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k-IDEALS AND *k*-$\{^+\}$-CONGRUENCES OF CORE REGULAR DOUBLE STONE ALGEBRAS

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Abstract

In this paper, the authors study many interesting properties of ideals and congruences of the class of a core regular double Stone algebra (briefly CRD-Stone algebra). We introduce and characterize the concepts of k-ideals and principal k-ideals of a core regular double Stone algebra with the core element k and establish the algebraic structures of such ideals. Also, we investigate k-$\{^+\}$-congruences and principal k-$\{^+\}$-congruences of a CRD-Stone algebra L which are induced by k-ideals and principal k-ideals of L, respectively. We obtain an isomorphism between the lattice of k-ideals (principal k-ideals) and the lattice of k-$\{^+\}$-congruences (principal k-$\{^+\}$-congruences) of a CRD-Stone algebra. We provide some examples to clarify the basic results of this article.

Keywords: stone algebras, double Stone algebras, regular double Stone algebras, core regular double Stone algebras, ideals, filters.

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1. Introduction

The concept of psudo-complement was considered in semi-lattices and distributive lattices by Frink [22] and Birkhof [12], respectively. The class **S** of Stone algebras was studied and characterized by several authors, like, Badawy [1], Chain and

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Grätzer [18, 19], Grätzer [23], Frink [22], Balbes [13] and Katrinák [25]. Regular double p-algebras and regular double Stone algebras are characterized by Katrinák [25] and Comer [21], respectively.

The intersection of the set $D(L)$ of dense elements and the set $\overline{D(L)}$ of dual dense elements of a double Stone algebra L is called the core of L and denoted by $K(L)$. In a regular double Stone algebra L, the core $K(L)$ is either an empty set or a singleton set, if a regular double Stone algebra L has a non-empty core, then such a core $K(L)$ has exactly only one element, which is denoted by k. Ravi Kumar *et al.* [27] introduced some properties of core regular double Stone algebra Srikanth *et al.* [28] and [29] studied many properties of ideals (filters) and congruences of a core regular double Stone algebras, respectively. Badawy *et al.* [9] constructed a double Stone algebra from a Stone quadruple. Badawy [3] constructed each core regular Stone algebra from a suitable Boolean algebra $B = (B; \vee, \wedge, ', 0, 1)$. The constructing CRD-Stone algebra $(B^{[2]}; \vee, \wedge, ^*, ^+, (0,0), (1,1))$ with the core element $(0,1)$, where

$$
\begin{aligned}
B^{[2]} &= \{(x, y) \in B^{[2]} : \ x \leq y\}, \\
(x, y) \wedge (x_1, y_1) &= (x \wedge x_1, y \wedge y_1), \\
(x, y) \vee (x_1, y_1) &= (x \vee x_1, y \vee y_1), \\
(x, y)^* &= (y', y'), \\
(x, y)^+ &= (x', x').
\end{aligned}
$$

In Section 2, We list the basic concepts and important results which are needed throughout this paper. Also, we provide some examples of RD-Stone algebras with core element k and RD-Stone algebras with empty core. We refer the reader to [4, 7, 8, 10, 15] and [16] for filters, ideals and [2, 6, 11] for congruences of lattices and p-algebras.

In Section 3, we introduce the k-ideals of a CRD-Stone algebra L and obtain many related properties. A set of equivalent conditions for an ideal I of a CRD-Stone algebra L to become a k-ideal is given. We observe that the class $I_k(L)$ of all k-ideals of L forms a bounded distributive lattice.

In Section 4, we define and characterize the concept of principal k-ideals of a CRD-Stone algebra L. We show that the class $I_k^p(L)$ of all principal k-ideals of L is a Boolean ring and so a Boolean algebra. Example 25 describes the Boolean algebra $I_k^p(L)$.

In Section 5, we investigate the k-$\{^+\}$-congruences via k-ideals of a CRD-Stone algebra L. Also, we observe that the set $Con_k^+(L)$ of all k-$\{^+\}$-congruences forms a bounded distributive lattice which is isomorphic to the lattice $I_k(L)$ of k-ideals.

In Section 6, we investigate and charaterize the principal k-$\{^+\}$-congruences of a CRD-Stone algebra L via principal k-ideals of L. Then, we study the properties and the algebraic structure of the class $Con_k^p(L)$ of all principal k-$\{^+\}$-

congruences of L. Moreover, we show that $I_k^p(L)$ and $Con_k^p(L)$ are isomorphic Boolean algebras. We give Example 42 to clarify the last result.

2. Preliminaries

In this section, we recall certain definitions and results which are used throughout the paper, which are taken from the references [1, 5, 14, 21, 23, 27, 28] and [30].

Definition 1 [1]**.** An algebra $(L;\wedge,\vee)$ of type $(2,2)$ is said to be a lattice if

(1) the operations $\wedge,\vee$ are idempotent, commutative and associative,

(2) the absorption identities hold on L, that is, $(a\wedge b)\vee a=a, (a\vee b)\wedge a=a$.

Definition 2 [14]**.** A lattice L is called a bounded if it has the greatest element 1 and the smallest element 0.

Definition 3 [1]**.** A lattice L is called a distributive lattice if it satisfies either of the following equivalent distributive laws:

(1) $a\wedge(b\vee c)=(a\wedge b)\vee(a\wedge c)$,

(2) $a\vee(b\wedge c)=(a\vee b)\wedge(a\vee c)$, for all $a,b,c\in L$.

Definition 4 [28]**.** A nonempty subset I of a lattice L is called an ideal if

(1) $x\vee y\in I$ for all $x,y\in I$,

(2) $x\in I$ and $z\in L$ be such that $z\leq x$ imply $z\in I$.

Definition 5 [23]**.** If $\phi\neq A\subseteq L$, then $(A]$ is the smallest ideal of a lattice L which contains A, where $(A]=\{x\in L: x\leq a_1\vee a_2\vee\cdots\vee a_n,\ a_i\in A,\ i=1,2,\dots,n\}$.

The case that $A=\{a\}$, we write $(a]$ instead of $(\{a\}]$ and $(a]$ is called the principal ideal of L generated by a, where $(a]=\{x\in L: x\leq a\}$.

Let $I(L)$ be the set of all ideals of a lattice L. Then $(I(L);\wedge,\vee)$ forms a lattice, where

$$I\wedge J=I\cap J \text{ and } I\vee J=\{x\in L: x\leq i\vee j: i\in I, j\in J\}.$$

Also, algebra $(I^p(L);\vee,\wedge)$ of all principal ideals of L is a sublattice of the lattice $I(L)$, where

$$(a]\vee(b]=(a\vee b] \text{ and } (a]\wedge(b]=(a\wedge b].$$

It is known that the lattice $I(L)$ is distributive if and only if L is distributive.

Definition 6 [1]**.** For any element a of a bounded lattice L, the dual pseudo-complement a^+ (the pseudo- complement a^*) of a is defined as follows

$$a\vee x=1\Leftrightarrow a^+\leq x\ \ (a\wedge x=0\Leftrightarrow x\leq a^*).$$

Definition 7 [23]**.** A distributive lattice L in which every element has a pseudocomplement is called a distributive pseudo-complemented lattice or a distributive p-algebra. Dually, a distributive lattice L in which every element has a dual pseudocomplement is called a distributive dual pseudocomplement lattice or dual distributive p-algebra.

Definition 8 [5]**.** A distributive p-algebra (distributive dual p-algebra) L is called a Stone algebra (dual Stone algebra) if $x^* \vee x^{**} = 1$ $(x^+ \wedge x^{++} = 0)$ for all $x \in L$.

Theorem 1 [1]**.** *Let L be a distributive p-algebra (distributive dual p-algebra). Then for any two elements a, b of L, we have*

(1) $0^{**} = 0$ *and* $1^{**} = 1$ $(0^{++} = 0$ *and* $1^{++} = 1)$,
(2) $a \wedge a^* = 0$ $(a \vee a^+ = 1)$,
(3) $a \leq b$ *implies* $b^* \leq a^*$ $(a \geq b$ *implies* $b^+ \geq a^+)$,
(4) $a \leq a^{**}$ $(a^{++} \leq a)$,
(5) $a^{***} = a^*$ $(a^{+++} = a^+)$,
(6) $(a \vee b)^* = a^* \wedge b^*$ $((a \wedge b)^+ = a^+ \vee b^+)$,
(7) $(a \wedge b)^* = a^* \vee b^*$ $((a \vee b)^+ = a^+ \wedge b^+)$,
(8) $(a \vee b)^{**} = a^{**} \vee b^{**}$ $((a \wedge b)^{++} = a^{++} \wedge b^{++})$,
(9) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$ $((a \vee b)^{++} = a^{++} \vee b^{++})$.

Definition 9 [30]**.** A Double Stone-algebra L is an algebra $\langle L,^*,^+ \rangle$, where

(i) $(L,^*)$ is a Stone algebra,
(ii) $(L,^+)$ is a dual Stone algebra.

Definition 10 [21]**.** A regular double Stone algebra (briefly RD-Stone algebra) L is a double Stone such that

$$x^{**} = y^{**} \text{ and } x^{++} = y^{++} \text{ imply } x = y.$$

Let L be a double Stone algebra. The element $a \in L$ is called a closed element of L if $a^{**} = a$ and the element $a \in L$ is called a dual closed element of L if $a^{++} = a$. An element $d \in L$ is called dense if $d^* = 0$ and an element $d \in L$ is called dual dense if $d^+ = 1$.

Lemma 2 [28]**.** *Let L be a double Stone algebra. Then*

(1) *the set* $D(L) = \{a \in L \mid a^* = 0\} = \{a \vee a^* \mid a \in L\}$ *of all dense elements of L is a filter of L,*
(2) *the set* $\overline{D(L)} = \{a \in L \mid a^+ = 1\} = \{a \wedge a^+ \mid a \in L\}$ *of all dual dense elements of L is an ideal of L,*

(3) *the set* $B(L) = \{a^* : a \in L\} = \{a^+ : a \in L\}$ *of all closed elements of* L *forms a Boolean subalgebra of* L*,*

(4) *the set* $K(L) = D(L) \cap \overline{D(L)}$ *is called the core of* L*, we have two cases of* $K(L)$*, namely,* $K(L) = \phi$ *or* $K(L) \neq \phi$*.*

It is easy to show the proof of the following two lemmas.

Lemma 3. *The non empty core* $K(L)$ *of a RD-Stone algebra* L *has exactly one element.*

Definition 11. A regular double Stone algebra with non empty core is called a core regular double Stone algebra (briefly CRD-Stone algebra).

Lemma 4. *Let* L *be a* CRD*-Stone algebra with the core* k*. Then*

(1) $D(L) = [k)$*, that is,* $D(L)$ *is a principal filter of* L *generated by* k*,*

(2) $\overline{D(L)} = (k]$*, that is,* $\overline{D(L)}$ *is a principal ideal of* L *generated by* k*.*

We use k for the core element of a CRD-Stone algebra L, that is, $K(L) = \{k\}$. Now, we give examples of CRD-Stone algebras and RD-Stone algebras with empty core.

Example 5. (1) Let $L = \{0, x, y, 1 : 0 < x < y < 1\}$ be the four element chain. It is clear that $\langle L,^*,^+\rangle$ is a double Stone algebra, where $x^* = y^* = 1^* = 0,\ 0^* = 1$ and $0^+ = x^+ = y^+ = 1,\ 1^+ = 0$. Then $K(L) = D(L) \cap \overline{D(L)} = \{x, y, 1\} \cap \{x, y, 0\} = \{x, y\}$ is a non empty core. We observe that L is not regular as $x^{++} = y^{++}$ and $x^{**} = y^{**}$, but $x \neq y$.

(2) The double Stone algbra $S_3 = \{0, k, 1 : 0 < k < 1\}$ is the smallest non trival core regular double Stone algebra with core k, (S_3 is called the discrete CRD-Stone algebra).

(3) Every Boolean algebra $(B; \vee, \wedge,', 0, 1)$ can be regarded as a RD-Stone algebra with empty core, where $x^* = x^+ = x'$, for all $x \in B$ and $K(B) = \{1\} \cap \{0\} = \phi$.

