

4 **k -IDEALS AND k - $\{+\}$ -CONGRUENCES OF CORE**
5 **REGULAR DOUBLE STONE ALGEBRAS**

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11 **Abstract**

12 In this paper, the authors study many interesting properties of ideals
13 and congruences of the class of a core regular double Stone algebra (briefly
14 *CRD*-Stone algebra). We introduce and characterize the concepts of k -ideals
15 and principal k -ideals of a core regular double Stone algebra with the core
16 element k and establish the algebraic structures of such ideals. Also, we
17 investigate k - $\{+\}$ -congruences and principal k - $\{+\}$ -congruences of a *CRD*-
18 Stone algebra L which are induced by k -ideals and principal k -ideals of
19 L , respectively. We obtain an isomorphism between the lattice of k -ideals
20 (principal k -ideals) and the lattice of k - $\{+\}$ -congruences (principal k - $\{+\}$ -
21 congruences) of a *CRD*-Stone algebra. We provide some examples to clarify
22 the basic results of this article.

23 **Keywords:** stone algebras, double Stone algebras, regular double Stone
24 algebras, core regular double Stone algebras, ideals, filters.

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26 1. INTRODUCTION

27 The concept of pseudo-complement was considered in semi-lattices and distributive
28 lattices by Frink [22] and Birkhof [12], respectively. The class \mathbf{S} of Stone algebras
29 was studied and characterized by several authors, like, Badawy [1], Chain and

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30 Grätzer [18, 19], Grätzer [23], Frink [22], Balbes [13] and Katrinák [25]. Reg-
 31 ular double p -algebras and regular double Stone algebras are characterized by
 32 Katrinák [25] and Comer [21], respectively.

33 The intersection of the set $D(L)$ of dense elements and the set $\overline{D(L)}$ of
 34 dual dense elements of a double Stone algebra L is called the core of L and
 35 denoted by $K(L)$. In a regular double Stone algebra L , the core $K(L)$ is ei-
 36 ther an empty set or a singleton set, if a regular double Stone algebra L has a
 37 non-empty core, then such a core $K(L)$ has exactly only one element, which is
 38 denoted by k . Ravi Kumar *et al.* [27] introduced some properties of core reg-
 39 ular double Stone algebra Srikanth *et al.* [28] and [29] studied many properties
 40 of ideals (filters) and congruences of a core regular double Stone algebras, re-
 41 spectively. Badawy *et al.* [9] constructed a double Stone algebra from a Stone
 42 quadruple. Badawy [3] constructed each core regular Stone algebra from a suit-
 43 able Boolean algebra $B = (B; \vee, \wedge, ', 0, 1)$. The constructing CRD -Stone algebra
 44 $(B^{[2]}; \vee, \wedge, *, +, (0, 0), (1, 1))$ with the core element $(0, 1)$, where

$$\begin{aligned}
 45 \quad B^{[2]} &= \{(x, y) \in B^{[2]} : x \leq y\}, \\
 46 \quad (x, y) \wedge (x_1, y_1) &= (x \wedge x_1, y \wedge y_1), \\
 47 \quad (x, y) \vee (x_1, y_1) &= (x \vee x_1, y \vee y_1), \\
 48 \quad (x, y)^* &= (y', y'), \\
 49 \quad (x, y)^+ &= (x', x').
 \end{aligned}$$

50 In Section 2, We list the basic concepts and important results which are
 51 needed throughout this paper. Also, we provide some examples of RD -Stone
 52 algebras with core element k and RD -Stone algebras with empty core. We refer
 53 the reader to [4, 7, 8, 10, 15] and [16] for filters, ideals and [2, 6, 11] for congruences
 54 of lattices and p -algebras.

55 In Section 3, we introduce the k -ideals of a CRD -Stone algebra L and obtain
 56 many related properties. A set of equivalent conditions for an ideal I of a CRD -
 57 Stone algebra L to become a k -ideal is given. We observe that the class $I_k(L)$ of
 58 all k -ideals of L forms a bounded distributive lattice.

59 In Section 4, we define and characterize the concept of principal k -ideals of a
 60 CRD -Stone algebra L . We show that the class $I_k^p(L)$ of all principal k -ideals of
 61 L is a Boolean ring and so a Boolean algebra. Example 25 describes the Boolean
 62 algebra $I_k^p(L)$.

63 In Section 5, we investigate the k - $\{+\}$ -congruences via k -ideals of a CRD -
 64 Stone algebra L . Also, we observe that the set $Con_k^+(L)$ of all k - $\{+\}$ -congruences
 65 forms a bounded distributive lattice which is isomorphic to the lattice $I_k(L)$ of
 66 k -ideals.

67 In Section 6, we investigate and characterize the principal k - $\{+\}$ -congruences
 68 of a CRD -Stone algebra L via principal k -ideals of L . Then, we study the
 69 properties and the algebraic structure of the class $Con_k^p(L)$ of all principal k - $\{+\}$ -

70 congruences of L . Moreover, we show that $I_k^p(L)$ and $Con_k^p(L)$ are isomorphic
 71 Boolean algebras. We give Example 42 to clarify the last result.

72 2. PRELIMINARIES

73 In this section, we recall certain definitions and results which are used throughout
 74 the paper, which are taken from the references [1, 5, 14, 21, 23, 27, 28] and [30].

75 **Definition 1** [1]. An algebra $(L; \wedge, \vee)$ of type $(2, 2)$ is said to be a lattice if

- 76 (1) the operations \wedge, \vee are idempotent, commutative and associative,
 77 (2) the absorption identities hold on L , that is, $(a \wedge b) \vee a = a, (a \vee b) \wedge a = a$.

78 **Definition 2** [14]. A lattice L is called a bounded if it has the greatest element
 79 1 and the smallest element 0.

