 6) of the present paper we identify a partial endomorphism of the dual space of a Stonean Hilbert algebra; we think that this partial endomorphism will play a very important roll in establishing a connection between the structure of Stonean Hilbert algebras and the monoid of their endomorphisms. Section 2 will be devoted to recall the necessary definitions and known results whereas in Section 4 we present some examples which serve to illustrate the main concepts considered in this paper.

## 2. Preliminaries

In this section we provide the main definitions and several rules of computation that will be used throughout the paper. They can be consulted mainly in [3, 7, 11, 22]. A Hilbert algebra is an algebraic structure $\mathbf{A} = \langle A, \to, 1\rangle$ of type (2,0) that satisfies, for all $a, b, c \in A$ the following

(1) $$a \to (b \to a) = 1;$$

(2) $$(a \to (b \to c)) \to ((a \to b) \to (a \to c)) = 1;$$

(3) $$a \to b = 1 \text{ and } b \to a = 1 \text{ imply } a = b.$$

Following [3], we denote the class of Hilbert algebras by $\mathcal{H}$. The binary relation $\leq$ defined on $A$ by the rule $a \leq b$ iff $a \to b = 1$ is a partial order on $A$ with last element 1. We call this order the natural order induced on $\mathbf{A}$ by the operation '$\to$'. The following rules valid in any Hilbert algebra will be used without special reference

(4) $a \to a = 1$;

(5) $1 \to a = a$;

(6) $a \leq b \to a$;

(7) $a \to (a \to b) = a \to b$;

(8) $a \to (b \to c) = b \to (a \to c)$;

(9) $a \to (b \to c) = (a \to b) \to (a \to c)$;

(10) $a \leq b$ implies $b \to c \leq a \to c$ and $c \to a \leq c \to b$;

(11) $a \to b \leq (b \to c) \to (a \to c)$.

A *bounded Hilbert algebra* or $H_0$-algebra is a Hilbert algebra $\mathbf{A} := \langle A; \to, 1\rangle$ of type $(2, 0)$ for which there exists an element $0 \in A$ such that $0 \to x = 1$ for all $x \in A$. We shall write $\neg x$ instead of $x \to 0$. The class of bounded Hilbert algebras shall be denoted by $\mathcal{H}_0$. It is not difficult to check that the following properties are satisfied for all elements $a, b, c$ in any bounded Hilbert algebra

(12) $a \leq \neg\neg a$;

(13) $a \leq b \Longrightarrow \neg b \leq \neg a$;

(14) $\neg a = \neg\neg\neg a$;

(15) $a \to b \leq \neg b \to \neg a$;

(16) $\neg a = a \to \neg a$;

(17) $\neg a \to a = \neg\neg a$;

(18) $a \to \neg b = b \to \neg a$;

(19) $\neg\neg(a \to b) \leq \neg\neg a \to \neg\neg b$;

(20) $a \leq \neg a \to b$ and $b \leq \neg a \to b$.

All these properties of bounded Hilbert algebras can be consulted in [7] and the reference therein.

A non-empty subset $D$ of a Hilbert algebra $\mathbf{A}$ is called a *deductive system* if

(i) $1 \in D$, and

(ii) $a, a \to b \in D$ imply $b \in D$.

Deductive systems are called in [21] *implicative filters* or simply *filters*. We denote the set of deductive systems of a bounded Hilbert algebra $\mathbf{A}$ as follows

$$\mathcal{D}_s(\mathbf{A}) := \text{ deductive systems of } \mathbf{A}.$$

A proper deductive system $D$ is said to be *irreducible* if from $D = D_1 \cap D_2$ with $D_1, D_2 \in \mathcal{D}_s(\mathbf{A})$ it always follows that $D_1 = D$ or $D_2 = D$. The set of all irreducible deductive system of $\mathbf{A}$ is denoted by $X(\mathbf{A})$.

$$X(\mathbf{A}) := \text{ irreducible deductive systems of } \mathbf{A}.$$

A proper deductive system is called *maximal* if it is not contained properly in any other deductive system. Every maximal deductive system is also an irreducible deductive system (see [1], Remark 1.2). A Hilbert algebra is called a *local Hilbert algebra* if it has just a maximal deductive system.

An element $a$ of a bounded Hilbert algebra $\mathbf{A}$ is called *dense* if $\neg a = 0$. The set

$$D(A) := \{a \in A : \neg a = 0\}$$

of dense elements of $\mathbf{A}$ is a deductive system (see [1, 2]).

**Proposition 1** ([21], Proposition 3.3)**.** *Let $\mathbf{A} \in \mathcal{H}_0$. Then, $\mathbf{A}$ is a local Hilbert algebra iff all its elements except* 0 *are dense, i.e., $D(A) = A \setminus \{0\}$.*

A *bounded Hilbert algebra with supremum* or $H_0^\vee$-algebra is an algebra $\mathbf{A} := \langle A; \to, \vee, 1\rangle$ of type $(2, 2, 0)$ such that the reduct $\langle A; \to, 1\rangle$ is a bounded Hilbert algebra, the reduct $\langle A; \vee, 1\rangle$ is a join semi-lattice with last element 1 and the identities

(21) $$a \to (a \vee b) = 1,$$

(22) $$(a \to b) \to ((a \vee b) \to b) = 1$$

are satisfied. The class of bounded Hilbert algebras with supremum shall be denoted by $\mathcal{H}_0^\vee$. Hilbert algebras with supremum are called in [21], *sH*-Hilbert algebras.