Example 6. (1) Consider the bounded distributive lattice S_9 in Figure 1. It is clear that L_1 is a core regular double Stone algebra with core element k, where $k^* = 1^* = y^* = x^* = 0,\ c^* = a^* = b,\ d^* = b^* = a,\ 1^* = 0$ and $k^+ = c^+ = d^+ = 0^+ = 1,\ b^+ = y^+ = a,\ x^+ = a^+ = b,\ 0^+ = 1$.

(2) Consider the bounded distributive lattice L_1 in Figure 2. We observe that L_1 is a regular double Stone algebra with empty core as $K(L) = D(L_1) \cap \overline{D(L_1)} = \{d, 1\} \cap \{0, y\} = \phi$, where $0^* = d^* = 1^*$, $c = x^*$, $x = c^* = y^*$, $1 = 0^*$ and $0 = 1^+$, $c = x^+ = d^+$, $x = c^+$, $1 = y^+ = 0^+$.

Lemma 7. *If* L *is a* CRD*-Stone algebra with core element* k*, then every element* x *of* L *can be written by each of the following formulas:*

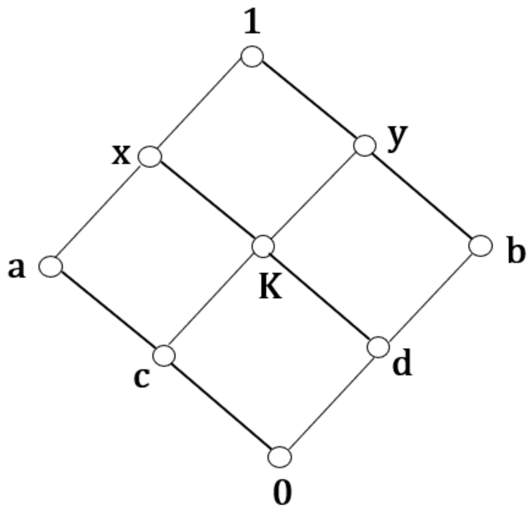

Figure 1. S_9 is a CRD-Stone algebra with core k.

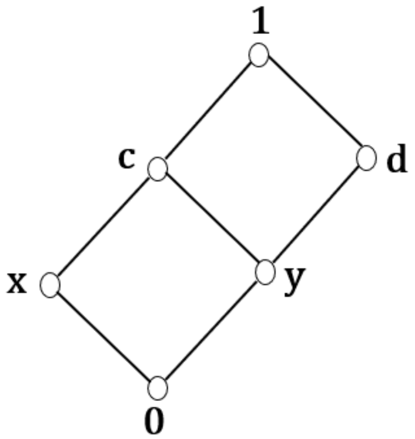

Figure 2. L_1 is a RD-Stone algebra with empty core.

(1) $x = x^{**} \wedge (x^{++} \vee k)$ *and its dual* $x = x^{++} \vee (x^{**} \wedge k)$,

(2) $x = x^{**} \wedge (x \vee k)$ *and its dual* $x = x^{++} \vee (x \wedge k)$.

Definition 12 [1]**.** An equivalent relation θ on a lattice L is called a lattice congruence on L if $(a, b) \in \theta$ and $(c, d) \in \theta$ implies $(a \vee c, b \vee d) \in \theta$ and $(a \wedge c, b \wedge d) \in \theta$.

Theorem 8 [23]**.** *An equivalent relation on a distributive lattice L is a lattice congruence on L if and only if $(a,b)\in\theta$ implies $(a\vee z, b\vee z)\in\theta$ and $(a\wedge z, b\wedge z)\in\theta$ for all $z\in L$.*

Definition 13. A lattice congruence θ on a dual Stone (Stone) algebra L is called a $\{^+\}$-congruence ($\{^*\}$-congruence) if $(a,b)\in\theta$ implies $(a^+,b^+)\in\theta$ ($(a,b)\in\theta$ implies $(a^*,b^*)\in\theta$).

Definition 14. A lattice congruence θ on a D-Stone algebra L is called a congruence (or $\{^*,^+\}$-congruence) if $(a,b)\in\theta$ implies $(a^*,b^*)\in\theta$ and $(a^+,b^+)\in\theta$.

A binary relation Ψ^+ defined a double Stone algebra L by

$$(x,y)\in\Psi^+ \Leftrightarrow x^+ = y^+$$

is a $\{^+\}$-congruence relation which is called the dual Glivenko congruence relation on L. It is known that the quotient lattice $L/\Psi = \{[x]\Psi : x\in L\}$ is a Boolean algebra and $L/\Psi \cong B(L)$, where $[x]\Psi = \{y\in L : y^+ = x^+\}$ is the congruence class of x modulo Ψ. Moreover, the element x^{++} is the smallest element of the congruence class $[x]\Psi$, $[0]\Psi = \overline{D(L)}$ and $[1]\Psi = \{1\}$.

For a double Stone algebra L, we use $Con(L)$ to denote the lattice of all congruence of L and $Con^+(L)$ to denote the lattice of all $\{^+\}$-congruence of a dual Stone algebra $(L,^+)$. Also, we use ∇_L and Δ_L for the universal congruence $L\times L$ and equality congruence $\{(x,x) : x\in L\}$ of L, respectively.

Definition 15 [14]. A lattice congruence θ on a lattice L is called a principal congruence and is doneted by $\theta(a,b)$ if θ is the smallest congruence on L containing a,b on the same class.

Theorem 9 [14]**.** *If L is a distributive lattice and $a,b\in L$ then the principal congruence $\theta(a,b)$ of L is given by*

(1) $(x,y)\in\theta(a,b) \Leftrightarrow x\vee a\vee b = y\vee a\vee b$ *and* $x\wedge a\wedge b = y\wedge a\wedge b$,

(2) *If* $a\le b$, *then* $(x,y)\in\theta(a,b) \Leftrightarrow x\vee b = y\vee b$ *and* $x\wedge a = y\wedge a$,

(3) $(x,y)\in\theta(0,b) \Leftrightarrow x\vee b = y\vee b$.

Throughout the paper, we will use L for a CRD-Stone algebra and k for the core element of L. For more information we refer the reader to [24, 31] for Stone algebras, [32] for double Stone algebras, [21] for regular double Stone algebras and [20, 27, 28, 29] for core regular double Stone algebras.

3. k-IDEALS OF CRD-STONE ALGEBRAS

In this section, we define the notion of k-ideal of a CRD-Stone algebra L and introduce many basic properties of such ideals. A characterization of a k-ideal of a CRD-Stone algebra L is given. Also, we observe that the class $I_k(L)$ of all k-ideals of L forms a bounded distributive lattice.

Definition 16. An ideal I of a CRD-Stone algebra L with core k is called a k-ideal if $k \in I$.

Let A be a non empty subset of a CRD-Stone algebra L. Consider $A^{\triangledown}$ as follows

$$A^{\triangledown} = \big\{x \in L : x^{++} \leq a^{++} \vee k, \text{ for some } a \in A\big\}.$$

Lemma 10. *Let A be a non empty subset of a CRD-Stone algebra L, which is closed under $\vee$. Then $A^{\triangledown}$ is a k-ideal of L containing A.*

Proof. Clearly $0, k \in (A)^{\triangledown}$. Let $x, y \in (A)^{\triangledown}$. Thus $x^{++} \leq a^{++} \vee k$, $y^{++} \leq b^{++} \vee k$ for some $a, b \in A$. Then $(x \vee y)^{++} \leq (a \vee b)^{++} \vee k$ and $a \vee b \in A$, imply $x \vee y \in (A)^{\triangledown}$. Now, let $x \in L, y \in (A)^{\triangledown}$ and $x \leq y$. Then $x^{++} \leq y^{++} \leq a^{++} \vee k$. So $x \in (A)^{\triangledown}$. Thus $(A)^{\triangledown}$ is k-ideal of L. Since, $a^{++} \leq a^{++} \vee k$, forall $a \in A$, then $A \in A^{\triangledown}$. ∎

Lemma 11. *Let A, B be two subsets of a CRD-Stone algebra L, which are closed under $\vee$. Then*

(1) $(A]^{\triangledown} = A^{\triangledown}$,

(2) $A \subseteq B \Rightarrow A^{\triangledown} \subseteq B^{\triangledown}$,

(3) $A^{\triangledown} = (A] \vee \overline{D(L)}$,

(4) $A^{\triangledown\triangledown} = A^{\triangledown}$.

Proof. (1) Since A is closed with respect to $\vee$, then for $a \in (A]$, we have $a \leq a_1 \vee a_2 \vee \cdots \vee a_n \in A$, $a_i \in A$, $i = 1, 2, \ldots, n$. Immediately, we get

$$\begin{aligned}(a]^{\triangledown} &= \{x \in L : x^{++} \leq a^{++} \vee k, \text{ for some } a \in (A]\} \\ &= \{x \in L : x^{++} \leq (a_1 \vee a_2 \vee \cdots \vee a_n)^{++} \vee k,\ a_1 \vee a_2 \vee \cdots \vee a_n \in A\} = A^{\triangledown}.\end{aligned}$$

(2) Suppose $A \subseteq B$ and $x \in A^{\triangledown}$. Then $x^{++} \leq a^{++} \vee k$ for some $a \in A \subseteq B$. It follows that $x \in B^{\triangledown}$. Thus $A^{\triangledown} \subseteq B^{\triangledown}$.

(3) Since $(A] \subseteq (A]^{\triangledown} = A^{\triangledown}$ by (1) and $\overline{D(L)} = (k] \subseteq A^{\triangledown}$, then $(A]^{\triangledown} \vee \overline{D(L)} \subseteq A^{\triangledown}$. Conversely, let $x \in A^{\triangledown}$. Then $x^{++} \leq a^{++} \vee k$ for some $a \in A$. We have

$$\begin{aligned}x = x^{++} \vee (x \wedge k) &\leq (a^{++} \vee k) \vee (x \wedge k) && \text{(by Lemma 7.(2))} \\ &= (a^{++} \vee k \vee x) \wedge (a^{++} \vee k) && \text{(by distributivity of } L) \\ &= a^{++} \vee k \leq a \vee k \in (a \vee k] \\ &\Rightarrow x \in (a \vee k] = (a] \vee (k] = (a] \vee \overline{D(L)} \subseteq (A] \vee \overline{D(L)} \\ & && \text{((as } (a] \subseteq (A]).)\end{aligned}$$

Therefore $A^{\triangledown} = (A] \vee \overline{D(L)}$.

(4) By the definition of $A^{\triangledown}$, we have

$$\begin{aligned} A^{\triangledown\triangledown} &= \{x \in L : x^{++} \le a_1^{++} \vee k, \text{ for some } a_1 \in A^{\triangledown}\} \\ &= \{x \in L : x^{++} \le a_1^{++} \vee k, a_1^{++} \le a^{++} \vee k \text{ for some } a \in A\} \\ &= \{x \in L : x^{++} \le a^{++} \vee k, \text{ for some } a \in A\} = A^{\triangledown}. \end{aligned}$$ ■

A characterization of k-ideals of a CRD-Stone algebra L is given in the following.

Theorem 12. *Let I be an ideal of a CRD-Stone algebra L with core k. Then the following statements are equivalent:*

(1) *I is a k-ideal of L,*

(2) *$\overline{D(L)} \subseteq I$,*

(3) *$x \wedge x^+ \in I$, for all $x \in L$,*

(4) *$I = I^{\triangledown}$.*

Proof. (1)⇒(2) Let I is a k-ideal of L. Then $k \in I$ implies $\overline{D(L)} = (k] \subseteq I$.

(2)⇒(3) Let $\overline{D(L)} \subseteq I$. Forall $x \in L$, we have $x \wedge x^+ \in \overline{D(L)} \subseteq I$.

(3)⇒(4) By Lemma 10, $I \subseteq I^{\triangledown}$. For the converse, let $y \in I^{\triangledown}$. Then $y^{++} \le i^{++} \vee k$, for some $i \in I$. Thus $y^{++} \le i^{++}$. By Lemma 7(2) $y = y^{++} \vee (y \wedge k) \le i^{++} \vee (y \wedge k)$. By (3), $k = k \wedge k^+ \in I$, where $k^+ = 1$. Since, $i^{++},\ y \wedge k \in I$, then $i^{++} \vee (y \wedge k) \in I$ and hence $y \in I$.