80 **Definition 3** [1]. A lattice L is called a distributive lattice if it satisfies either
 81 of the following equivalent distributive laws:

- 82 (1) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,
 83 (2) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$, for all $a, b, c \in L$.

84 **Definition 4** [28]. A nonempty subset I of a lattice L is called an ideal if

- 85 (1) $x \vee y \in I$ for all $x, y \in I$,
 86 (2) $x \in I$ and $z \in L$ be such that $z \leq x$ imply $z \in I$.

87 **Definition 5** [23]. If $\phi \neq A \subseteq L$, then $(A]$ is the smallest ideal of a lattice L which
 88 contains A , where $(A] = \{x \in L : x \leq a_1 \vee a_2 \vee \dots \vee a_n, a_i \in A, i = 1, 2, \dots, n\}$.

89 The case that $A = \{a\}$, we write $(a]$ instead of $(\{a\}]$ and $(a]$ is called the
 90 principal ideal of L generated by a , where $(a] = \{x \in L : x \leq a\}$.

91 Let $I(L)$ be the set of all ideals of a lattice L . Then $(I(L); \wedge, \vee)$ forms a
 92 lattice, where

93
$$I \wedge J = I \cap J \text{ and } I \vee J = \{x \in L : x \leq i \vee j : i \in I, j \in J\}.$$

94 Also, algebra $(I^p(L); \vee, \wedge)$ of all principal ideals of L is a sublattice of the lattice
 95 $I(L)$, where

96
$$(a] \vee (b] = (a \vee b] \text{ and } (a] \wedge (b] = (a \wedge b].$$

97 It is known that the lattice $I(L)$ is distributive if and only if L is distributive.

Definition 6 [1]. For any element a of a bounded lattice L , the dual pseudo-
 complement a^+ (the pseudo- complement a^*) of a is defined as follows

$$a \vee x = 1 \Leftrightarrow a^+ \leq x \quad (a \wedge x = 0 \Leftrightarrow x \leq a^*).$$

98 **Definition 7** [23]. A distributive lattice L in which every element has a pseu-
 99 docomplement is called a distributive pseudo-complemented lattice or a distribu-
 100 tive p -algebra. Dually, a distributive lattice L in which every element has a dual
 101 pseudocomplement is called a distributive dual pseudocomplement lattice or dual
 102 distributive p -algebra.

103 **Definition 8** [5]. A distributive p -algebra (distributive dual p -algebra) L is called
 104 a Stone algebra (dual Stone algebra) if $x^* \vee x^{**} = 1$ ($x^+ \wedge x^{++} = 0$) for all $x \in L$.

105 **Theorem 1** [1]. *Let L be a distributive p -algebra (distributive dual p -algebra).*
 106 *Then for any two elements a, b of L , we have*

- 107 (1) $0^{**} = 0$ and $1^{**} = 1$ ($0^{++} = 0$ and $1^{++} = 1$),
 108 (2) $a \wedge a^* = 0$ ($a \vee a^+ = 1$),
 109 (3) $a \leq b$ implies $b^* \leq a^*$ ($a \geq b$ implies $b^+ \geq a^+$),
 110 (4) $a \leq a^{**}$ ($a^{++} \leq a$),
 111 (5) $a^{***} = a^*$ ($a^{+++} = a^+$),
 112 (6) $(a \vee b)^* = a^* \wedge b^*$ ($(a \wedge b)^+ = a^+ \vee b^+$),
 113 (7) $(a \wedge b)^* = a^* \vee b^*$ ($(a \vee b)^+ = a^+ \wedge b^+$),
 114 (8) $(a \vee b)^{**} = a^{**} \vee b^{**}$ ($(a \wedge b)^{++} = a^{++} \wedge b^{++}$),
 115 (9) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$ ($(a \vee b)^{++} = a^{++} \vee b^{++}$).

116 **Definition 9** [30]. A Double Stone-algebra L is an algebra $\langle L, *, + \rangle$, where

- 117 (i) $(L, *)$ is a Stone algebra,
 118 (ii) $(L, +)$ is a dual Stone algebra.

119 **Definition 10** [21]. A regular double Stone algebra (briefly RD -Stone algebra)
 120 L is a double Stone such that

121
$$x^{**} = y^{**} \text{ and } x^{++} = y^{++} \text{ imply } x = y.$$

122 Let L be a double Stone algebra. The element $a \in L$ is called a closed
 123 element of L if $a^{**} = a$ and the element $a \in L$ is called a dual closed element of
 124 L if $a^{++} = a$. An element $d \in L$ is called dense if $d^* = 0$ and an element $d \in L$
 125 is called dual dense if $d^+ = 1$.

126 **Lemma 2** [28]. *Let L be a double Stone algebra. Then*

- 127 (1) *the set $D(L) = \{a \in L \mid a^* = 0\} = \{a \vee a^* \mid a \in L\}$ of all dense elements of*
 128 *L is a filter of L ,*
 129 (2) *the set $\overline{D(L)} = \{a \in L \mid a^+ = 1\} = \{a \wedge a^+ \mid a \in L\}$ of all dual dense ele-*
 130 *ments of L is an ideal of L ,*

- 131 (3) the set $B(L) = \{a^* : a \in L\} = \{a^+ : a \in L\}$ of all closed elements of L
 132 forms a Boolean subalgebra of L ,
 133 (4) the set $K(L) = D(L) \cap \overline{D(L)}$ is called the core of L , we have two cases of
 134 $K(L)$, namely, $K(L) = \phi$ or $K(L) \neq \phi$.

135 It is easy to show the proof of the following two lemmas.

136 **Lemma 3.** The non empty core $K(L)$ of a RD -Stone algebra L has exactly one
 137 element.