The following identity is valid in any Hilbert algebra with supremum (see [11])

(23) $$(a \to c) \to ((b \to c) \to ((a \vee b) \to c)) = 1.$$

**Notation.** Let $\langle X, \leq\rangle$ a poset and $S \subseteq X$. Then $(S] := \{x \in X : x \leq s, \text{ some } s \in S\}$ and $[S) := \{x \in X : s \leq x, \text{ some } s \in S\}$.

**Definition 1** ([4], Definition 3.1)**.** A *Hilbert space* or *H-space* is a ordered topological space $\mathbf{X} := \langle X, \leq, \tau_{\mathcal{K}}\rangle$ such that

(i) $\mathcal{K}$ is a base of compact-open and decreasing subsets of $X$ for the topology $\tau_{\mathcal{K}}$ on $X$;

(ii) For every $A, B \in \mathcal{K}$, $(A \cap B^{\complement}] \in \mathcal{K}$. So, $\emptyset \in \mathcal{K}$;

(iii) For $x, y \in X$, $x \not\leq y$ implies that there exists $U \in \mathcal{K}$ such that $x \notin U$ and $y \in U$;

(iv) If $Y$ is a closed subset and $L \subseteq \mathcal{K}$ is dually directed set (i.e., for any $A, B \in L, \exists\, C \in L$ such that $C \subseteq A$ and $C \subseteq B$) such that $Y \cap U \neq \emptyset$ for all $U \in L$ then $\bigcap\{U : U \in L\} \cap Y \neq \emptyset$.

**Definition 2.** $\mathbf{X}$ is called a $H^{\vee}$-*space* if $\mathbf{X}$ is a $H$-*space* such that

(v) $U \cap V \in \mathcal{K}$ for all $U, V \in \mathcal{K}$.

**Definition 3.** A bounded $H^{\vee}$-*space* or $H_0^{\vee}$-*space* is a $H^{\vee}$-*space* such that $X \in \mathcal{K}$.

The set of increasing subsets of $X(\mathbf{A})$ ordered by inclusion (including there the empty set) is denoted by $\mathcal{P}_i(X(A))$.

It is shown in [8] (see also [6]) that

$$\mathcal{P}_i(X(\mathbf{A})) := \langle \mathcal{P}_i(X(A)); \rightarrow, \cup, X \rangle,$$

where the operation $\rightarrow$ is defined by the rule

$$U \rightarrow V := \left(U \cap V^{\complement}\right]^{\complement} \tag{24}$$

is a $H_0^{\vee}$-algebra and, if $\mathbf{A}$ is a $H_0^{\vee}$-algebra, then the mapping $\varphi : A \longrightarrow \mathcal{P}_i(X(A))$ given by

$$\varphi(a) = \big\{P \in X(A) : a \in P\big\} \tag{25}$$

is an injective homomorphism of $H_0^{\vee}$-algebras ([4], Lemma 5.1). Moreover,

$$\mathcal{K}_A := \left\{\varphi(a)^{\complement} : a \in A\right\} \tag{26}$$

is a basis for a topology $\tau_{\mathcal{K}_A}$ on $X(A)$ and $\mathbf{X}(A) := \langle X(A), \subseteq, \tau_{\mathcal{K}_A}\rangle$ is a $H^{\vee}$-space ([4], Theorem 5.6).

If $\mathbf{X} := \langle X, \leq, \tau_{\mathcal{K}}\rangle$ is an $H_0^{\vee}$-space then $D(\mathbf{X}) := \langle D(X); \rightarrow, \cup, X\rangle$, where

$$D(X) := \left\{U^{\complement} : U \in \mathcal{K}\right\}$$

and the operation $\rightarrow$ given by the formula (24) is a $H_0^{\vee}$-algebra (see [4], Proposition 5.3). The image of the mapping $\varphi$ given by the equality (25) is $D(X(A))$ so that

$$\varphi : \mathbf{A} \cong D(X(\mathbf{A})).$$

Observe that if $\mathbf{A} \in \mathcal{H}_0^{\vee}$, $\varphi(0) = \{P \in X(A) : 0 \in P\} = \emptyset = X^{\complement} \in D(X(A))$. As a consequence of the preceding discussion we have the following theorem.