(4)⇒(1) Since $k \in I^{\triangledown}$, Lemma 10. Then by (4), $k \in I$ and hence I is a k-ideal of a CRD-Stone algebra L. ■

As a consequence of Lemma 11 and Theorem 12, we invistigate the following Corollary 13 and Lemma 14, respectively.

Corollary 13. *For any two ideals I, J of a CRD-Stone algebra L, we have the following:*

(1) *$I \subseteq J \Rightarrow I^{\triangledown} \subseteq J^{\triangledown}$,*

(2) *$I^{\triangledown\triangledown} = I^{\triangledown}$.*

Lemma 14. *Let L be a CRD-Stone algebra L. Then*

(1) *$I^{\triangledown} = I \vee \overline{D(L)}$,*

(2) *$\overline{D(L)}$ is the smallest k-ideal of L,*

(3) *Every k-ideal of L can be expresed in the form $I^{\triangledown}$ for some $I \in I(L)$.*

Let $I_k(L)$=$\{I$: *I is a k–ideal of L*$\}$= $\{I^{\triangledown} : I \in I(L)\}$ be the set of all k-ideals of L.

Theorem 15. *Let L be a CRD-Stone algebra L. Then for all $I, J \in I(L)$*

(1) $(I \vee J)^{\triangledown} = I^{\triangledown} \vee J^{\triangledown}$,

(2) $(I \cap J)^{\triangledown} = I^{\triangledown} \cap J^{\triangledown}$.

Proof. (1) Since $I, J \subseteq I \vee J$. Then by Corollary 13(1), $I^{\triangledown}, J^{\triangledown} \subseteq (I \vee J)^{\triangledown}$. Thus, $(I \vee J)^{\triangledown}$ is an upper bound of $I^{\triangledown}$ and $J^{\triangledown}$. Let $H^{\triangledown}$ be an upper bound of both $I^{\triangledown}$ and $J^{\triangledown}$ for some $H \in I_k(L)$. Then $I^{\triangledown}, J^{\triangledown} \subseteq H^{\triangledown}$ implies $I, J \subseteq H^{\triangledown}$. Hence, $I \vee J \subseteq H^{\triangledown}$. Therefore, by Corollary 13(1) and (2), we get $(I \vee J)^{\triangledown} \subseteq H^{\triangledown\triangledown} = H^{\triangledown}$. This deduce that $(I \vee J)^{\triangledown}$ is the least upper bound of both $I^{\triangledown}$ and $J^{\triangledown}$ in $I_k(L)$. Then $(I \vee J)^{\triangledown} = I^{\triangledown} \vee J^{\triangledown}$.

(2) Obviously, $(I \cap J)^{\triangledown} \subseteq I^{\triangledown} \cap J^{\triangledown}$. Conversely, let $x \in I^{\triangledown} \cap J^{\triangledown}$. Then $x^{++} \leq i^{++} \vee k$ and $x^{++} \leq j^{++} \vee k$ for some $i \in I$ and $j \in J$. Hence $x^{++} \leq (i^{++} \vee k) \wedge (j^{++} \vee k) = (i^{++} \wedge j^{++}) \vee k = (i \wedge j)^{++} \vee k$. It yields that $x \in (I \cap J)^{\triangledown}$ as $i \wedge j \leq i, j$ imples $i \wedge j \in I \cap J$. Therefore $I^{\triangledown} \cap J^{\triangledown} \subseteq (I \cap J)^{\triangledown}$. ■

Theorem 16. *The class $I_k(L)$ of all k-ideals of a CRD-Stone algebra L forms a bounded distributive lattice and $\{1\}$-sublattice of $I(L)$.*

Proof. From Theorem 15, $(I_k(L); \vee, \wedge)$ is a sublattice of the lattice $I(L)$, where

$$(I \vee J)^{\triangledown} = I^{\triangledown} \vee J^{\triangledown} \text{ and } (I \cap J)^{\triangledown} = I^{\triangledown} \cap J^{\triangledown} \text{ for all } I, J \in I(L).$$

Then $(I_k(L); \vee, \wedge)$ is sublattice of $I(L)$. Since $I(L)$ is a distributive lattice, then $I_k(L)$ is also distributive. Since $\overline{D(L)}$ and L are the smallest and the greatest members of $I_k(L)$, respectively. Then $(I_k(L); \vee, \wedge, \overline{D(L)}, L)$ is a bounded distributive lattice on its own and hence a $\{1\}$-sublattice of $I(L)$. ■

4. Principal k-ideals of a CRD-Stone algebra

In this section, we introduce the concept of principal k-ideals of a CRD-Stone algebra L and investigate many elegant properties of such ideals. A characterization of a k-ideal of L is given via the principal k-ideals. It is observed the set of all principal k-ideals of a CRD-Stone algebra L is a Boolean ring and so a Boolean algebra.

Now, let $A = \{a\}$ be a subset of a CRD-Stone L. Then ready is seen that

$$\{a\}^{\triangledown} = \{x \in L : x^{++} \leq a^{++} \vee k\}.$$

For brevity, set $(a)^{\triangledown}$ instead of $\{a\}^{\triangledown}$. Clearly, $(0)^{\triangledown} = \overline{D(L)}$ and $(1)^{\triangledown} = L$, are the smallest and the greatest k-ideals of L, respectively.

Definition 17. A k-ideal I of a CRD-Stone algebra L is called a principal k-ideal of L if I is a principal ideal of L.

Theorem 17. *Let L be a CRD-Stone algebra. Then for any $x, y \in L$, we get*

(1) $y \in (x)^{\triangledown} \Leftrightarrow y^+ \vee x = 1$,

(2) $(x)^{\triangledown} = (x^{++} \vee k] = (x^{++}] \vee \overline{D(L)}$, *this is,* $(x)^{\triangledown}$ *is a principal k-ideal of L,*

(3) $x \in \overline{D(L)} \Leftrightarrow (x)^{\triangledown} = \overline{D(L)}$.

Proof. (1) Let $y \in (x)^{\triangledown}$. Then, we have

$$\begin{aligned} y^{++} \leq x^{++} \vee k &\Leftrightarrow y^+ \geq x^+ \\ &\Leftrightarrow y^+ \vee x = 1 \qquad \text{(by Definition 6)} \end{aligned}$$

(2) For all $x \in L$, we get

$$\begin{aligned} (x)^{\triangledown} &= \{y \in L : y^{++} \leq x^{++} \vee k\} \\ &= \{y \in L : y^{++} \vee (y \wedge k) \leq x^{++} \vee k \vee (y \wedge k)\} \\ &= \{y \in L : y \leq x^{++} \vee k\} \qquad \text{(by Lemma 7(2) and Definition 1(2))} \\ &= (x^{++} \vee k] \\ &= (x^{++}] \vee (k] = (x^{++}] \vee \overline{D(L)}. \end{aligned}$$

(3) Let $x \in \overline{D(L)}$. Then $x^+ = 1$. Now,

$$\begin{aligned} (x)^{\triangledown} &= (x^{++} \vee k] \qquad \text{(by(2))} \\ &= (0 \vee k] = (k] = \overline{D(L)}. \end{aligned}$$

The second implication is clear. ∎

More interesting properties of principal k-ideals are given in the following two lemmas.

Lemma 18. *Let L be a CRD-Stone algebra L. Then for any $x, y \in L$, we have*

(1) $(x)^{\triangledown\triangledown} = (x)^{\triangledown}$,

(2) $(x]^{\triangledown} = (x)^{\triangledown}$,

(3) $x \in (y)^{\triangledown} \Leftrightarrow (x)^{\triangledown} \subseteq (y)^{\triangledown}$,

(4) $x \leq y \Rightarrow (x)^{\triangledown} \subseteq (y)^{\triangledown}$.

Lemma 19. *Let L be a CRD-Stone algebra L. For any $x, y \in L$, we have*

(1) $(x)^{\triangledown} = (x^{++})^{\triangledown}$,

(2) $(x \wedge y)^{\triangledown} = (x)^{\triangledown} \cap (y)^{\triangledown}$,

(3) $(x \vee y)^{\triangledown} = (x)^{\triangledown} \vee (y)^{\triangledown}$,

(4) $(x \vee x^+)^{\triangledown} = (1)^{\triangledown} = L$,

(5) $(x \wedge x^{+})^{\bigtriangledown} = \overline{D(L)}$.

Proof. (1) $(x)^{\bigtriangledown} = \{y \in L : y^{++} \leq x^{++} \vee k = (x^{++})^{++} \vee k\} = (x^{++})^{\bigtriangledown}$, as $x^{++++} = x^{++}$.

(2) By Theorem 17.(2), we get

$$\begin{aligned}(x \wedge y)^{\bigtriangledown} &= ((x \wedge y)^{++}] \vee \overline{D(L)} \\ &= ((x^{++} \wedge y^{++})] \vee \overline{D(L)} \\ &= ((x^{++}] \cap (y^{++}]) \vee \overline{D(L)} \\ &= ((x^{++}] \vee \overline{D(L)}) \cap ((y^{++})] \vee \overline{D(L)}) \qquad \text{(by distributivity of I(L))} \\ &= (x)^{\bigtriangledown} \cap (y)^{\bigtriangledown}.\end{aligned}$$

(3) By Theorem 17(2), we get

$$\begin{aligned}(x \vee y)^{\bigtriangledown} &= ((x \vee y)^{++}] \vee \overline{D(L)} \\ &= ((x^{+} \wedge y^{+})^{+}] \vee \overline{D(L)} \\ &= (x^{++} \vee y^{++}] \vee \overline{D(L)} \\ &= ((x^{++}] \vee (y^{++}]) \vee \overline{D(L)} \\ &= ((x^{++}] \vee \overline{D(L)}) \vee ((y^{++})] \vee \overline{D(L)}) \qquad \text{(by distributivity of I(L))} \\ &= (x)^{\bigtriangledown} \vee (y)^{\bigtriangledown}.\end{aligned}$$

(4) Since $x \vee x^{+}$, we get $(x \vee x^{+})^{\bigtriangledown} = (1] = L$.

(5) Since $x \wedge x^{+} \in \overline{D(L)}$, then by Theorem 17(3), $(x \wedge x^{+})^{\bigtriangledown} = \overline{D(L)}$. ■

Lemma 20. *Let L be a CRD-Stone algebra L. For any $x, y \in L$, we have*

(1) $(x)^{\bigtriangledown} = (y)^{\bigtriangledown} \Leftrightarrow x^{++} = y^{++} \Leftrightarrow x^{+} = y^{+}$,

(2) $(x)^{\bigtriangledown} = (y)^{\bigtriangledown} \Rightarrow (x \wedge z)^{\bigtriangledown} = (y \wedge z)^{\bigtriangledown}, \ \forall z \in L$,

(3) $(x)^{\bigtriangledown} = (y)^{\bigtriangledown} \Rightarrow (x \vee z)^{\bigtriangledown} = (y \vee z)^{\bigtriangledown}, \ \forall z \in L$.

Now, we introduce the following important result.

Theorem 21. *Every principal k-ideal of L can be expressed as $(x)^{\bigtriangledown}$ for some $x \in L$.*

Proof. Let $(x]$ be a principal k-ideal of L. We claim that $(x] = (x)^{\bigtriangledown}$. Since $x \in (x)^{\bigtriangledown}$ *then* $(x] \subseteq (x)^{\bigtriangledown}$. For the converse, let $y \in (x)^{\bigtriangledown}$. Then

$$\begin{aligned}y \in (x)^{\bigtriangledown} &\Rightarrow y^{++} \leq x^{++} \vee k \\ &\Rightarrow y^{++} \vee (y \wedge k) \leq (x^{++} \vee k) \vee (y \wedge k) = (x^{++} \vee k \vee y) \wedge (x^{++} \vee k) \\ &\qquad = x^{++} \vee k \leq x \vee k \\ &\Rightarrow y \leq x \vee k \quad \text{as } y = y^{++} \vee (y \wedge k) \\ &\Rightarrow y \in (x \vee k] \subseteq (x] \quad \text{as } k \leq x.\end{aligned}$$

Therefore $(x)^\triangledown \subseteq (x]$ and hence $(x)^\triangledown = (x]$. ■

A characterization of a k-ideal via the principal k-ideal is given in the following theorem.