138 **Definition 11.** A regular double Stone algebra with non empty core is called a
 139 core regular double Stone algebra (briefly CRD -Stone algebra).

140 **Lemma 4.** Let L be a CRD -Stone algebra with the core k . Then

- 141 (1) $D(L) = [k]$, that is, $D(L)$ is a principal filter of L generated by k ,
 142 (2) $\overline{D(L)} = (k)$, that is, $\overline{D(L)}$ is a principal ideal of L generated by k .

143 We use k for the core element of a CRD -Stone algebra L , that is, $K(L) = \{k\}$.
 144 Now, we give examples of CRD -Stone algebras and RD -Stone algebras with
 145 empty core.

146 **Example 5.** (1) Let $L = \{0, x, y, 1 : 0 < x < y < 1\}$ be the four element chain. It
 147 is clear that $\langle L, *, ^+ \rangle$ is a double Stone algebra, where $x^* = y^* = 1^* = 0$, $0^* = 1$
 148 and $0^+ = x^+ = y^+ = 1$, $1^+ = 0$. Then $K(L) = D(L) \cap \overline{D(L)} = \{x, y, 1\} \cap$
 149 $\{x, y, 0\} = \{x, y\}$ is a non empty core. We observe that L is not regular as
 150 $x^{++} = y^{++}$ and $x^{**} = y^{**}$, but $x \neq y$.

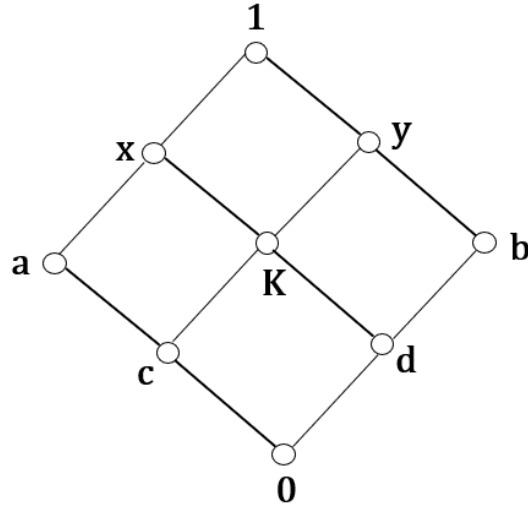
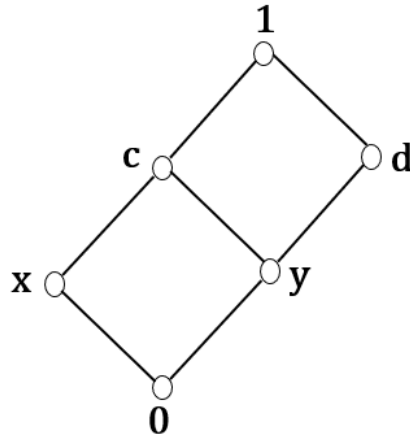
151 (2) The double Stone algebra $S_3 = \{0, k, 1 : 0 < k < 1\}$ is the smallest non
 152 trival core regular double Stone algebra with core k , (S_3 is called the discrete
 153 CRD -Stone algebra).

154 (3) Every Boolean algebra $(B; \vee, \wedge, ', 0, 1)$ can be regarded as a RD -Stone
 155 algebra with empty core, where $x^* = x^+ = x'$, for all $x \in B$ and $K(B) =$
 156 $\{1\} \cap \{0\} = \phi$.

157 **Example 6.** (1) Consider the bounded distributive lattice S_9 in Figure 1. It
 158 is clear that L_1 is a core regular double Stone algebra with core element k ,
 159 where $k^* = 1^* = y^* = x^* = 0$, $c^* = a^* = b$, $d^* = b^* = a$, $1^* = 0$ and
 160 $k^+ = c^+ = d^+ = 0^+ = 1$, $b^+ = y^+ = a$, $x^+ = a^+ = b$, $0^+ = 1$.

161 (2) Consider the bounded distributive lattice L_1 in Figure 2. We observe that
 162 L_1 is a regular double Stone algebra with empty core as $K(L) = D(L_1) \cap \overline{D(L_1)} =$
 163 $\{d, 1\} \cap \{0, y\} = \phi$, where $0^* = d^* = 1^*$, $c = x^*$, $x = c^* = y^*$, $1 = 0^*$ and $0 = 1^+$,
 164 $c = x^+ = d^+$, $x = c^+$, $1 = y^+ = 0^+$.

165 **Lemma 7.** If L is a CRD -Stone algebra with core element k , then every element
 166 x of L can be written by each of the following formulas:

Figure 1. S_9 is a *CRD*-Stone algebra with core k .Figure 2. L_1 is a *RD*-Stone algebra with empty core.

167 (1) $x = x^{**} \wedge (x^{++} \vee k)$ and its dual $x = x^{++} \vee (x^{**} \wedge k)$,

168 (2) $x = x^{**} \wedge (x \vee k)$ and its dual $x = x^{++} \vee (x \wedge k)$.

169 **Definition 12** [1]. An equivalent relation θ on a lattice L is called a lattice
 170 congruence on L if $(a, b) \in \theta$ and $(c, d) \in \theta$ implies $(a \vee c, b \vee d) \in \theta$ and $(a \wedge c, b \wedge d)$
 171 $\in \theta$.

172 **Theorem 8** [23]. *An equivalent relation on a distributive lattice L is a lattice*
 173 *congruence on L if and only if $(a, b) \in \theta$ implies $(a \vee z, b \vee z) \in \theta$ and $(a \wedge z, b \wedge z) \in \theta$*
 174 *for all $z \in L$.*

175 **Definition 13.** A lattice congruence θ on a dual Stone (Stone) algebra L is called
 176 a $\{^+\}$ -congruence ($\{^*\}$ -congruence) if $(a, b) \in \theta$ implies $(a^+, b^+) \in \theta$ ($(a, b) \in \theta$
 177 implies $(a^*, b^*) \in \theta$).