**Theorem 2** ([6], Theorem 2.8)**.** *Let* $\mathbf{A} \in \mathcal{H}_0^{\vee}$*. Then, there exists a poset* $\mathbf{X} := \langle X, \leq \rangle$ *with maximum such that* $\mathbf{A}$ *is isomorphic to a subalgebra of*

$$\mathcal{P}_i(\mathbf{X}) := \langle \mathcal{P}_i(X); \rightarrow, \cup, X \rangle.$$

**Lemma 3** ([7], Lemma 9)**.** *Let* $\mathbf{A} \in \mathcal{H}_0^{\vee}$ *and* $P \in \mathcal{D}_s(\mathbf{A})$*. Then, the following conditions are equivalent:*

(i) $P$ *is maximal.*

(ii) $\forall\, a \in A,\ (a \notin P \Longrightarrow \neg a \in P)$.

(iii) $\forall\, a \in A,\ (a \notin P \Longrightarrow \neg\neg a \notin P)$.

(iv) $P \in X(\mathbf{A})$ *and* $D(A) \subseteq P$.

Diego in [8] proves that if $\mathbf{A} \in \mathcal{H}$, $P \in X(\mathbf{A})$ iff for every $a, b \in A$ such that $a, b \notin P$ there exists $c \notin P$ such that $a, b \leq c$. From this, the following result follows easily.

**Proposition 4.** *Let* $\mathbf{A} \in \mathcal{H}_0^{\vee}$ *and* $P \in \mathcal{D}_s(\mathbf{A})$*. Then,* $P \in X(\mathbf{A})$ *iff* $\forall\, a, b \in A$*,* $a \vee b \in P \Longrightarrow a \in P$ *or* $b \in P$*.*

Let $\mathbf{A} \in \mathcal{H}_0^{\vee}$. $\mathbf{A}$ is said to be an *Stone* $H_0^{\vee}$*-algebra* if it satisfies the *Stone identity*

$$\neg a \vee \neg\neg a = 1. \tag{27}$$

Stone $H_0^{\vee}$ algebras are called in [20], *Stonean Hilbert algebras.* It follows from Proposition 1 that a local bounded Hilbert algebra with supremum is necessarily a Stonean Hilbert algebra.

Several characterizations of this kind of $H_0^{\vee}$-algebras are given in [7]; for our purpose, we mention next two of them.

**Proposition 5** ([7], Theorem 26)**.** *Let* $\mathbf{A} \in \mathcal{H}_0^{\vee}$*. Then* $\mathbf{A}$ *is a Stone* $H_0^{\vee}$*-algebra iff for increasing subsets* $U, V$ *of* $X(A)$*, we have* $(U] \cap (V] = (U \cap V]$ *iff each irreducible deductive system of* $\mathbf{A}$ *is contained in a unique maximal deductive system.*

## 3. Sub-directly irreducible Stonean Hilbert algebras

**Proposition 6.** *For* $\mathbf{A} \in \mathcal{H}_0^{\vee}$ *and* $a \in A$*, the relation* $x \sim_a y$ *iff* $a \rightarrow x = a \rightarrow y$ *is a congruence relation on* $\mathbf{A}$*.*

***Proof.*** It is proved in [15] that $\sim_a$ is a equivalence relation on $A$ that preserves $\rightarrow$, i.e., $\sim_a$ is a congruence relation on $\langle A, \rightarrow, 1 \rangle$. Next we prove that $\sim_a$ also preserves $\vee$: suppose that $x \sim_a y$ and $z \sim_a w$, i.e., $a \to x = a \to y$ and $a \to z = a \to w$. It follows that $a \leq x \to y, y \to x, z \to w, w \to z$. We want $a \to (x \vee z) = a \to (y \vee w)$. We show first that $(a \to (x \vee z) \leq a \to (y \vee w)$. Set $c = y \vee w$. By (23) we have

$$(x \to c) \to ((z \to c) \to ((x \vee z) \to c)) = 1.$$

As $y \leq c$, we have $x \to y \leq x \to c$. Then we have

$$(x \to y) \to ((z \to c) \to ((x \vee z) \to c)) = 1$$

or, equivalently,

$$(z \to c) \to ((x \to y) \to ((x \vee z) \to c)) = 1.$$

As $w \leq c$, we have $z \to w \leq z \to c$ and therefore we obtain

$$(z \to w) \to ((x \to y) \to ((x \vee z) \to c)) = 1.$$

It follows from $a \leq z \to w$ and the above equation that

$$a \to ((x \to y) \to ((x \vee z) \to c)) = 1$$

or, equivalently,

$$(x \to y) \to (a \to ((x \vee z) \to c)) = 1$$

and, finally, since $a \leq x \to y$ we obtain

$$a \to ((x \vee z) \to c)) = a \to (a \to ((x \vee z) \to c)) = 1$$

and, from this, we get $a \to (x \vee z) \leq a \to c = a \to (y \vee w)$. In a similar way we obtain the reverse inequality. So, $a \to (x \vee z) = a \to (y \vee w)$. ■

**Proposition 7.** *$\mathbf{A} \in \mathcal{H}_0^{\vee}$ is sub-directly irreducible iff $\mathbf{A}$ has a unique co-atom, i.e., there exists $e \in A$ such that $e < 1$ and for all $x \in A$, if $x \neq 1$ then $x \leq e$.*