Theorem 22. *Let I be an ideal of a CRD-Stone algebra L. Then the following statements are equivalent:*

(1) *I is a k-ideal,*

(2) *$x^{++} \in I \Rightarrow x \in I$,*

(3) *for all $x, y \in L$, $(x)^\triangledown = (y)^\triangledown$ and $y \in I \Rightarrow x \in I$,*

(4) *$I = \bigcup_{x\in I}(x)^\triangledown$,*

(5) *$x \in I \Rightarrow (x)^\triangledown \subseteq I$.*

Proof. (1)⇒(2) Let I be a k-ideal of L and $x^{++} \in I$. Then $k \in I$ implies $x \wedge k \in I$. Now, x^{++}, $x \wedge k \in I$ imply that $x = x^{++} \vee (x \wedge k) \in I$.

(2)⇒(3) Let $(x)^\triangledown = (y)^\triangledown$, $y \in I$. Thus $x \in (y)^\triangledown$. Then, $x^{++} \leq y^{++} \vee k$ implies $x^{++} \leq y^{++} \leq y \in I$. Thus, $x^{++} \in I$. By (2), we get $x \in I$.

(3)⇒(4) For any $x \in I$, we have $x \in (x)^\triangledown \subseteq \bigcup_{x\in I}(x)^\triangledown$. Then $I \subseteq \bigcup_{x\in I}(x)^\triangledown$. Conversely, let $y \in \bigcup_{x\in I}(x)^\triangledown$. Then $y \in (z)^\triangledown$ for some $z \in I$. Hence, $(y)^\triangledown \subseteq (z)^\triangledown$, by Lemma 18(3). It follows that $(y)^\triangledown = (y)^\triangledown \cap (z)^\triangledown = (y \wedge z)^\triangledown$. Since $y \wedge z \in I$, then by (3), we get $y \in I$. Therefore, $\bigcup_{x\in I}(x)^\triangledown \subseteq I$ and hence $\bigcup_{x\in I}(x)^\triangledown = I$.

(4)⇒(5) Assume (4). Let $x \in I$. Then by (4), we get $x \in (i)^\triangledown$ for some $i \in I$. Suppose $t \in (x)^\triangledown$. Then it concludes $t \in (x)^\triangledown \subseteq (i)^\triangledown$with $i \in I$. Then $t \in \bigcup_{i\in I}(i)^\triangledown = I$ and hence $(x)^\triangledown \subseteq I$.

(5)⇒(1) Assume (5). Since $k \in (x)^\triangledown$, $\forall x \in I$, then by (5), $k \in (x)^\triangledown \subseteq I$. This proves that I is a k-ideal of L. ■

Let $I_k^p(L) = \{(x)^\triangledown : x \in L\}$ be the set of all principal k-ideal of a CRD-Stone algebra L.

Theorem 23. *Let L be a CRD-Stone algebra. Then $(I_k^p(L); +, \bullet, (0)^\triangledown, (1)^\triangledown)$ forms a Boolean ring, where $+$ the addition operation and $\bullet$ the multiplication operation are defined as follows:*

$$(x)^\triangledown + (y)^\triangledown = ((x \wedge y^+) \vee (y \wedge x^+))^\triangledown,$$
$$(x)^\triangledown \bullet (y)^\triangledown = (x \wedge y)^\triangledown.$$

Proof. Let $(x)^\triangledown, (y)^\triangledown, (z)^\triangledown \in I_k^p(L)$. Then we deduce the following properties:

(i) Associativity of $+$,

$$\begin{aligned}
&(x)^{\bigtriangledown} + ((y)^{\bigtriangledown} + (z)^{\bigtriangledown}) \\
&= (x)^{\bigtriangledown} + ((y \wedge z^{+}) \vee (z \wedge y^{+}))^{\bigtriangledown} \\
&= ((x \wedge \{(y \wedge z^{+}) \vee (z \wedge y^{+})\}^{+}) \vee (x^{+} \wedge \{(y \wedge z^{+}) \vee (z \wedge y^{+})\}))^{\bigtriangledown} \\
&= (\{x \wedge y^{+} \wedge z^{+}\} \vee \{x \wedge z^{++} \wedge y^{++}\} \vee \{x^{+} \wedge y \wedge z^{+}\} \vee \{x^{+} \wedge z \wedge y^{+}\})^{\bigtriangledown}
\end{aligned}$$

where

$$\begin{aligned}
&(x \wedge \{(y \wedge z^{+}) \vee (z \wedge y^{+})\}^{+}) \\
&= (x \wedge \{(y \wedge z^{+})^{+} \wedge (z \wedge y^{+})^{+}\}) && \text{(by Theorem 1(7))} \\
&= x \wedge \{(y^{+} \vee z^{++}) \wedge (z^{+} \vee y^{++})\} && \text{(by Theorem 1(6))} \\
&= \{(x \wedge y^{+}) \vee (x \wedge z^{++})\} \wedge (z^{+} \vee y^{++}) && \text{(by distributivity of } L) \\
&= \{(x \wedge y^{+}) \wedge (z^{+} \vee y^{++})\} \vee \{(x \wedge z^{++}) \wedge (z^{+} \vee y^{++})\} && \text{(by distributivity of } L) \\
&= (x \wedge y^{+} \wedge z^{+}) \vee (x \wedge y^{+} \wedge y^{++}) \vee (x \wedge z^{++} \wedge z^{+}) \vee (x \wedge z^{++} \wedge y^{++}) \\
&= (x \wedge y^{+} \wedge z^{+}) \vee (x \wedge z^{++} \wedge y^{++}) \ \textit{as}\ x^{+} \wedge x^{++} = 0,\ \forall x \in L.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&((x)^{\bigtriangledown} + (y)^{\bigtriangledown}) + (z)^{\bigtriangledown} \\
&= (((x \wedge y^{+}) \vee (y \wedge x^{+}))^{\bigtriangledown} + z^{\bigtriangledown}) \\
&= ((\{(x \wedge y^{+}) \vee (y \wedge x^{+})\} \wedge z^{+}) \vee (\{(x \wedge y^{+}) \vee (y \wedge x^{+})\}^{+} \wedge z))^{\bigtriangledown} \\
&= (\{x \wedge y^{+} \wedge z^{+}\} \vee \{x^{+} \wedge y \wedge z^{+}\} \vee \{x^{++} \wedge y^{++} \wedge z\} \vee \{x^{+} \wedge y^{+} \wedge z\})^{\bigtriangledown}
\end{aligned}$$

where

$$\begin{aligned}
&(\{(x \wedge y^{+}) \vee (y \wedge x^{+})\}^{+} \wedge z) \\
&= (\{(x \wedge y^{+})^{+} \wedge (y \wedge x^{+})^{+}\} \wedge z) && \text{(by Theorem 1(7))} \\
&= (\{(x^{+} \vee y^{++}) \wedge (y^{+} \vee x^{++})\} \wedge z) && \text{(by Theorem 1(6))} \\
&= (\{((x^{+} \vee y^{++}) \wedge y^{+}) \vee ((x^{+} \vee y^{++}) \wedge x^{++})\} \wedge z) && \text{(by distributivity of } L) \\
&= \{(x^{+} \vee y^{++}) \wedge y^{+} \wedge z\} \vee \{(x^{+} \vee y^{++}) \wedge x^{++} \wedge z\} && \text{(by distributivity of } L) \\
&= (x^{+} \wedge y^{+} \wedge z) \vee (y^{++} \wedge y^{+} \wedge z) \vee (x^{+} \wedge x^{++} \wedge z) \vee (y^{++} \wedge x^{++} \wedge z) \\
&= (x^{+} \wedge y^{+} \wedge z) \vee (y^{++} \wedge x^{++} \wedge z) \ \textit{as}\ x^{+} \wedge x^{++} = 0,\ \forall x \in L.
\end{aligned}$$

Now, we use the fact $(x)^{\bigtriangledown} = (y)^{\bigtriangledown} \Leftrightarrow x^{++} = y^{++} \Leftrightarrow x^{+} = y^{+}$, see Lemma 20(1). It is easy to check that

$$\begin{aligned}
&\{\{x \wedge y^{+} \wedge z^{+}\} \vee \{x \wedge z^{++} \wedge y^{++}\} \vee \{x^{+} \wedge y \wedge z^{+}\} \vee \{x^{+} \wedge z \wedge y^{+}\}\}^{+} \\
&= \{\{x \wedge y^{+} \wedge z^{+}\} \vee \{x^{+} \wedge y \wedge z^{+}\} \vee \{x^{++} \wedge y^{++} \wedge z\} \vee \{x^{+} \wedge y^{+} \wedge z\}\}^{+} \\
&= \{x^{+} \vee y^{++} \vee z^{++}\} \wedge \{x^{+} \vee z^{+} \vee y^{+}\} \wedge \{x^{++} \vee y^{+} \vee z^{++}\} \wedge \{x^{++} \vee z^{+} \vee y^{++}\}.
\end{aligned}$$

Therefore, $(\{x\wedge y^+\wedge z^+\}\vee\{x\wedge z^{++}\wedge y^{++}\}\vee\{x^+\wedge y\wedge z^+\}\vee\{x^+\wedge z\wedge y^+\})^\triangledown=(\{x\wedge y^+\wedge z^+\}\vee\{x^+\wedge y\wedge z^+\}\vee\{x^{++}\wedge y^{++}\wedge z\}\vee\{x^+\wedge y^+\wedge z\})^\triangledown$ implies $((x)^\triangledown+(y)^\triangledown)+(z)^\triangledown=(x)^\triangledown+((y)^\triangledown+(z)^\triangledown)$.

(ii) Since $(x)^\triangledown+(0)^\triangledown=((x\wedge 0^+)\vee(x^+\wedge 0))^\triangledown=(x\vee 0)^\triangledown=(x)^\triangledown$,then $(0)^\triangledown$ is the additive identity on $I_k^p(L)$.

(iii) Commutativity of $+$ and $\bullet$,

$$\begin{aligned}(x)^\triangledown+(y)^\triangledown&=(x\wedge y^+)\vee(y\wedge x^+)^\triangledown\\&=(y\wedge x^+)\vee(y^+\wedge x)^\triangledown\\&=(y)^\triangledown+(x)^\triangledown,\\(x)^\triangledown\bullet(y)^\triangledown&=(x\wedge y)^\triangledown\\&=(y\wedge x)^\triangledown\\&=(y)^\triangledown\bullet(x)^\triangledown.\end{aligned}$$

(iv) It is clear that the additive inverse of $(x)^\triangledown\in I_K^p(L)$ is $(x)^\triangledown$ itself, that is, $-(x)^\triangledown=(x)^\triangledown$.

(v) The multiplicative identity of $I_k^p(L)$ is $(1)^\triangledown$.

(vii) The distributive law on $I_k^p(L)$,

$$\begin{aligned}(x)^\triangledown\bullet\{(y)^\triangledown+(z)^\triangledown\}&=(x)^\triangledown\bullet((y\wedge z^+)\vee(z\wedge y^+))^\triangledown\\&=(x\wedge\{(y\wedge z^+)\vee(z\wedge y^+)\})^\triangledown\\&=(\{x\wedge y\wedge z^+\}\vee\{x\wedge z\wedge y^+\})^\triangledown,\end{aligned}$$

and

$$\begin{aligned}&\{(x)^\triangledown\bullet(y)^\triangledown\}+\{(x)^\triangledown\bullet(z)^\triangledown\}\\&=(x\wedge y)^\triangledown+(x\wedge z)^\triangledown\\&=(\{(x\wedge y)\wedge(x\wedge z)^+\}\vee\{(x\wedge y)^+\wedge(x\wedge z)\})^\triangledown\\&=(\{(x\wedge y)\wedge(x^+\vee z^+)\}\vee\{(x^+\vee y^+)\wedge(x\wedge z)\})^\triangledown\\&=(\{x\wedge y\wedge x^+\}\vee\{x\wedge y\wedge z^+\}\vee\{x^+\wedge x\wedge z\}\vee\{y^+\wedge x\wedge z\})^\triangledown.\end{aligned}$$

Then by Lemma 20(1), we get $(\{x\wedge y\wedge z^+\}\vee\{x\wedge z\wedge y^+\})^\triangledown=(\{x\wedge y\wedge x^+\}\vee\{x\wedge y\wedge z^+\}\vee\{x^+\wedge x\wedge z\}\vee\{y^+\wedge x\wedge z\})^\triangledown$.