178 **Definition 14.** A lattice congruence θ on a D -Stone algebra L is called a con-
 179 gruence (or $\{^*, ^+\}$ -congruence) if $(a, b) \in \theta$ implies $(a^*, b^*) \in \theta$ and $(a^+, b^+) \in \theta$.

180 A binary relation Ψ^+ defined a double Stone algebra L by

$$181 \quad (x, y) \in \Psi^+ \Leftrightarrow x^+ = y^+$$

182 is a $\{^+\}$ -congruence relation which is called the dual Glivenko congruence relation
 183 on L . It is known that the quotient lattice $L/\Psi = \{[x]\Psi : x \in L\}$ is a Boolean
 184 algebra and $L/\Psi \cong B(L)$, where $[x]\Psi = \{y \in L : y^+ = x^+\}$ is the congruence
 185 class of x modulo Ψ . Moreover, the element x^{++} is the smallest element of the
 186 congruence class $[x]\Psi$, $[0]\Psi = \overline{D(L)}$ and $[1]\Psi = \{1\}$.

187 For a double Stone algebra L , we use $Con(L)$ to denote the lattice of all
 188 congruence of L and $Con^+(L)$ to denote the lattice of all $\{^+\}$ -congruence of a
 189 dual Stone algebra $(L, ^+)$. Also, we use ∇_L and Δ_L for the universal congruence
 190 $L \times L$ and equality congruence $\{(x, x) : x \in L\}$ of L , respectively.

191 **Definition 15** [14]. A lattice congruence θ on a lattice L is called a principal con-
 192 gruenence and is denoted by $\theta(a, b)$ if θ is the smallest congruence on L containing
 193 a, b on the same class.

194 **Theorem 9** [14]. *If L is a distributive lattice and $a, b \in L$ then the principal*
 195 *congruence $\theta(a, b)$ of L is given by*

$$196 \quad (1) \quad (x, y) \in \theta(a, b) \Leftrightarrow x \vee a \vee b = y \vee a \vee b \text{ and } x \wedge a \wedge b = y \wedge a \wedge b,$$

$$197 \quad (2) \quad \text{If } a \leq b, \text{ then } (x, y) \in \theta(a, b) \Leftrightarrow x \vee b = y \vee b \text{ and } x \wedge a = y \wedge a,$$

$$198 \quad (3) \quad (x, y) \in \theta(0, b) \Leftrightarrow x \vee b = y \vee b.$$

199 Throughout the paper, we will use L for a CRD -Stone algebra and k for the
 200 core element of L . For more information we refer the reader to [24, 31] for Stone
 201 algebras, [32] for double Stone algebras, [21] for regular double Stone algebras
 202 and [20, 27, 28, 29] for core regular double Stone algebras.

203 3. k -IDEALS OF CRD -STONE ALGEBRAS

204 In this section, we define the notion of k -ideal of a CRD -Stone algebra L and
 205 introduce many basic properties of such ideals. A characterization of a k -ideal

206 of a *CRD*-Stone algebra L is given. Also, we observe that the class $I_k(L)$ of all
207 k -ideals of L forms a bounded distributive lattice.

208 **Definition 16.** An ideal I of a *CRD*-Stone algebra L with core k is called a
209 k -ideal if $k \in I$.

210 Let A be a non empty subset of a *CRD*-Stone algebra L . Consider A^∇ as
211 follows

$$212 \quad A^\nabla = \{x \in L : x^{++} \leq a^{++} \vee k, \text{ for some } a \in A\}.$$

213 **Lemma 10.** Let A be a non empty subset of a *CRD*-Stone algebra L , which is
214 closed under \vee . Then A^∇ is a k -ideal of L containing A .

215 **Proof.** Clearly $0, k \in (A)^\nabla$. Let $x, y \in (A)^\nabla$. Thus $x^{++} \leq a^{++} \vee k, y^{++} \leq$
216 $b^{++} \vee k$ for some $a, b \in A$. Then $(x \vee y)^{++} \leq (a \vee b)^{++} \vee k$ and $a \vee b \in A$, imply
217 $x \vee y \in (A)^\nabla$. Now, let $x \in L, y \in (A)^\nabla$ and $x \leq y$. Then $x^{++} \leq y^{++} \leq a^{++} \vee k$.
218 So $x \in (A)^\nabla$. Thus $(A)^\nabla$ is k -ideal of L . Since, $a^{++} \leq a^{++} \vee k$, for all $a \in A$,
219 then $A \in A^\nabla$. \blacksquare

220 **Lemma 11.** Let A, B be two subsets of a *CRD*-Stone algebra L , which are closed
221 under \vee . Then

- 222 (1) $(A]^\nabla = A^\nabla$,
- 223 (2) $A \subseteq B \Rightarrow A^\nabla \subseteq B^\nabla$,
- 224 (3) $A^\nabla = (A] \vee \overline{D(L)}$,
- 225 (4) $A^{\nabla\nabla} = A^\nabla$.

226 **Proof.** (1) Since A is closed with respect to \vee , then for $a \in (A]$, we have $a \leq$
227 $a_1 \vee a_2 \vee \dots \vee a_n \in A, a_i \in A, i = 1, 2, \dots, n$. Immediately, we get

$$\begin{aligned} (a]^\nabla &= \{x \in L : x^{++} \leq a^{++} \vee k, \text{ for some } a \in (A]\} \\ &= \{x \in L : x^{++} \leq (a_1 \vee a_2 \vee \dots \vee a_n)^{++} \vee k, a_1 \vee a_2 \vee \dots \vee a_n \in A\} = A^\nabla. \end{aligned}$$

228 (2) Suppose $A \subseteq B$ and $x \in A^\nabla$. Then $x^{++} \leq a^{++} \vee k$ for some $a \in A \subseteq B$.
229 It follows that $x \in B^\nabla$. Thus $A^\nabla \subseteq B^\nabla$.