***Proof.*** Suppose that $A$ is sub-directly irreducible and let $\Upsilon$ be the monolith of $\mathbf{A}$. First we observe that $\sim_x = \Delta$ iff $x = 1$. Clearly, $\Upsilon = \mathrm{Cg}(e, b)$ (the smallest congruence containing the pair $(e, f)$) for some $e, b \in A$. If $\Delta \notin \{\sim_e, \sim_b\}$ then $\Upsilon = \mathrm{Cg}(e, b) \subseteq \sim_e \cap \sim_b$. But this means that $1 = e \to e = e \to b$ and $1 = b \to b = b \to e$, i.e., $e = b$, a contradiction. Then, say $\sim_b = \Delta$, i.e., $b = 1$, so $\Upsilon = \mathrm{Cg}(e, 1)$. Let $x \in A \setminus \{1\}$. As $\Upsilon \subseteq \sim_x$ we have that $(e, 1) \in \sim_x$, i.e., $x \to e = x \to 1 = 1$ and this means that $x \leq e$.

Conversely, suppose that $\mathbf{A}$ has a unique co-atom $e$. Let $\theta \in \mathrm{Con}(A) \setminus \{\Delta\}$. Let $x, y, x \neq y$ in $A$ such that $(x, y) \in \theta$. As $x \neq y$ we have that, say, $x \to y < 1$ so that $(x \to y) \to e = 1$. Observe now that $(x \to x = 1, x \to y) \in \theta$; consequently, $(1 \to e = e, (x \to y) \to e = 1) \in \theta$. Then, as $\theta \in \mathrm{Cong}(A) \setminus \{\Delta\}$ was arbitrary, we have proved that $\mathrm{Cg}(e, 1)$ is the monolith of $\mathbf{A}$ and, consequently, $\mathbf{A}$ is sub-directly irreducible. ■

The set of congruences of $\mathbf{A} \in \mathcal{H}^{\vee}$ is denoted by $Con(\mathbf{A})$. If $\theta \in Con(\mathbf{A})$, $[1]_\theta \in \mathcal{D}_s(\mathbf{A})$ ($[1]_\theta$ denote the congruence class of 1). If $D \in \mathcal{D}_s(\mathbf{A})$ then $\theta(D) = \{(a, b) \in A^2 : a \to b, b \to a \in D\} \in Con(\mathbf{A})$. If $\theta_1, \theta_2 \in Con(\mathbf{A})$, $\theta_1 \subseteq \theta_2 \Longrightarrow [1]_{\theta_1} \subseteq [1]_{\theta_2}$ and if $D_1, D_2 \in \mathcal{D}_s(\mathbf{A})$, $D_1 \subseteq D_2 \Longrightarrow \theta(D_1) \subseteq \theta(D_1)$ (see [3]). In the previous proposition, it is evident that $\{e, 1\}$ is a irreducible deductive system. Indeed, $\{e, 1\}$ is the smallest irreducible deductive system which, at the same time, is the smallest non-trivial deductive system of $\mathbf{A}$; then, having in mind Proposition 1 and Proposition 5 we have the following corollary.

**Corollary 8.** *A Stonean Hilbert algebra is sub-directly irreducible iff it is a local bounded Hilbert algebra with supremum that has a smallest non-trivial deductive system which is also the smallest irreducible deductive system.*

From a result of Idziak (see [16]) it follows that the class of Hilbert algebras with supremum is a variety. Here, we need to consider the class of Stone $H_0^{\vee}$-algebras as a variety. Indeed, this class of Hilbert algebras with supremum is closed under the formation of homomorphic images and direct product but it is not closed under the formation of sub-algebras, as the Stonean Hilbert algebra $\mathbf{A}_0$ (taking from [20]) shows:

$\mathbf{A}_0$:=

| → | 0 | a | b | c | d | e | f | g | 1 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| a | f | 1 | 1 | f | 1 | 1 | f | 1 | 1 |
| b | f | g | 1 | f | g | 1 | f | g | 1 |
| c | b | b | b | 1 | 1 | 1 | 1 | 1 | 1 |
| d | 0 | b | b | f | 1 | 1 | f | 1 | 1 |
| e | 0 | a | b | f | g | 1 | f | g | 1 |
| f | b | b | b | e | e | e | 1 | 1 | 1 |
| g | 0 | b | b | c | e | e | f | 1 | 1 |
| 1 | 0 | a | b | c | d | e | f | g | 1. |

We see that $\{a, b, c, d, e, f, g, 1\}$ is a subalgebra of such a bounded Hilbert algebra with supremum which is not even Stonean since it does not have a minimum. So, in order to consider the class of Stonean Hilbert algebra as a variety, the minimum 0 has to be considered as a nullary operation. This automatically implies that the unary operation $\neg$ is preserved by Hilbert algebra homomorphisms which, by the way, being them order preserving maps, have to send minimums to minimums, i.e., they have to preserve the least element (see [10]). Now, since there is a one to one an onto correspondence between congruences and homomorphic images then any sub-directly irreducible Stonean Hilbert algebra is also sub-directly irreducible as a Hilbert algebra.