Therefore, $(x)^\triangledown\bullet\{(y)^\triangledown+(z)^\triangledown\}=\{(x)^\triangledown\bullet(y)^\triangledown\}+\{(x)^\triangledown\bullet(z)^\triangledown\}$.

(viii) $(x)^\triangledown\bullet(x)^\triangledown=(x\wedge x)^\triangledown=(x)^\triangledown$. Consequently $(I_k^p(L);+,\bullet,(0)^\triangledown,(1)^\triangledown)$ is a Boolean ring. ■

It is known that there is a one-to-one correspondence between Boolean algebras and Boolean rings (see [17]). Then we can convert the Boolean ring $I_k^p(L)$ into a Boolean algebra as follows.

Corollary 24. *Let $(I_k^p(L);+,\bullet,(0)^\triangledown,(1)^\triangledown)$ be a Boolean ring of all principal k-ideals of a CRD-Stone algebra L. Then $(I_k^p(L);\vee,\wedge,',(0)^\triangledown,(1)^\triangledown)$ is a Boolean algebra, where*

$$(x)^\triangledown \vee (y)^\triangledown = (x)^\triangledown + (y)^\triangledown + \{(x)^\triangledown \bullet (y)^\triangledown\} = (x \wedge y)^\triangledown,$$
$$(x)^\triangledown \cap (y)^\triangledown = (x)^\triangledown \bullet (y)^\triangledown = (x \wedge y)^\triangledown,$$
$$(x)^{\triangledown\prime} = (x^+)^\triangledown.$$

Now, we give an example to clarify the basic properties of the class of all principal k-ideals of a certain CRD-Stone algebra L.

Example 25. Consider the CRD-Stone algebra S_9 which is given in Example 6(1) (see Figure 1). The principal k-ideals of S_9 are given as follows.

$(0)^\triangledown = (c)^\triangledown = (d)^\triangledown = (k)^\triangledown = (k]$, $(a)^\triangledown = (x)^\triangledown = (x]$, $(b)^\triangledown = (y)^\triangledown = (y]$ and $(1)^\triangledown = L = (1]$. We determine the algebras $(I_k^p(L),+)$ and $(I_k^p(L),\bullet)$ as in the following tables.

$+$	$(0)^\triangledown$	$(a)^\triangledown$	$(b)^\triangledown$	$(1)^\triangledown$
$(0)^\triangledown$	$(0)^\triangledown$	$(a)^\triangledown$	$(b)^\triangledown$	$(1)^\triangledown$
$(a)^\triangledown$	$(a)^\triangledown$	$(0)^\triangledown$	$(b)^\triangledown$	$(1)^\triangledown$
$(b)^\triangledown$	$(b)^\triangledown$	$(1)^\triangledown$	$(0)^\triangledown$	$(a)^\triangledown$
$(1)^\triangledown$	$(1)^\triangledown$	$(b)^\triangledown$	$(a)^\triangledown$	$(0)^\triangledown$

$\bullet$	$(0)^\triangledown$	$(a)^\triangledown$	$(b)^\triangledown$	$(1)^\triangledown$
$(0)^\triangledown$	$(0)^\triangledown$	$(0)^\triangledown$	$(0)^\triangledown$	$(0)^\triangledown$
$(a)^\triangledown$	$(0)^\triangledown$	$(a)^\triangledown$	$(0)^\triangledown$	$(a)^\triangledown$
$(b)^\triangledown$	$(0)^\triangledown$	$(0)^\triangledown$	$(b)^\triangledown$	$(b)^\triangledown$
$(1)^\triangledown$	$(0)^\triangledown$	$(a)^\triangledown$	$(b)^\triangledown$	$(1)^\triangledown$

From the above tables, we abserve that $(I_k^p(L);+,\bullet)$ forms a Boolean ring. Also, Figure 3. Shows that $(I_k^p(L);\vee,\wedge,',(0)^\triangledown,(1)^\triangledown)$ forms a Boolean algebra which is isomorphic to $B(L)$, where $'$ is given as, $(0)^{\triangledown\prime} = (1)^\triangledown$, $(a)^{\triangledown\prime} = (b)^\triangledown$, $(b)^{\triangledown\prime} = (a)^\triangledown$, $(1)^{\triangledown\prime} = (0)^\triangledown$.

Theorem 26. *Let L be a CRD-Stone algebra. Then*

(1) $(I_k(L);\vee,\wedge,\overline{D(L)},L)$ *is a* $\{1\}$*-sublattice of* $I(L)$,

(2) $(I_k^p(L);\vee,\wedge,(0)^\triangledown,(1)^\triangledown)$ *is a bounded sublattice of* $I_k(L)$,

(3) $B(L)$ *is isomorphic to* $I_k^p(L)$.

Proof. (1) Let $I,J \in I_k(L)$. Since $k \in I,J$, then $I \cap J$ and $\overline{I \vee J}$ are k-ideals. Since $k \in L = (1]$, then L is the greatest k-ideal of L, but $\overline{D(L)} = (k]$ is the smallest k-ideal of L. Then $I_k(L)$ is a $\{1\}$-sublattice of the lattice $I(L)$.

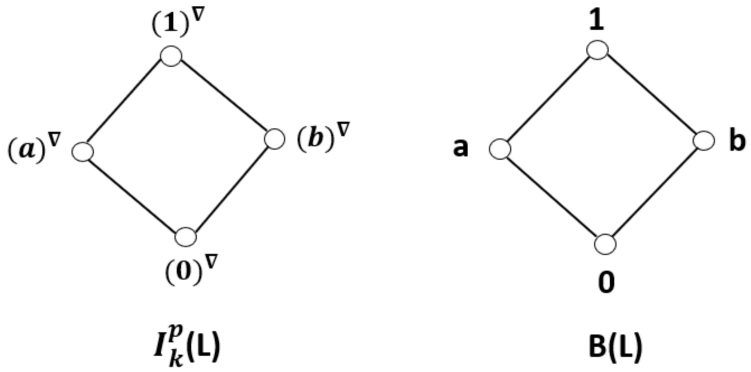

Figure 3. $I_k^p(L)$ and $B(L)$ are isomorphic Boolean algebras.

(2) We have $(x \vee y)^{\triangledown} = (x)^{\triangledown} \vee (y)^{\triangledown}$ and $(x \wedge y)^{\triangledown} = (x)^{\triangledown} \wedge (y)^{\triangledown}$ for all $(x)^{\triangledown}, (y)^{\triangledown} \in I_k^p(L)$. It is observed that $(0)^{\triangledown} = \overline{D(L)}$, $(1)^{\triangledown} = L$ are the smallest and the greatest members of $I_k^p(L)$, respectively. Therefore, $(I_k^p(L); \vee, \wedge, (0)^{\triangledown}, (1)^{\triangledown})$ is a bounded sublattice of the lattice $I_k(L)$.

(3) Define mapping: $f : B(L) \longrightarrow I_k^p(L)$ by $f(x) = (x)^{\triangledown}$, for all $x \in B(L)$. To prove that f is a homomorphism, let $x, y \in B(L)$,

$$\begin{aligned} f(x \vee y) &= (x \vee y)^{\triangledown} \\ &= (x)^{\triangledown} \vee (y)^{\triangledown} && \text{(by Lemma 19(3))} \\ &= f(x) \vee f(y) \end{aligned}$$

Thus $f(x \vee y) = f(x) \vee f(y)$. Similarly, we can get $f(x \wedge y) = f(x) \wedge f(y)$. Then f is homomorphism. Let $f(x) = f(y)$. Then $(x)^{\triangledown} = (y)^{\triangledown}$ and hence $x = x^{++} = y^{++} = y$. Then f is an injective map. For all $(x)^{\triangledown} \in I_k^p(L)$, we have $(x)^{\triangledown} = (x^{++})^{\triangledown} = f(x^{++})$, $x^{++} \in B(L)$. Then f is a surjective map. Therefore f is an isomorphism and $B(L) \cong I_k^p(L)$. ■

5. k-$\{^+\}$-CONGRUENCES ON A CRD-STONE ALGEBRA

In this section, we study the relationships between k-ideals and k-$\{^+\}$-congruences of a CRD-Stone algebra L. Also, we describe the lattice $Con_k^+(L)$ of all k-$\{^+\}$-congruences of L.

Definition 18. A $\{^+\}$-congruence θ on a CRD-Stone algebra L is called a k-$\{^+\}$-congruence if $k \in Ker\ \theta$, where $Ker\ \theta = \{x \in L : (x, 0) \in \theta\} = [0]_\theta$

Proposition 27. *Define a binary relation θ on a core regular double Stone L as follows:*

$$(x,y) \in \theta \Leftrightarrow (x)^{\triangledown} = (y)^{\triangledown}.$$

Then θ is a k-$\{^+\}$-congruence on L. Moreover, $\theta = \psi^+$.

Let I be a k-ideal of CRD-Stone algebra L. Define a binary relation θ_I on L as follows:

$$\theta_I = \{(a,b) \in L \times L : a \vee i \vee k = b \vee i \vee k, \ for \ some \ i \in I\}.$$

Theorem 28. *Let I be a k-ideal of CRD-Stone algebra L. Then θ_I is a k-$\{^+\}$-congerence on L such that $Ker\ \theta_I = I$.*

Proof. It is Clear that θ_I is an equivalent relation on L. Let $(a,b) \in \theta_I$. Then $a \vee i \vee k = b \vee i \vee k$ for some $i \in I$. Now for all $c \in L$, then by distributivity of L, we get

$$(a \wedge c) \vee i \vee k = (b \wedge c) \vee i \vee k,$$
$$(a \vee c) \vee i \vee k = (b \vee c) \vee i \vee k.$$

Therefore $(a \wedge c, b \wedge c), (a \vee c, b \vee c) \in \theta_I$. So by **Theorem 8**, θ_I is a lattice congruence on L. It remains to show that $(a,b) \in \theta_I$ implies $(a^+, b^+) \in \theta_I$.

$$\begin{aligned}
(a,b) \in \theta_I &\Rightarrow a \vee i \vee k = b \vee i \vee k \\
&\Rightarrow a^+ \wedge i^+ \wedge k^+ = b^+ \wedge i^+ \wedge k^+ \\
&\Rightarrow a^+ \wedge i^+ = b^+ \wedge i^+ \ as \ k^+ = 1 \\
&\Rightarrow \left(a^+ \wedge i^+\right) \vee i = \left(b^+ \wedge i^+\right) \vee i \\
&\Rightarrow \left(a^+ \vee i\right) \wedge \left(i^+ \vee i\right) = \left(b^+ \vee i\right) \wedge \left(i^+ \vee i\right) \quad \text{(by distributivity of L)} \\
&\Rightarrow \left(a^+ \vee i\right) \wedge 1 = \left(b^+ \vee i\right) \wedge 1 \quad \text{(by Theorem 1(2))} \\
&\Rightarrow a^+ \vee i = b^+ \vee i \\
&\Rightarrow \left(a^+, b^+\right) \in \theta_I
\end{aligned}$$

Then θ_I is a $\{^+\}$-congruence on L.