230 (3) Since $(A] \subseteq (A]^\nabla = A^\nabla$ by (1) and $\overline{D(L)} = (k] \subseteq A^\nabla$, then $(A]^\nabla \vee \overline{D(L)} \subseteq$
231 A^∇ . Conversely, let $x \in A^\nabla$. Then $x^{++} \leq a^{++} \vee k$ for some $a \in A$. We have

$$\begin{aligned} x &= x^{++} \vee (x \wedge k) \leq (a^{++} \vee k) \vee (x \wedge k) && \text{(by Lemma 7.(2))} \\ &= (a^{++} \vee k \vee x) \wedge (a^{++} \vee k) && \text{(by distributivity of } L) \\ &= a^{++} \vee k \leq a \vee k \in (a \vee k] \\ &\Rightarrow x \in (a \vee k] = (a] \vee (k] = (a] \vee \overline{D(L)} \subseteq (A] \vee \overline{D(L)} \\ & && \text{((as } (a] \subseteq (A]).) \end{aligned}$$

232 Therefore $A^\nabla = (A] \vee \overline{D(L)}$.

233 (4) By the definition of A^∇ , we have

$$\begin{aligned} A^{\nabla\nabla} &= \{x \in L : x^{++} \leq a_1^{++} \vee k, \text{ for some } a_1 \in A^\nabla\} \\ &= \{x \in L : x^{++} \leq a_1^{++} \vee k, a_1^{++} \leq a^{++} \vee k \text{ for some } a \in A\} \\ 234 \quad &= \{x \in L : x^{++} \leq a^{++} \vee k, \text{ for some } a \in A\} = A^\nabla. \quad \blacksquare \end{aligned}$$

235 A characterization of k -ideals of a CRD -Stone algebra L is given in the
236 following.

237 **Theorem 12.** *Let I be an ideal of a CRD -Stone algebra L with core k . Then*
238 *the following statements are equivalent:*

- 239 (1) I is a k -ideal of L ,
- 240 (2) $\overline{D(L)} \subseteq I$,
- 241 (3) $x \wedge x^+ \in I$, for all $x \in L$,
- 242 (4) $I = I^\nabla$.

243 **Proof.** (1) \Rightarrow (2) Let I is a k -ideal of L . Then $k \in I$ implies $\overline{D(L)} = (k] \subseteq I$.

244 (2) \Rightarrow (3) Let $\overline{D(L)} \subseteq I$. For all $x \in L$, we have $x \wedge x^+ \in \overline{D(L)} \subseteq I$.

245 (3) \Rightarrow (4) By Lemma 10, $I \subseteq I^\nabla$. For the converse, let $y \in I^\nabla$. Then $y^{++} \leq$
246 $i^{++} \vee k$, for some $i \in I$. Thus $y^{++} \leq i^{++}$. By Lemma 7(2) $y = y^{++} \vee (y \wedge k) \leq$
247 $i^{++} \vee (y \wedge k)$. By (3), $k = k \wedge k^+ \in I$, where $k^+ = 1$. Since, i^{++} , $y \wedge k \in I$, then
248 $i^{++} \vee (y \wedge k) \in I$ and hence $y \in I$.

249 (4) \Rightarrow (1) Since $k \in I^\nabla$, Lemma 10. Then by (4), $k \in I$ and hence I is a
250 k -ideal of a CRD -Stone algebra L . \blacksquare

251 As a consequence of Lemma 11 and Theorem 12, we investigate the following
252 Corollary 13 and Lemma 14, respectively.

253 **Corollary 13.** *For any two ideals I, J of a CRD -Stone algebra L , we have the*
254 *following:*

- 255 (1) $I \subseteq J \Rightarrow I^\nabla \subseteq J^\nabla$,
- 256 (2) $I^{\nabla\nabla} = I^\nabla$.

257 **Lemma 14.** *Let L be a CRD -Stone algebra L . Then*

- 258 (1) $I^\nabla = I \vee \overline{D(L)}$,
- 259 (2) $\overline{D(L)}$ is the smallest k -ideal of L ,
- 260 (3) Every k -ideal of L can be expressed in the form I^∇ for some $I \in I(L)$.

261 Let $I_k(L) = \{I : I \text{ is a } k\text{-ideal of } L\} = \{I^\nabla : I \in I(L)\}$ be the set of all
262 k -ideals of L .

263 **Theorem 15.** *Let L be a CRD-Stone algebra L . Then for all $I, J \in I(L)$*

264 (1) $(I \vee J)^\nabla = I^\nabla \vee J^\nabla,$

265 (2) $(I \cap J)^\nabla = I^\nabla \cap J^\nabla.$

266 **Proof.** (1) Since $I, J \subseteq I \vee J$. Then by Corollary 13(1), $I^\nabla, J^\nabla \subseteq (I \vee J)^\nabla$.
 267 Thus, $(I \vee J)^\nabla$ is an upper bound of I^∇ and J^∇ . Let H^∇ be an upper bound of
 268 both I^∇ and J^∇ for some $H \in I_k(L)$. Then $I^\nabla, J^\nabla \subseteq H^\nabla$ implies $I, J \subseteq H^\nabla$.
 269 Hence, $I \vee J \subseteq H^\nabla$. Therefore, by Corollary 13(1) and (2), we get $(I \vee J)^\nabla \subseteq$
 270 $H^{\nabla\nabla} = H^\nabla$. This deduce that $(I \vee J)^\nabla$ is the least upper bound of both I^∇ and
 271 J^∇ in $I_k(L)$. Then $(I \vee J)^\nabla = I^\nabla \vee J^\nabla$.