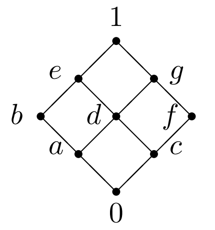


Figure 1. The natural order of $\mathbf{A}_0$.

## 4. EXAMPLES

**Example I.**

$\mathbf{A}_1 :=$

| $\to$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 5 | 5 | 5 | 5 | 5 | 5 |
| 1 | 0 | 5 | 5 | 5 | 5 | 5 |
| 2 | 0 | 3 | 5 | 3 | 5 | 5 |
| 3 | 0 | 2 | 2 | 5 | 5 | 5 |
| 4 | 0 | 1 | 2 | 3 | 5 | 5 |
| 5 | 0 | 1 | 2 | 3 | 4 | 5 |

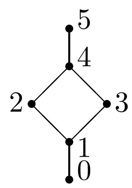


Figure 2. The natural order of $\mathbf{A}_1$.

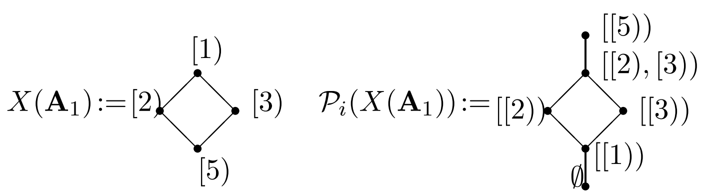


Figure 3. The order of the irreducible deductive systems of $\mathbf{A}_1$.

Observe that $\mathbf{A}_1 \cong \mathcal{P}_i(X(\mathbf{A}_1)))$. Observe also that $B := \{0,2,3,4,5\}$ is a subuniverse of $\mathbf{A}_1$ and that $X(\mathbf{A}_1) = X(\mathbf{B})$, $\mathbf{B}$ being a proper subalgebra of $\mathcal{P}_i(X(\mathbf{B}))$. Notice that $\mathbf{A}_1$ as well as $\mathbf{B}$ are local sub-directly irreducible Stonean Hilbert algebras.

**Example II.**

$\mathbf{A}_2 :=$

| $\to$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 3 | 3 | 3 | 3 |
| 1 | 2 | 3 | 2 | 3 |
| 2 | 1 | 1 | 3 | 3 |
| 3 | 0 | 1 | 2 | 3 |

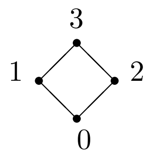


Figure 4. The natural order of $\mathbf{A}_2$.

$X(A_2) = \left\{(1]^{\complement}, (2]^{\complement}\right\} = \{[2), [1)\}$. In this example, $X(\mathbf{A}_2)$ is an anti-chain, does not have a maximum and does not have a minimum.

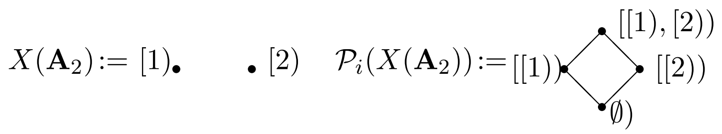


Figure 5. The order of the irreducible deductive systems of $\mathbf{A}_2$.

Observe that $\mathbf{A}_2$ is a Stonean Hilbert algebra, neither local nor subdirectly irreducible.

**Example III.**

$\mathbf{A}_3 :=$

| $\to$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 4 | 4 | 4 | 4 | 4 |
| 1 | 0 | 4 | 4 | 4 | 4 |
| 2 | 0 | 3 | 4 | 3 | 4 |
| 3 | 0 | 2 | 2 | 4 | 4 |
| 4 | 0 | 1 | 2 | 3 | 4 |

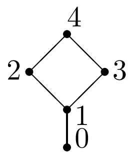


Figure 6. The natural order of $\mathbf{A}_3$.

$X(\mathbf{A}_3) = \{(0]^{\complement}, (2]^{\complement}, (3]^{\complement}\} = \{[1), [3), [2)\}$. In this example, $X(\mathbf{A}_3)$ has a maximum but it does not have a minimum.

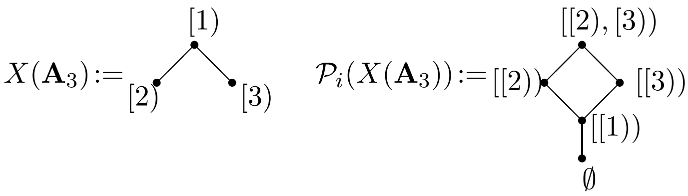


Figure 7. The order of $X(\mathbf{A}_3)$.