Now, we prove that $Ker\ \theta_I = I$.

$$\begin{aligned}
Ker\ \theta_I &= \{x \in L : (0,x) \in \theta_I\} \\
&= \{x \in L : 0 \vee i \vee k = x \vee i \vee k, i \in I\} \\
&= \{x \in L : i \vee k = x \vee i \vee k\} \\
&= \{x \in L : x \leq i \vee k\} \\
&= \{x \in L : x^{++} \leq i^{++} \leq i^{++} \vee k\} \\
&= \{x : x \in I^{\triangledown} = I\} = I.
\end{aligned}$$

Since $k \in I = Ker\ \theta_I$, then θ_I is a a k-$\{^+\}$-congruence on L. ∎

Theorem 29. *For any k-ideals I, J of a CRD-Stone algebra L, we have*

(1) $I \subseteq J \Leftrightarrow \theta_I \subseteq \theta_J$,

(2) $\psi^+ \subseteq \theta_I$, *where ψ^+ is the dual Glivenko congruence on L,*

(3) $\theta_{\overline{D(L)}} = \psi^+$,

(4) $\theta_L = \nabla_L$,

(5) *the quotient lattice L/θ_I forms a Boolean algebra.*

Proof. (1) Suppose $I \subseteq J$ and $(a,b) \in \theta_I$. Then there exists $i \in I$ such that $a \vee i \vee k = b \vee i \vee k$. Since $I \subseteq J$, then $(a,b) \in \theta_J$. Thus $\theta_I \subseteq \theta_J$. Conversely, let $\theta_I \subseteq \theta_J$. Then by the above **Theorem 28**, $I = Ker\ \theta_I \subseteq Ker\ \theta_J = J$.

(2) Let $(a,b) \in \psi^+$. Then $a^+ = b^+$ *implies* $a^{++} = b^{++}$. Now, we have

$$\begin{aligned} a \vee i \vee k &= (a^{++} \vee (a \wedge k)) \vee i \vee k && \text{(by Lemma 7(2))} \\ &= a^{++} \vee i \vee ((a \wedge k) \vee k) \\ &= a^{++} \vee i \vee k && \text{(by Definition 1(2))} \\ &= b^{++} \vee i \vee k \\ &= b^{++} \vee i \vee ((b \wedge k) \vee k) \\ &= (b^{++} \vee (b \wedge k)) \vee i \vee k \\ &= b \vee i \vee k. \end{aligned}$$

Thus $(a,b) \in \theta_I$ and hence $\psi^+ \subseteq \theta_I$.

(3) Since, $i^+ = 1$, for all $i \in \overline{D(L)}$, we get

$$\begin{aligned} \theta_{\overline{D(L)}} &= \{(a,b) \in L \times L : a \vee i \vee k = b \vee i \vee k,\ i \in \overline{D(L)}\} \\ &= \{(a,b) \in L \times L : a^+ \wedge i^+ \wedge k^+ = b^+ \wedge i^+ \wedge k^+\} \\ &= \{(a,b) \in L \times L : a^+ = b^+\} = \psi^+ \quad (as\ i^+ = k^+ = 1). \end{aligned}$$

(4) Since $a \vee 1 \vee k = b \vee 1 \vee k$ for all $a, b \in L$, then $(a,b) \in \theta_L$ and hence $\theta_L = \nabla_L$.

(5) The quotient set L/θ_I is $\{[a]\theta_I : a \in L\}$, where $[a]\theta_I$ is the congruence class of an element $a \in L$ modulo θ_I. It is known that $L/\theta_I = (L/\theta_I; \vee, \wedge, [1]\theta_I, [0]\theta_I)$ is a bounded distributive lattice, where $[0]_I = I$, $[1]\theta_I$ are the bounds of L/θ_I and $[a]\theta_I \wedge [b]\theta_I = [a \wedge b]\theta_I$, $[a]\theta_I \vee [b]\theta_I = [a \vee b]\theta_I$. Define L/θ_I by $[a]'\theta_I = [a^+]\theta_I$, since $[a]\theta_I \wedge [a^+]\,\theta_I = [a \wedge a^+]\theta_I = [0]\theta_I$, $[a]\theta_I \vee [a^+]\theta_I = [a \vee a^+]\theta_I = [1]\theta_I$ and $[a]''\theta_I = [a^+]'\theta_I = [a^{++}]\theta_I = [a]\theta_I$. Then $(L/\theta_I; \vee, \wedge, ', [0]\theta_I, [1]\theta_I)$ is a Boolean algebra. ■

Let $Con_k^+(L) = \{\theta_I : I \in I_k(L)\}$ be the set of all k-$\{^+\}$-congruences on L which are induced by the k-ideals of L. Using Theorem 29. We can show the following results.

Theorem 30. *For any θ_I and θ_J of $Con_k^+(L)$, we have the following:*

(1) $\theta_I \cap \theta_J = \theta_{(I \cap J)}$,

(2) $\theta_I \vee \theta_J = \theta_{(I \vee J)}$,

(3) $\big(Con_k^+(L); \vee, \wedge, \theta_{\overline{D(L)}}, \theta_L\big)$ *forms a bounded lattice and a sublattice of* $Con^+(L)$.

Proof. (1) Since $I \cap J \subseteq I, J$, by Theorem 29 $\theta_{(I\cap J)} \subseteq \theta_I, \theta_J$ implies $\theta_{(I\cap J)} \subseteq \theta_I \cap \theta_J$. Conversely, let $(a, b) \in \theta_I \cap \theta_J$. We get

$$\begin{aligned}(a, b) \in \theta_I \cap \theta_J &\Rightarrow (a, b) \in \theta_I \text{ and } (a, b) \in \theta_J \\ &\Rightarrow a \vee i \vee k = b \vee i \vee k \text{ for some } i \in I \text{ and } a \vee j \vee k = b \vee j \vee \\ &\quad\; k \text{ for some } j \in J \\ &\Rightarrow (a \vee i \vee k) \wedge (a \vee j \vee k) = (b \vee i \vee k) \wedge (a \vee j \vee k) \\ &\Rightarrow (a \vee k \vee i) \wedge (a \vee k \vee j) = (b \vee k \vee i) \wedge (a \vee k \vee j) \\ &\Rightarrow a \vee k \vee (i \wedge j) = b \vee k \vee (i \wedge j) \\ &\Rightarrow (a, b) \in \theta_{(I\cap J)} \text{ as } (i \wedge j) \in (I \cap J).\end{aligned}$$

Then $\theta_I \cap \theta_J \subseteq \theta_{(I\cap J)}$ and hence $\theta_I \cap \theta_J = \theta_{(I\cap J)}$.

(2) Since $I, J \subseteq I \vee J$, then by Theorem 29, $\theta_I, \theta_J \subseteq \theta_{(I\vee J)}$. Thus, $\theta_{(I\vee J)}$ is an upper bound of θ_I,θ_J. Conversely, let θ_k be an upper bound of θ_I and θ_J, for $k \in I_k(L)$. Then θ_I, $\theta_J \subseteq \theta_k$. Hence I, $J \subseteq k$ as $I \vee J$ is the least upper bound of I, J on $I_k(L)$. By Theorem 29, θ_I, $\theta_J \subseteq \theta_k$. Therefore $\theta_{(I\vee J)}$ is the least upper bound of θ_I,θ_J. This proves that $\theta_I \vee \theta_J = \theta_{(I\vee J)}$.

(3) From (1) and (2), it is clear that $(Con_k^+(L); \vee, \wedge)$ forms a sublattice of $Con^+(L)$. Since $\theta_{\overline{D(L)}}$ and θ_L are the smallest and the greatest members of $Con_k^+(L)$, respectively. Then $(Con_k^+(L); \vee, \wedge, \theta_{\overline{D(L)}}, \theta_L)$ is a bounded lattice. ∎

Now, we introduce the following interesting results.

Theorem 31. *For every k-$\{^+\}$-congruence θ on a CRD-Stone algebra L, we have*

(1) $[0]\,\theta$ *is a k-ideal of L,*

(2) θ *can be expressed as θ_I for some k-ideal I of L.*

Proof. (1) It is clear that $[0]\theta = \{x \in L : (x, 0) \in \theta)\} = Ker\ \theta$. It is known that the $Ker\ \theta$ is an ideal of L. Since θ is a k-$\{^+\}$-congruence, then $k \in Ker\ \theta$. Therefore $[0]\theta$ is a k-ideal of L.

(2) We claim that $\theta = \theta_{[0]\theta}$. Let $(x, y) \in \theta$. Since $(k, k) \in \theta$ hence $(x \wedge k, y \wedge k) \in \theta$. Since $[0]\theta$ is a k-ideal of L, then $x \wedge k,\ y \wedge k \in [0]\theta$. Hence

$(x \wedge k,\ y \wedge k) \in \theta_{[0]\theta}$. Now, we prove that $(x^{++}, y^{++}) \in \theta_{[0]\theta}$.

$$\begin{aligned}
\left(x^{+}, y^{+}\right) \in \theta \Rightarrow\ & \left(x^{+} \wedge x^{++}, y^{+} \wedge x^{++}\right) \in \theta \text{ and } \left(x^{+} \wedge y^{++}, y^{+} \wedge y^{++}\right) \in \theta \\
\Rightarrow\ & \left(0, y^{+} \wedge x^{++}\right) \in \theta \text{ and } \left(x^{+} \wedge y^{++}, 0\right) \in \theta \text{ (by Definition 8)} \\
\Rightarrow\ & x^{+} \wedge y^{++}, y^{+} \wedge x^{++} \in [0]\theta \\
\Rightarrow\ & \left(x^{+} \wedge y^{++}, y^{+} \wedge x^{++}\right) \in \theta_{[0]\theta} \\
\Rightarrow\ & \left(x^{+} \vee \left(x^{+} \wedge y^{++}\right), x^{+} \vee \left(y^{+} \wedge x^{++}\right)\right) = \left(x^{+}, x^{+} \vee y^{+}\right) \theta_{[0]\theta} \\
& \quad \text{(by Definition 1(2))} \\
& \text{and } \left(y^{+} \vee \left(x^{+} \wedge y^{++}\right), y^{+} \vee \left(y^{+} \wedge x^{++}\right)\right) = \left(x^{+} \vee y^{+}, y^{+}\right) \in \theta_{[0]\theta} \\
\Rightarrow\ & \left(x^{+}, y^{+}\right) \in \theta_{[0]\theta} \\
\Rightarrow\ & \left(x^{++}, y^{++}\right) \in \theta_{[0]\theta}.
\end{aligned}$$

Now, $(x^{++}, y^{++}) \in \theta_{[0]\theta}$ and $(x \wedge k, y \wedge k) \in \theta_{[0]\theta}$ imply that $(x, y) = (x^{++} \vee (x \wedge k), y^{++} \vee (y \wedge k)) = (x^{++}, y^{++}) \vee (x \wedge k, y \wedge k) \in \theta_{[0]\theta}$. Then $\theta \subseteq \theta_{[0]\theta}$. For the converse, let $(x, y) \in \theta_{[0]\theta}$. Then $(x \wedge k, y \wedge k) \in \theta_{[0]\theta}$. Since $x \wedge k, y \wedge k \in [0]\theta$, then $(x \wedge k, y \wedge k) \in \theta$.

Now, we prove that $(x^{++}, y^{++}) \in \theta$ for all $(x, y) \in \theta_{[0]\theta}$

$$\begin{aligned}
& (x, y) \in \theta_{[0]\theta} \\
& \Rightarrow \left(x^{+}, y^{+}\right) \in \theta_{[0]\theta} \\
& \Rightarrow \left(x^{+} \wedge x^{++}, y^{+} \wedge x^{++}\right), \left(x^{+} \wedge y^{++}, y^{+} \wedge y^{++}\right) \in \theta_{[0]\theta} \\
& \Rightarrow \left(0, y^{+} \wedge x^{++}\right), \left(x^{+} \wedge y^{++}, 0\right) \in \theta_{[0]\theta} \text{ as } x^{+} \wedge x^{++} = 0, y^{+} \wedge y^{++} = 0 \\
& \Rightarrow x^{+} \wedge y^{++}, y^{+} \wedge x^{++} \in [0]\theta \\
& \Rightarrow \left(x^{+} \wedge y^{++}, y^{+} \wedge x^{++}\right) \in [0]\theta \\
& \Rightarrow \left(x^{+} \vee \left(x^{+} \wedge y^{++}\right), x^{+} \vee \left(y^{+} \wedge x^{++}\right)\right), \left(y^{+} \vee \left(x^{+} \wedge y^{++}\right), y^{+} \vee \left(y^{+} \wedge x^{++}\right)\right) \in \theta \\
& \Rightarrow \left(x^{+}, \left(x^{+} \vee y^{+}\right) \wedge \left(x^{+} \vee x^{++}\right)\right), \left(\left(y^{+} \vee x^{+}\right) \wedge \left(y^{+} \vee y^{++}\right), y^{+}\right) \in \theta \\
& \quad \text{(by Definition 1(2))} \\
& \Rightarrow \left(x^{+}, x^{+} \vee y^{+}\right), \left(x^{+} \vee y^{+}, y^{+}\right) \in \theta \text{ (by Definition 8)} \\
& \Rightarrow \left(x^{+}, y^{+}\right) \in \theta \\
& \Rightarrow \left(x^{++}, y^{++}\right) \in [0]\theta.
\end{aligned}$$

Now, $(x^{++}, y^{++}) \in \theta$ and $(x \wedge k, y \wedge k) \in [0]\theta$ imply that $(x, y) = (x^{++}, y^{++}) \vee (x \wedge k, y \wedge k) \in \theta$. Therefore $\theta_{[0]\theta} \subseteq \theta$ and $\theta = \theta_{[0]\theta}$. ■

According to Theorem 30 and Theorem 31, we observe that there is a one to one correspondence between the elements of the lattice $I_k(L)$ of all k-ideals of a CRD-Stone algebra L and the elements of the lattice $Con_k^{+}(L)$ of all k-$\{^+\}$-

Congruences of L. In fact, this deduces that the lattices $I_k(L)$ and $Con_k^+(L)$ are isomorphic and hence the lattice $Con_k^+(L)$ is a distributive lattice.