272 (2) Obviously, $(I \cap J)^\nabla \subseteq I^\nabla \cap J^\nabla$. Conversely, let $x \in I^\nabla \cap J^\nabla$. Then
 273 $x^{++} \leq i^{++} \vee k$ and $x^{++} \leq j^{++} \vee k$ for some $i \in I$ and $j \in J$. Hence $x^{++} \leq$
 274 $(i^{++} \vee k) \wedge (j^{++} \vee k) = (i^{++} \wedge j^{++}) \vee k = (i \wedge j)^{++} \vee k$. It yields that $x \in (I \cap J)^\nabla$
 275 as $i \wedge j \leq i, j$ implies $i \wedge j \in I \cap J$. Therefore $I^\nabla \cap J^\nabla \subseteq (I \cap J)^\nabla$. ■

276 **Theorem 16.** *The class $I_k(L)$ of all k -ideals of a CRD-Stone algebra L forms*
 277 *a bounded distributive lattice and $\{1\}$ -sublattice of $I(L)$.*

278 **Proof.** From Theorem 15, $(I_k(L); \vee, \wedge)$ is a sublattice of the lattice $I(L)$, where

279 $(I \vee J)^\nabla = I^\nabla \vee J^\nabla$ and $(I \cap J)^\nabla = I^\nabla \cap J^\nabla$ for all $I, J \in I(L)$.

280 Then $(I_k(L); \vee, \wedge)$ is sublattice of $I(L)$. Since $I(L)$ is a distributive lattice,
 281 then $I_k(L)$ is also distributive. Since $\overline{D(L)}$ and L are the smallest and the great-
 282 est members of $I_k(L)$, respectively. Then $(I_k(L); \vee, \wedge, \overline{D(L)}, L)$ is a bounded
 283 distributive lattice on its own and hence a $\{1\}$ -sublattice of $I(L)$. ■

284 4. PRINCIPAL k -IDEALS OF A CRD-STONE ALGEBRA

285 In this section, we introduce the concept of principal k -ideals of a CRD-Stone
 286 algebra L and investigate many elegant properties of such ideals. A characteri-
 287 zation of a k -ideal of L is given via the principal k -ideals. It is observed the set
 288 of all principal k -ideals of a CRD-Stone algebra L is a Boolean ring and so a
 289 Boolean algebra.

290 Now, let $A = \{a\}$ be a subset of a CRD-Stone L . Then ready is seen that

291 $\{a\}^\nabla = \{x \in L : x^{++} \leq a^{++} \vee k\}.$

292 For brevity, set $(a)^\nabla$ instead of $\{a\}^\nabla$. Clearly, $(0)^\nabla = \overline{D(L)}$ and $(1)^\nabla = L$, are
 293 the smallest and the greatest k -ideals of L , respectively.

294 **Definition 17.** A k -ideal I of a CRD-Stone algebra L is called a principal k -ideal
 295 of L if I is a principal ideal of L .

296 **Theorem 17.** *Let L be a CRD-Stone algebra. Then for any $x, y \in L$, we get*

- 297 (1) $y \in (x)^\nabla \Leftrightarrow y^+ \vee x = 1$,
 298 (2) $(x)^\nabla = (x^{++} \vee k] = (x^{++}] \vee \overline{D(L)}$, this is, $(x)^\nabla$ is a principal k -ideal of L ,
 299 (3) $x \in \overline{D(L)} \Leftrightarrow (x)^\nabla = \overline{D(L)}$.

300 **Proof.** (1) Let $y \in (x)^\nabla$. Then, we have

$$\begin{aligned} y^{++} \leq x^{++} \vee k &\Leftrightarrow y^+ \geq x^+ \\ &\Leftrightarrow y^+ \vee x = 1 \end{aligned} \quad (\text{by Definition 6})$$

301 (2) For all $x \in L$, we get

$$\begin{aligned} (x)^\nabla &= \{y \in L : y^{++} \leq x^{++} \vee k\} \\ &= \{y \in L : y^{++} \vee (y \wedge k) \leq x^{++} \vee k \vee (y \wedge k)\} \\ &= \{y \in L : y \leq x^{++} \vee k\} \quad (\text{by Lemma 7(2) and Definition 1(2)}) \\ &= (x^{++} \vee k] \\ &= (x^{++}] \vee (k] = (x^{++}] \vee \overline{D(L)}. \end{aligned}$$

302 (3) Let $x \in \overline{D(L)}$. Then $x^+ = 1$. Now,

$$\begin{aligned} (x)^\nabla &= (x^{++} \vee k] \quad (\text{by(2)}) \\ &= (0 \vee k] = (k] = \overline{D(L)}. \end{aligned}$$

303 The second implication is clear. ■

304 More interesting properties of principal k -ideals are given in the following
 305 two lemmas.

306 **Lemma 18.** *Let L be a CRD-Stone algebra L . Then for any $x, y \in L$, we have*

- 307 (1) $(x)^{\nabla\nabla} = (x)^\nabla$,
 308 (2) $(x]^\nabla = (x)^\nabla$,
 309 (3) $x \in (y)^\nabla \Leftrightarrow (x)^\nabla \subseteq (y)^\nabla$,
 310 (4) $x \leq y \Rightarrow (x)^\nabla \subseteq (y)^\nabla$.