**Example IV.** A bounded Hilbert algebra with supremum which is not Stonean.

$\mathbf{A}_4$:=

| $\to$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 5 | 5 | 5 | 5 | 5 | 5 |
| 1 | 2 | 5 | 2 | 5 | 5 | 5 |
| 2 | 1 | 1 | 5 | 3 | 5 | 5 |
| 3 | 0 | 1 | 2 | 5 | 5 | 5 |
| 4 | 0 | 1 | 2 | 3 | 5 | 5 |
| 5 | 0 | 1 | 2 | 3 | 4 | 5 |

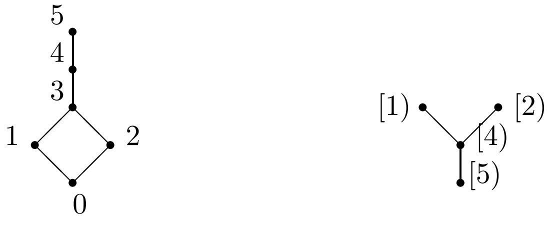


Figure 8. The natural order of $\mathbf{A}_4$ and $X(\mathbf{A}_4)$.

## 5. Dual space of a Stonean Hilbert algebra

Next we characterize Stone $H_0^{\vee}$-algebra in terms of its dual $H_0^{\vee}$-space, more precisely, in terms of the inclusion order of its irreducible deductive systems.

**Theorem 9.** *The dual $H_0^{\vee}$-algebra of a $H_0^{\vee}$-space $\mathbf{X} := \langle X, \leq, \tau_{\mathcal{K}} \rangle$ is a Stone $H_0^{\vee}$-algebra iff in case $X$ has a minimum then it has a maximum and, in case $X$ does not have a minimum then the poset $X \smallsetminus \{m\}$ ($m$, if any, is the top element of $X$) is a direct sum of a family $\{X_i : i \in I\}$ of disjoint posets ($X_i \cap X_j = \emptyset$ for $i, j \in I$ with $i \neq j$) such that each $X_i$ has a maximum $m_i$. This means that for $x, y \in X$ to be comparable, they have to belong to the same $X_i$. In symbols,*

$$X = \dot{\bigcup}_{i \in I} X_i, \quad \mathbf{X} := \bigoplus_{i \in I} \mathbf{X}_i.$$

***Proof.*** For the sufficiency part, based on Theorem 2, it is enough to prove the result for the $H_0^{\vee}$-algebra $\mathcal{P}_i(\mathbf{X})$. So let $U \in \mathcal{P}_i(X)$. We consider two cases:

*Case* 1. *$X$ does not have a top element.* Clearly,

$$U = \dot{\bigcup}_{j \in J} (U \cap X_j) \text{ some } J \subseteq I.$$

Then

$$\neg U = U \to \emptyset = \left( U \cap \emptyset^{\mathsf{C}} \right]^{\mathsf{C}} = (U \cap X]^{\mathsf{C}} = (U]^{\mathsf{C}} = \left( \dot{\bigcup}_{j \in J} (U \cap X_j) \right]^{\mathsf{C}} = \dot{\bigcup}_{j \notin J} X_j$$

and

$$\neg\neg U = \neg U \to \emptyset = \left( \dot{\bigcup}_{j \notin J} X_j \right]^{\mathsf{C}} = \left( \dot{\bigcup}_{j \notin J} X_j \right)^{\mathsf{C}} = \dot{\bigcup}_{j \in J} X_j.$$

So, $\neg U \cup \neg\neg U = \dot{\bigcup}_{i \in I} X_i = X$.

*Case* 2. *$X$ has a top element $m$.* In this case it is enough to observe that, as $m \in U \in \mathcal{P}_i(X)$ then $\neg U = U \to \emptyset = (U \cap \emptyset^{\mathsf{C}}]^{\mathsf{C}} = (U]^{\mathsf{C}} = X^{\mathsf{C}} = \emptyset$ and obviously, $\neg\neg U = X$ so, $\neg U \cup \neg\neg U = \emptyset \cup X = X$.

For the necessity we have into account Proposition 5. Just observe that if the order on $X(\mathbf{A})$ does not look like the direct sum just described then there exist two distinct co-atoms $x, y \in X(A)$ and a third element $z \in X(A)$ such that $z \leq x, y$. Then, $([x) \cap [y)] = \emptyset$ whereas $\emptyset \neq (z] \subseteq ([x)] \cap ([y)]$. ■

**Corollary 10.** *The $H^{\vee}$-space described in the previous theorem has a top element $m$ iff the corresponding Stone $H_0^{\vee}$-algebra is local.*

**Corollary 11.** *Let $\mathbf{A} \in \mathcal{H}_0^{\vee}$. Then $A$ is Stonean iff, $\forall P \in X(A)$ one of the following things occurs:*

(i) $P \nsubseteq D(A)$,

(ii) $P \subseteq D(A)$.