Theorem 32. *Let L be a CRD-Stone algebra. Then the lattices $I_k(L)$ and $Con_k^+(L)$ are isomorphic and hence $Con_k^+(L)$ is a distributive lattice.*

Proof. Define a map h: $I_k(L) \longrightarrow Con_k^+(L)$ by $h(I) = \theta_I$, for all $I \in I_k(L)$. From Theorem 30, for $I, J \in I_k(L)$, we have

$$h(I \vee J) = \theta_I \vee \theta_J = \theta_{(I \vee J)} = h(I) \vee h(J),$$
$$h(I \cap J) = \theta_I \cap \theta_J = \theta_{(I \cap J)} = h(I) \cap h(J),$$
$$h(\overline{D(L)}) = \theta_{\overline{D(L)}} = \psi^+,$$
$$h(L) = \theta_L = \nabla_L.$$

Then h is (0,1)-lattice homomorphism. Let $h(I) = h(J)$. Then $\theta_I = \theta_J$ implies $I = J$. Thus h is an injective map. For each $\theta \in Con_k^+(L)$, by Theorem 31(2), we have $\theta = \theta_I$ for some $I \in I_k(L)$. Then $h(I) = \theta_I = \theta$ implies that h is a surjective. Therefore, h is an isomorphism and hence $I_k(L)$ and $Con_k^+(L)$ are isomorphic lattices. Since $I_k(L)$ is a distributive lattice (see Theorem 16), then also, $Con_k^+(L)$ a distributive lattice. ■

6. PRINCIPAL k-$\{^+\}$-CONGRUENCES ON A CRD-STONE ALGEBRA

In this section, we describe the principal k-$\{^+\}$-Congruences on a CRD-Stone algebra L which are induced by the principal k-ideals of L. Also, we describe the algebraic structure of the class $Con_k^p(L)$ all principal k-$\{^+\}$-ideals of L.

Proposition 33. *Let L be a CRD-Stone algebra L and $I = (x)^{\triangledown}$. Then $\theta_{(x)^{\triangledown}}$ is given as follows:*

$$\theta_{(x)^{\triangledown}} = \{(a,b) \in L \times L : a \vee x \vee k = b \vee x \vee k\} \textit{ and } Ker\ \theta_{(x)^{\triangledown}} = (x)^{\triangledown}.$$

Proof. Let $I = (x)^{\triangledown}$. Then

$$\theta_I = \theta_{(x)^{\triangledown}} = \left\{(a,b) \in L \times L : a \vee i \vee k = b \vee i \vee k, \text{ for some } i \in (x)^{\triangledown}\right\}.$$

Let $(a,b) \in \theta_I$. Since $I = (x)^{\triangledown}$, thus $a \vee i \vee k = b \vee i \vee k$, for some $i \in (x)^{\triangledown}$ and hence $a^{++} \vee i^{++} = b^{++} \vee i^{++}$. Since $i \in (x)^{\triangledown}$, then $i^{++} \leq x^{++} \vee k$ and we have $i^{++} \leq x^{++}$.

$$\begin{aligned}
a \vee x \vee k &= (a^{++} \vee (a \wedge k)) \vee (x^{++} \vee (x \wedge k)) \vee k && \text{(by Lemma 7(2))}\\
&= (a^{++} \vee (a \wedge k)) \vee x^{++} \vee ((x \wedge k) \vee k)\\
&= (a^{++} \vee (a \wedge k)) \vee x^{++} \vee k && \text{(by Definition 1(2))}\\
&= a^{++} \vee x^{++} \vee ((a \wedge k) \vee k)\\
&= a^{++} \vee x^{++} \vee k && \text{(by Definition 1(2))}\\
&= b^{++} \vee x^{++} \vee k\\
&= b^{++} \vee x^{++} \vee (x \wedge k) \vee (b \wedge k) \vee k\\
&= (b^{++} \vee (b \wedge k)) \vee (x^{++} \vee (x \wedge k)) \vee k\\
&= b \vee x \vee k.
\end{aligned}$$

Then, we have $(a, b) \in \theta_{(x)^\triangledown}$ if and only if $a \vee x \vee k = b \vee x \vee k$ and hence $\theta_{(x)^\triangledown} = \{(a, b) \in L \times L : a \vee x \vee k = b \vee x \vee k\}$. From Theorem 28, $Ker\ \theta_{(x)^\triangledown} = (x)^\triangledown$. ■

Definition 19. A k-$\{^+\}$-congruence θ on a CRD-Stone algebra L is called a principal k-$\{^+\}$-congruence if θ is a principal $\{^+\}$-congruence on L.

Proposition 34. *For any element x of a CRD-Stone algebra L, define $\theta(0, x^{++} \vee k)$ on L as follows*

$$\theta(0, x^{++} \vee k) = \{(a, b) \in L \times L : a \vee x^{++} \vee k = b \vee x^{++} \vee k\}.$$

Then $\theta(0, x^{++} \vee k)$ is a principal k-$\{^+\}$-congruence on L and $Ker\ \theta(0, x^{++} \vee k) = (x^{++} \vee k] = (x)^\triangledown$.

Proof. It is known that $\theta(0, x^{++} \vee k)$ is a principal lattice congruence on L (see Theorem 9(3)).

Let $(a, b) \in \theta(0, x^{++} \vee k)$. Then, we get

$$\begin{aligned}
&a \vee x^{++} \vee k = b \vee x^{++} \vee k\\
&\Rightarrow a^+ \wedge x^+ \wedge k^+ = b^+ \wedge x^+ \wedge k^+\\
&\Rightarrow a^+ \wedge x^+ = b^+ \wedge x^+ \text{ as } k^+ = 1\\
&\Rightarrow (a^+ \wedge x^+) \vee (x^{++} \vee k) = (b^+ \wedge x^+) \vee (x^{++} \vee k)\\
&\Rightarrow (a^+ \vee x^{++} \vee k) \wedge (x^+ \vee x^{++} \vee k) = (b^+ \vee x^{++} \vee k) \wedge (x^+ \vee x^{++} \vee k)\\
&\Rightarrow a^+ \vee x^{++} \vee k = b^+ \vee x^{++} \vee k \text{ as } x^+ \vee x^{++} = 1.
\end{aligned}$$

Then $(a^+, b^+) \in \theta(0, x^{++} \vee k)$. Thus $\theta(0, x^{++} \vee k)$ a principal $\{^+\}$-congruence on L. Since $0 \vee x^{++} \vee k = k \vee x^{++} \vee k$, then $(0, k) \in \theta(0, x^{++} \vee k)$. Then $k \in Ker\ \theta(0, x^{++} \vee k)$ and hence θ is a principal k-$\{^+\}$-congruence on L.

Now, for every for all $x \in L$, we prove $Ker\ \theta(0, x^{++} \vee k) = (x^{++} \vee k]$.

$$\begin{aligned} Ker\ \theta(0, x^{++} \vee k) &= \{y \in L : (0, y) \in \theta(0, x^{++} \vee k)\} \\ &= \{y \in L : x^{++} \vee k = y \vee x^{++} \vee k\} \\ &= \{y \in L : y \leq x^{++} \vee k\} \\ &= (x^{++} \vee k] \\ &= (x)^{\bigtriangledown}. \end{aligned}$$

■

Theorem 35. *Let x be an element of a CRD-Stone algebra L. Then*

$$\theta(0, x^{++} \vee k) = \theta_{(x)^{\bigtriangledown}}.$$

Proof. Let $(a, b) \in \theta(0, x^{++} \vee k)$. Then

$$\begin{aligned} a \vee x^{++} \vee k = b \vee x^{++} \vee k &\Rightarrow a \vee x^{++} \vee x \vee k = b \vee x^{++} \vee x \vee k \\ &\Rightarrow a \vee x \vee k = b \vee x \vee k \\ &\Rightarrow (a, b) \in \theta_{(x)^{\bigtriangledown}}. \end{aligned}$$

Thus $\theta(0, x^{++} \vee k) \subseteq \theta_{(x)^{\bigtriangledown}}$. Conversely, let $(a, b) \in \theta_{(x)^{\bigtriangledown}}$. Then we get

$$\begin{aligned} & a \vee x \vee k = b \vee x \vee k \\ & \Rightarrow a \vee (x^{++} \vee (x \wedge k)) \vee x \vee k = b \vee (x^{++} \vee (x \wedge k)) \vee x \vee k \text{ (by Lemma 7(2))} \\ & \Rightarrow a \vee x^{++} \vee ((x \wedge k) \vee k) = b \vee x^{++} \vee ((x \wedge k) \vee k) \text{ (by Definition 1(2))} \\ & \Rightarrow a \vee x^{++} \vee k = b \vee x^{++} \vee k \\ & \Rightarrow (a, b) \in \theta(0, x^{++} \vee k). \end{aligned}$$

Then $\theta_{(x)^{\bigtriangledown}} \subseteq \theta(0, x^{++} \vee k)$ and hence $\theta_{(x)^{\bigtriangledown}} = \theta(0, x^{++} \vee k)$. ■

Corollary 36. *Let L be a CRD-Stone algebra. Then*

$$Ker\ \theta_{(x)^{\bigtriangledown}} = Ker\ \theta(0, x^{++} \vee k) = (x^{++} \vee k] = (x)^{\bigtriangledown}.$$

A charclerization of a principle k-$\{^{+}\}$-congruence on a CRD-Stone algebra L is given in the following two theorems.