311 **Lemma 19.** *Let L be a CRD-Stone algebra L . For any $x, y \in L$, we have*

- 312 (1) $(x)^\nabla = (x^{++})^\nabla$,
 313 (2) $(x \wedge y)^\nabla = (x)^\nabla \cap (y)^\nabla$,
 314 (3) $(x \vee y)^\nabla = (x)^\nabla \vee (y)^\nabla$,
 315 (4) $(x \vee x^+)^\nabla = (1)^\nabla = L$,

$$316 \quad (5) \quad (x \wedge x^+)^\nabla = \overline{D(L)}.$$

317 **Proof.** (1) $(x)^\nabla = \{y \in L : y^{++} \leq x^{++} \vee k = (x^{++})^{++} \vee k\} = (x^{++})^\nabla$, as
 318 $x^{++++} = x^{++}$.

319 (2) By Theorem 17.(2), we get

$$\begin{aligned} (x \wedge y)^\nabla &= ((x \wedge y)^{++}] \vee \overline{D(L)} \\ &= ((x^{++} \wedge y^{++})] \vee \overline{D(L)} \\ &= ((x^{++}] \cap (y^{++})] \vee \overline{D(L)} \\ &= ((x^{++}] \vee \overline{D(L)}) \cap ((y^{++})] \vee \overline{D(L)}) \quad (\text{by distributivity of } I(L)) \\ &= (x)^\nabla \cap (y)^\nabla. \end{aligned}$$

320 (3) By Theorem 17(2), we get

$$\begin{aligned} (x \vee y)^\nabla &= ((x \vee y)^{++}] \vee \overline{D(L)} \\ &= ((x^+ \vee y^+)^{++}] \vee \overline{D(L)} \\ &= (x^{++} \vee y^{++})] \vee \overline{D(L)} \\ &= ((x^{++}] \vee (y^{++})] \vee \overline{D(L)} \\ &= ((x^{++}] \vee \overline{D(L)}) \vee ((y^{++})] \vee \overline{D(L)}) \quad (\text{by distributivity of } I(L)) \\ &= (x)^\nabla \vee (y)^\nabla. \end{aligned}$$

321 (4) Since $x \vee x^+$, we get $(x \vee x^+)^\nabla = (1) = L$.

322 (5) Since $x \wedge x^+ \in \overline{D(L)}$, then by Theorem 17(3), $(x \wedge x^+)^\nabla = \overline{D(L)}$. ■

323 **Lemma 20.** *Let L be a CRD-Stone algebra L . For any $x, y \in L$, we have*

$$324 \quad (1) \quad (x)^\nabla = (y)^\nabla \Leftrightarrow x^{++} = y^{++} \Leftrightarrow x^+ = y^+,$$

$$325 \quad (2) \quad (x)^\nabla = (y)^\nabla \Rightarrow (x \wedge z)^\nabla = (y \wedge z)^\nabla, \quad \forall z \in L,$$

$$326 \quad (3) \quad (x)^\nabla = (y)^\nabla \Rightarrow (x \vee z)^\nabla = (y \vee z)^\nabla, \quad \forall z \in L.$$

327 Now, we introduce the following important result.

328 **Theorem 21.** *Every principal k -ideal of L can be expressed as $(x)^\nabla$ for some*
 329 $x \in L$.

330 **Proof.** Let $(x]$ be a principal k -ideal of L . We claim that $(x] = (x)^\nabla$. Since
 331 $x \in (x)^\nabla$ then $(x] \subseteq (x)^\nabla$. For the converse, let $y \in (x)^\nabla$. Then

$$\begin{aligned} y \in (x)^\nabla &\Rightarrow y^{++} \leq x^{++} \vee k \\ &\Rightarrow y^{++} \vee (y \wedge k) \leq (x^{++} \vee k) \vee (y \wedge k) = (x^{++} \vee k \vee y) \wedge (x^{++} \vee k) \\ &= x^{++} \vee k \leq x \vee k \\ &\Rightarrow y \leq x \vee k \quad \text{as } y = y^{++} \vee (y \wedge k) \\ &\Rightarrow y \in (x \vee k] \subseteq (x] \quad \text{as } k \leq x. \end{aligned}$$

332 Therefore $(x)^\nabla \subseteq (x]$ and hence $(x)^\nabla = (x]$. ■

333 A characterization of a k -ideal via the principal k -ideal is given in the follow-
334 ing theorem.

335 **Theorem 22.** *Let I be an ideal of a CRD-Stone algebra L . Then the following*
336 *statements are equivalent:*

- 337 (1) I is a k -ideal,
338 (2) $x^{++} \in I \Rightarrow x \in I$,
339 (3) for all $x, y \in L$, $(x)^\nabla = (y)^\nabla$ and $y \in I \Rightarrow x \in I$,
340 (4) $I = \bigcup_{x \in I} (x)^\nabla$,
341 (5) $x \in I \Rightarrow (x)^\nabla \subseteq I$.

342 **Proof.** (1) \Rightarrow (2) Let I be a k -ideal of L and $x^{++} \in I$. Then $k \in I$ implies
343 $x \wedge k \in I$. Now, x^{++} , $x \wedge k \in I$ imply that $x = x^{++} \vee (x \wedge k) \in I$.

344 (2) \Rightarrow (3) Let $(x)^\nabla = (y)^\nabla$, $y \in I$. Thus $x \in (y)^\nabla$. Then, $x^{++} \leq y^{++} \vee k$
345 implies $x^{++} \leq y^{++} \leq y \in I$. Thus, $x^{++} \in I$. By (2), we get $x \in I$.