*In the first case,* **A** *is not a local Hilbert algebra. In the second case,* **A** *is a local Hilbert algebra.*

***Proof.*** It is known that $D(A) = \bigcap Max(A)$ where $Max(A)$ denotes the set of all maximal filters of **A** (see [1]). If **A** is local, it is off course Stonean and in this case, $D(A)$ is the unique maximal filter of **A**. ∎

**Proposition 12.** *A Stone $H_0^{\vee}$-algebra* **A** *is sub-directly irreducible iff its dual space* **X** *has a minimum and consequently, $\mid I \mid= 1$.*

***Proof.*** It follows at once from Theorem 9 and Corollary 8. ∎

**Proposition 13.** *Any sub-directly irreducible Stonean Hilbert algebra* **A** *is local.*

***Proof.*** By Proposition 7, **A** has a co-atom, name it $e$. If $A$ is not local, then $D(A) \neq A \setminus \{0\}$. Choose $a \notin D(A)$ such that $a \neq 0$. Note that $\neg a = 1 \Longrightarrow a \leq \neg\neg a = 0 \Longrightarrow a = 0$. So $\neg a \leq e$. Note also that $\neg\neg a = 1 \Longrightarrow \neg a = 0$. So $1 = \neg a \vee \neg\neg a \leq e$, a contradiction. ∎

**Remark.** The converse of the previous proposition is not true, the Stone-$H_0^{\vee}$ algebra $\mathbf{A}_3$ of example III is local but not sub-directly irreducible.

It is clear that Stonean Hilbert algebras form a subvariety of the variety of bounded Hilbert algebra with supremum. We call the $\mathcal{H}_0^{\vee}$-spaces referred in Theorem 9, Stone H-spaces and we denote this class of $H$-spaces by $H_{st}$-spaces.

Summing up, we have that the $H_{st}$-space that corresponds to a local Stonean Hilbert algebra has to have a maximum and if it corresponds to a sub-directly irreducible Stonean Hilbert algebra must have a minimum. The $H_{st}$-space of a non-local Stonean Hilbert algebra must be the disjoint union (direct sum) of at least two $H_{st}$-spaces corresponding to local Stonean Hilbert algebras. In particular, it possesses neither maximum nor minimum. In case it possess minimum but not maximum, it does not even correspond to a Stonean Hilbert algebra.

## 6. $H$-PARTIAL FUNCTIONS

We begin this section extending the concept of $H$-partial function for $H^{\vee}$-spaces given in [4] to $H_0^{\vee}$-spaces. Let $\mathbf{X}_1 := \langle X_1; \leq, \tau_{\mathcal{K}_1} \rangle$ and $\mathbf{X}_2 := \langle X_2; \leq, \tau_{\mathcal{K}_2} \rangle$ be two $H_0^{\vee}$-spaces.

**Definition 4.** A partial map $f : X_1 \longrightarrow X_2$ with domain denoted by $dom(f)$ is said to be an $H_0$-*partial function* if the following conditions are satisfied:

(i) $[f(x)) = f([x))$ for each $x \in dom(f)$;

(ii) $[x) \cap dom(f) = \emptyset$ for each $x \notin dom(f)$ and $(x] \subseteq dom(f)$ if $x \in dom(f)$;

(iii) $(f^{-1}(U)] \in \mathcal{K}_1$ for each $U \in \mathcal{K}_2 \smallsetminus \{X_2\}$;

(iv) $f^{-1}(X_2) = X_1$.

Conditions (i) and (iii) of this definition are conditions (1) and (3) of Definition 6.1 in [4]; our condition (ii) make more precise condition (2) of the mentioned definition.

The variety $\mathcal{H}_0^{\vee}$ may be viewed as the category with objects the $H_0^{\vee}$-algebras and morphisms the algebraic homomorphisms (they must preserve 0). Following the ideas of Celani and Montangie in [4], it is easy to show that this category and the category with objects the $H_0^{\vee}$-spaces and morphisms the $H_0$-partial functions are dually equivalent. The details of this duality can be consulted in [4]. Here we will describe the dual space of a given $H_0^{\vee}$-algebra and the dual algebra of a given $H_0^{\vee}$-space.

Let $\mathbf{A} \in \mathcal{H}_0^{\vee}$. For $a \in A$ define

$$\varphi(a) := \{P \in X(\mathbf{A}) : a \in P\}.$$

It has been shown that $\mathcal{K}_A := \{\varphi(a)^{\complement} : a \in A\}$ is a basis for a topology $\tau_{\mathcal{K}_A}$ on $X(\mathbf{A})$ and $\mathbf{X}(\mathbf{A}) := \langle X(\mathbf{A}); \subseteq, \mathcal{K}_A\rangle$ is an $H_0^{\vee}$-space called the dual space of $\mathbf{A}$. For a given $H_0^{\vee}$-space $\mathbf{X} = \langle X; \leq, \mathcal{K}\rangle$ consider the set $D(\mathbf{X}) := \{U^{\complement} : U \in \mathcal{K}\}$. Then, $\mathbf{D}(\mathbf{X}) := \langle D(\mathbf{X}); \Rightarrow, \cup, X\rangle$ with the operation $\Rightarrow$ given by the formula

$$U \Rightarrow V := \left(U \cap V^{\complement}\right]^{\complement} = \{x \in X : [x) \cap U \subseteq V\}$$

is an $H_0^{\vee}$-algebra which is called the dual $H_0^{\vee}$-algebra of $\mathbf{X}$.