Theorem 37. *Let θ be a principle $\{^{+}\}$-congruence of L. Then $\theta(0, a)$ is principle k-$\{^{+}\}$-congruence if and only if $k \leq a$.*

Proof. If θ is a principle k-$\{^{+}\}$-congruence, then $k \in Ker\ \theta(0, a)$ implies $(k, 0) \in \theta(0, a)$ and hence $k \vee a = 0 \vee a = a$. Thus $k \leq a$. Conversely, let $k \leq a$ and $\theta(0, a)$ is a principal k-$\{^{+}\}$-congruence. Then $(k, 0) \in \theta(0, a)$. Since $k \in Ker\ \theta(0, a)$, thus $\theta(0, a)$ is a k-$\{^{+}\}$-congruence on L. ■

Theorem 38. *Let $\theta(0, a)$ be principle k-$\{^+\}$-congruence on L. Then $\theta(0, a) = \theta_{(a)^\triangledown}$ if and only if $k \leq a$.*

Proof. Let $\theta(a, b)$ be a k-$\{^+\}$-congruence on L and $\theta(0, a) = \theta_{(a)}$

$$\begin{aligned}\theta(0, a) = \theta_{(a)^\triangledown} &\Rightarrow k \in Ker\ \theta(0, a) = Ker\ \theta_{(a)^\triangledown}\\ &\Rightarrow (k, 0) = \theta(0, a)\\ &\Rightarrow k \vee a = 0 \vee a = a\\ &\Rightarrow k \leq a.\end{aligned}$$

Conversely, let $k \leq a$ and $(x, y) \in \theta(0, a)$.

$$\begin{aligned}(x, y) \in \theta(0, a) &\Rightarrow x \vee a = y \vee a\\ &\Rightarrow x \vee a \vee k = y \vee a \vee k\\ &\Rightarrow (x, y) \in \theta_{(a)^\triangledown}.\end{aligned}$$

Then $\theta(0, a) \subseteq \theta_{(a)^\triangledown}$. Let $(x, y) \in \theta_{(a)^\triangledown}$. Then we have

$$\begin{aligned}(x, y) \in \theta_{(a)^\triangledown} &\Rightarrow x \vee a \vee k = y \vee a \vee k\\ &\Rightarrow x \vee a = y \vee a\\ &\Rightarrow (x, y) \in \theta(0, a).\end{aligned}$$

Then $\theta_{(a)^\triangledown} \subseteq \theta(0, a)$ and hence $\theta_{(a)^\triangledown} = \theta(0, a)$. ∎

Corollary 39. *Every principle k-$\{^+\}$-congruence $\theta(0, a)$ on CRD-Stone algebra L can be expressed as $\theta(0, a^{++} \vee k)$.*

Let $\mathrm{Con}_k^p(L) = \left\{\theta_{(x)^\triangledown} : x \in L\right\}$ be the class of all principal k-$\{^+\}$-congerences which are induced by the principal k-ideals of L. Theorem 40 shows that the class $\mathrm{Con}_k^p(L)$ forms a Boolean ring which is isomorphic to the Boolean ring $I_k^p(L)$.

Theorem 40. *Let L be a CRD-Stone algebra. Then $(\mathrm{Con}_k^p(L); \oplus, \odot, \theta_{(1)^\triangledown}, \theta_{(0)^\triangledown})$ forms a Boolean ring, where*

$$\begin{aligned}\theta_{(x)^\triangledown} \oplus \theta_{(y)^\triangledown} &= \theta_{(x)^\triangledown + (y)^\triangledown},\\ \theta_{(x)^\triangledown} \odot \theta_{(y)^\triangledown} &= \theta_{(x)^\triangledown \bullet (y)^\triangledown}.\end{aligned}$$

Moreover, $\mathrm{Con}_k^p(L)$ and $I_k^p(L)$ are isomorphic Boolean rings.

Proof. According to Theorem 23, $\left(I_k^p(L); +, \bullet, (0)^\triangledown, (1)^\triangledown\right)$ is a Boolean ring. Consequently, for any $\theta_{(x)^\triangledown}, \theta_{(y)^\triangledown}, \theta_{(z)^\triangledown} \in Con_k^\triangledown(L)$, we use the properties of the ring $\left(I_k^p(L), +, \bullet\right)$ to show the following properties.

(i) The associativity of $\oplus$ and $\odot$.

$$\begin{aligned}\theta_{(x)\triangledown} \oplus \left\{\theta_{(y)\triangledown} \oplus \theta_{(z)\triangledown}\right\} &= \theta_{(x)\triangledown} \oplus \theta_{(y)\triangledown+(z)\triangledown} \\ &= \theta_{(x)\triangledown+\{(y)\triangledown+(z)\triangledown\}} \\ &= \theta_{\{(x)\triangledown+(y)\triangledown\}+(z)\triangledown} \text{ by associativity of } + \\ &= \theta_{(x)\triangledown+(y)\triangledown} \oplus \theta_{(z)\triangledown} \\ &= \left\{\theta_{(x)\triangledown} \oplus \theta_{(y)\triangledown}\right\} \oplus \theta_{(z)\triangledown},\end{aligned}$$

and

$$\begin{aligned}&\theta_{(x)\triangledown} \odot \left\{\theta_{(y)\triangledown} \odot \theta_{(z)\triangledown}\right\} = \theta_{(x)\triangledown} \odot \theta_{(y)\triangledown\bullet(z)\triangledown} \\ &= \theta_{(x)\triangledown\bullet\{(y)\triangledown\bullet(z)\triangledown\}} \\ &= \theta_{\{(x)\triangledown\bullet(y)\triangledown\}\bullet(z)\triangledown} \text{ by associativity of } \bullet \\ &= \theta_{(x)\triangledown\bullet(y)\triangledown} \odot \theta_{(z)\triangledown} \\ &= \left\{\theta_{(x)\triangledown} \odot \theta_{(y)\triangledown}\right\} \odot \theta_{(z)\triangledown}.\end{aligned}$$

(ii) The additive identity and the multiplicative identity in $\mathrm{Con}_k^p(L)$ are $\theta_{(1)\triangledown}$ and $\theta_{(0)\triangledown}$, respectively.

(iii) The commutativity of $\oplus$ and $\odot$.

$$\begin{aligned}\theta_{(x)\triangledown} \oplus \theta_{(y)\triangledown} &= \theta_{(x)\triangledown+(y)\triangledown} \\ &= \theta_{(y)\triangledown+(x)\triangledown} \text{ as } + \text{ is commutative in } I_k^p(L) \\ &= \theta_{(y)\triangledown} \oplus \theta_{(x)\triangledown}, \\ \theta_{(x)\triangledown} \odot \theta_{(y)\triangledown} &= \theta_{(b)\triangledown\bullet(y)\triangledown} \\ &= \theta_{(y)\triangledown\bullet(x)\triangledown} \text{ as } \bullet \text{ is commutative in } I_k^p(L) \\ &= \theta_{(y)\triangledown} \odot \theta_{(x)\triangledown}.\end{aligned}$$

(iv) The additive inverse of $\theta_{(x)\triangledown}$ is $\theta_{(x)\triangledown}$ itself.

(v) The distributive law holds as

$$\begin{aligned}\theta_{(x)\triangledown} \odot \left\{\theta_{(y)\triangledown} \oplus \theta_{(z)\triangledown}\right\} &= \theta_{(x)\triangledown} \odot \theta_{\{(y)\triangledown+(z)\triangledown\}} \\ &= \theta_{(x)\triangledown\bullet\{(y)\triangledown+(z)\triangledown\}} \\ &= \theta_{\{(x)\triangledown\bullet(y)\triangledown\}+\{(x)\triangledown\bullet(z)\triangledown\}} \text{ by distributivity of } I_k^p(L) \\ &= \theta_{\{(x)\triangledown\bullet(y)\triangledown\}} \oplus \theta_{\{(x)\triangledown\bullet(z)\triangledown\}} \\ &= \left\{\theta_{(x)\triangledown} \odot \theta_{(y)\triangledown}\right\} \oplus \left\{\theta_{(x)\triangledown} \odot \theta_{(z)\triangledown}\right\}.\end{aligned}$$

(vii) $\left[\theta_{(x)^{\triangledown}}\right]^2 = \theta_{(x)^{\triangledown}} \odot \theta_{(x)^{\triangledown}} = \theta_{(x)^{\triangledown} \bullet (x)^{\triangledown}} = \theta_{(x)^{\triangledown}}$.

Therefore $(\mathrm{Con}_k^p(L); \oplus, \odot, \theta_{(1)^{\triangledown}}, \theta_{(0)^{\triangledown}})$ is a Boolean ring. It is observed that the two rings $I_k^p(L)$ and $\mathrm{Con}_k^p(L)$ are isomorphic under the isomorphism $(x)^{\triangledown} \mapsto \theta_{(x)^{\triangledown}}$. ■

Combining the above Theorem 40 and Corollary 24, we will investigate the following interesting result.

Corollary 41. *Let* $(\mathrm{Con}_k^p(L); \oplus, \odot, \theta_{(1)^{\triangledown}}, \theta_{(0)^{\triangledown}})$ *be the Boolean ring of all principal* k-$\{^+\}$*-congerences on a* CRD*-Stone algebra* L*. Then* $(\mathrm{Con}_k^p(L); \vee, \cap, ', \theta_{(1)^{\triangledown}}, \theta_{(0)^{\triangledown}})$ *is a Boolean algebra, where*

$$\begin{aligned} \theta_{(x)^{\triangledown}} \vee \theta_{(y)^{\triangledown}} &= \theta_{(x \vee y)^{\triangledown}}, \\ \theta_{(x)^{\triangledown}} \cap \theta_{(y)^{\triangledown}} &= \theta_{(x \wedge y)^{\triangledown}}, \\ \theta'_{(x)^{\triangledown}} &= \theta_{(x^+)^{\triangledown}}. \end{aligned}$$

Example 42. Consider the CRD-Stone algebra S_9 as in Figure 1. The principal k-$\{^+\}$-congerences of S_9 are gives as follows:

$$\begin{aligned} &\theta(0,0) = \theta(0,c) = \theta(0,d) = \theta(0,k) = \triangle_L, \\ &\theta(0,a) = \theta(0,x) = \{\{0,d,c,k,a,x\},\{b,y,1\}\}, \\ &\theta(0,b) = \theta(0,y) = \{\{0,d,c,k,b,y\},\{a,x,1\}\}, \\ &\theta(0,1) = \triangledown_L. \end{aligned}$$

Then the following two tables show that $(\mathrm{Con}_k^p(L); \oplus, \odot)$ is a Boolean ring, where $\mathrm{Con}_k^p(L) = \{\theta(0,0), \theta(0,a), \theta(0,b), \theta(0,1)\} = \{\theta_{(0)^{\triangledown}}, \theta_{(a)^{\triangledown}}, \theta_{(b)^{\triangledown}}, \theta_{(1)^{\triangledown}}\}$.

$\oplus$	$\theta(0,0)$	$\theta(0,a)$	$\theta(0,b)$	$\theta(0,1)$
$\theta(0,0)$	$\theta(0,0)$	$\theta(0,a)$	$\theta(0,b)$	$\theta(0,1)$
$\theta(0,a)$	$\theta(0,a)$	$\theta(0,0)$	$\theta(0,1)$	$\theta(0,b)$
$\theta(0,b)$	$\theta(0,b)$	$\theta(0,1)$	$\theta(0,0)$	$\theta(0,a)$
$\theta(0,1)$	$\theta(0,1)$	$\theta(0,b)$	$\theta(0,a)$	$\theta(0,0)$

$\odot$	$\theta(0,0)$	$\theta(0,a)$	$\theta(0,b)$	$\theta(0,1)$
$\theta(0,0)$	$\theta(0,0)$	$\theta(0,0)$	$\theta(0,0)$	$\theta(0,0)$
$\theta(0,a)$	$\theta(0,0)$	$\theta(0,a)$	$\theta(0,0)$	$\theta(0,a)$
$\theta(0,b)$	$\theta(0,0)$	$\theta(0,0)$	$\theta(0,b)$	$\theta(0,b)$
$\theta(0,1)$	$\theta(0,0)$	$\theta(0,a)$	$\theta(0,b)$	$\theta(0,1)$

Figure 4. Shows that $(\mathrm{Con}_k^p(L); \oplus, \odot, \theta_{(1)^{\triangledown}}, \theta_{(0)^{\triangledown}})$ forms a Boolean algebra which is isomorphic to the Boolean algebra $I_k^p(L)$.

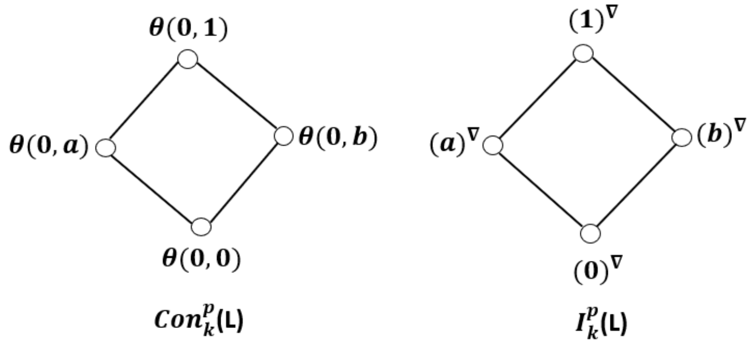

Figure 4. $\mathrm{Con}_k^p(L)$ and $I_k^p(L)$ are isomorphic Boolean algebras.

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