Let $h : \mathbf{A}_1 \longrightarrow \mathbf{A}_2$ be an homomorphism of $H_0^{\vee}$-algebras. Then, the map $h_X : \mathbf{X}(\mathbf{A}_2) \longrightarrow \mathbf{X}(\mathbf{A}_1)$ given by the formula

$$h_X(P) = h^{-1}(P)$$

is an $H_0$-partial function with domain $\{P \in X(\mathbf{A}_2) : h^{-1}(P) \in X(\mathbf{A}_1)\}$ called the dual $H_0$-partial function of $h$.

Let $f : \mathbf{X}_1 \longrightarrow \mathbf{X}_2$ be an $H_0$-partial function. Then, the map $f_D : \mathbf{D}(\mathbf{X}_2) \longrightarrow \mathbf{D}(\mathbf{X}_1)$ given by the formula

$$f_D(U) = \left(f^{-1}\left(U^{\complement}\right)\right]^{\complement}$$

is a homomorphism of $H_0^{\vee}$-algebras called the dual homomorphism of $f$.

The following results were proved in [14] for $\mathbf{X}$ a $H^{\vee}$-space; they remain valid when considering $\mathbf{X}$ to be a $H_0^{\vee}$-space.

**Proposition 14** (Proposition 8, [14]). *Let* $\mathbf{X} := \langle X; \leq, \tau_{\mathcal{K}} \rangle$ *be a* $H_0^{\vee}$*-space. Then, the image* $im(g)$ *of an* $H_0$*-partial endomorphism* $g$ *of* $\mathbf{X}$ *is an increasing set and if* $g$ *is idempotent then its domain,* $dom(g)$*, is equal to* $(im(g)]$*. Consequently, if* $t \in dom(g), t \leq g(t)$.

**Corollary 15** (Corollary 9, [14]). *Let* $\mathbf{X} := \langle X; \leq, \tau_{\mathcal{K}} \rangle$ *be a* $H_0^{\vee}$*-space. Let* $f$ *and* $g$ *be idempotent* $H_0$*-partial endomorphisms of* $\mathbf{X}$*. Then* $f = g$ *iff* $im(f) = im(g)$.

**Proposition 16.** *Let* $\mathbf{X} = \langle X, \leq \tau_{\mathcal{K}} \rangle$ *be a Stone* $H$*-space (*$H_{st}$*-space) and let* $U \in \mathcal{K}$*. For* $x \in X$ *such that* $U^{\complement} \cap [x) \neq \emptyset$*, define* $f_U(x) = m$ *where* $m$ *the unique maximal element of* $X \setminus \{u\}$ *(*$u$ *is the maximum of* $X$ *if any) above* $x$*. Then* $f_U$ *is an idempotent* $H_0$*-partial endomorphism.*

***Proof.*** To see that $f_U$ is well defined we recall that $U^{\complement}$ is an increasing set and since $U^{\complement} \cap [x) \neq \emptyset$, then taking into account the order structure of $\mathbf{X}$ described in Theorem 9, there is a unique maximal element $m$ of $X$ such that $x \leq m$. Clearly, $f_U$ is idempotent and if $x \in dom(f_U)$, $f_U([x)) = [f_u(x)) = \{m\}$. Let $x \notin dom(f_U)$ and $t \in [x)$. Then $[t) \cap U^{\complement} \subseteq [x) \cap U^{\complement} = \emptyset$ which means that $t \notin dom(f_U)$. This shows that $[x) \cap dom(f_U) = \emptyset$. Finally, it is easy to check that if $V \in \mathcal{K}$ then $f_U^{-1}(V) = (U^{\complement} \cap V] \in \mathcal{K}$. ■

**Conclusion and future research.** In this paper we have characterized the sub-directly irreducible Stonean Hilbert algebras and we have described the dual $H_0^{\vee}$-space of a Stonean Hilbert algebra. The relation between a universal algebra and the monoid of its endomorphisms was considered first in [19]. A bounded Hilbert algebra with supremum generated by finite chains is determined by the monoid of their endomorphisms (see [14]). In achieving such a result, the equivalence between the category of $H^{\vee}$-spaces with morphisms $H$-partial functions and the category of bounded Hilbert algebras with morphisms the algebraic homomorphisms was a powerful tool. It follows from Theorem 9 that Hilbert algebras generated by finite chains are Stonean Hilbert algebras. The class of $H_0$-partial endomorphisms of Stone $H$-spaces exhibited in Proposition 16 we think will be very useful to determine up to what extent a Stonean Hilbert algebra is determined by the monoid of its endomorphisms; for instance, it follows from Proposition 14 and Corollary 15 that if $\mathbf{A}$ is a Stonean Hilbert algebra, the constant map with image $\{1\}$ is an (idempotent) endomorphism iff $\mathbf{A}$ is subdirectly irreducible.

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