346 (3) \Rightarrow (4) For any $x \in I$, we have $x \in (x)^\nabla \subseteq \bigcup_{x \in I} (x)^\nabla$. Then $I \subseteq \bigcup_{x \in I} (x)^\nabla$.
347 Conversely, let $y \in \bigcup_{x \in I} (x)^\nabla$. Then $y \in (z)^\nabla$ for some $z \in I$. Hence, $(y)^\nabla \subseteq$
348 $(z)^\nabla$, by Lemma 18(3). It follows that $(y)^\nabla = (y)^\nabla \cap (z)^\nabla = (y \wedge z)^\nabla$. Since
349 $y \wedge z \in I$, then by (3), we get $y \in I$. Therefore, $\bigcup_{x \in I} (x)^\nabla \subseteq I$ and hence
350 $\bigcup_{x \in I} (x)^\nabla = I$.

351 (4) \Rightarrow (5) Assume (4). Let $x \in I$. Then by (4), we get $x \in (i)^\nabla$ for some
352 $i \in I$. Suppose $t \in (x)^\nabla$. Then it concludes $t \in (x)^\nabla \subseteq (i)^\nabla$ with $i \in I$. Then
353 $t \in \bigcup_{i \in I} (i)^\nabla = I$ and hence $(x)^\nabla \subseteq I$.

354 (5) \Rightarrow (1) Assume (5). Since $k \in (x)^\nabla$, $\forall x \in I$, then by (5), $k \in (x)^\nabla \subseteq I$.
355 This proves that I is a k -ideal of L . ■

356 Let $I_k^p(L) = \{(x)^\nabla : x \in L\}$ be the set of all principal k -ideal of a CRD-Stone
357 algebra L .

358 **Theorem 23.** *Let L be a CRD-Stone algebra. Then $(I_k^p(L); +, \bullet, (0)^\nabla, (1)^\nabla)$*
359 *forms a Boolean ring, where $+$ the addition operation and \bullet the multiplication*
360 *operation are defined as follows:*

$$361 \quad (x)^\nabla + (y)^\nabla = ((x \wedge y^+) \vee (y \wedge x^+))^\nabla,$$

$$362 \quad (x)^\nabla \bullet (y)^\nabla = (x \wedge y)^\nabla.$$

363 **Proof.** Let $(x)^\nabla, (y)^\nabla, (z)^\nabla \in I_k^p(L)$. Then we deduce the following properties:

364 (i) Associativity of $+$,

$$\begin{aligned}
& (x)^\nabla + ((y)^\nabla + (z)^\nabla) \\
&= (x)^\nabla + ((y \wedge z^+) \vee (z \wedge y^+))^\nabla \\
&= ((x \wedge \{(y \wedge z^+) \vee (z \wedge y^+)\}^+) \vee (x^+ \wedge \{(y \wedge z^+) \vee (z \wedge y^+)\}^\nabla))^\nabla \\
&= (\{x \wedge y^+ \wedge z^+\} \vee \{x \wedge z^{++} \wedge y^{++}\} \vee \{x^+ \wedge y \wedge z^+\} \vee \{x^+ \wedge z \wedge y^+\})^\nabla
\end{aligned}$$

365 where

$$\begin{aligned}
& (x \wedge \{(y \wedge z^+) \vee (z \wedge y^+)\}^+) \\
&= (x \wedge \{(y \wedge z^+)^+ \wedge (z \wedge y^+)^+\}) && \text{(by Theorem 1(7))} \\
&= x \wedge \{(y^+ \vee z^{++}) \wedge (z^+ \vee y^{++})\} && \text{(by Theorem 1(6))} \\
&= \{(x \wedge y^+) \vee (x \wedge z^{++})\} \wedge (z^+ \vee y^{++}) && \text{(by distributivity of } L) \\
&= \{(x \wedge y^+) \wedge (z^+ \vee y^{++})\} \vee \{(x \wedge z^{++}) \wedge (z^+ \vee y^{++})\} && \text{(by distributivity of } L) \\
&= (x \wedge y^+ \wedge z^+) \vee (x \wedge y^+ \wedge y^{++}) \vee (x \wedge z^{++} \wedge z^+) \vee (x \wedge z^{++} \wedge y^{++}) \\
&= (x \wedge y^+ \wedge z^+) \vee (x \wedge z^{++} \wedge y^{++}) \text{ as } x^+ \wedge x^{++} = 0, \forall x \in L.
\end{aligned}$$

366 On the other hand, we have

$$\begin{aligned}
& ((x)^\nabla + (y)^\nabla) + (z)^\nabla \\
&= (((x \wedge y^+) \vee (y \wedge x^+))^\nabla + z^\nabla) \\
&= ((\{(x \wedge y^+) \vee (y \wedge x^+)\} \wedge z^+) \vee (\{(x \wedge y^+) \vee (y \wedge x^+)\}^+ \wedge z))^\nabla \\
&= (\{x \wedge y^+ \wedge z^+\} \vee \{x^+ \wedge y \wedge z^+\} \vee \{x^{++} \wedge y^{++} \wedge z\} \vee \{x^+ \wedge y^+ \wedge z\})^\nabla
\end{aligned}$$

367 where

$$\begin{aligned}
& (\{(x \wedge y^+) \vee (y \wedge x^+)\}^+ \wedge z) \\
&= (\{(x \wedge y^+)^+ \wedge (y \wedge x^+)^+\} \wedge z) && \text{(by Theorem 1(7))} \\
&= (\{(x^+ \vee y^{++}) \wedge (y^+ \vee x^{++})\} \wedge z) && \text{(by Theorem 1(6))} \\
&= (\{((x^+ \vee y^{++}) \wedge y^+) \vee ((x^+ \vee y^{++}) \wedge x^{++})\} \wedge z) && \text{(by distributivity of } L) \\
&= \{(x^+ \vee y^{++}) \wedge y^+ \wedge z\} \vee \{(x^+ \vee y^{++}) \wedge x^{++} \wedge z\} && \text{(by distributivity of } L) \\
&= (x^+ \wedge y^+ \wedge z) \vee (y^{++} \wedge y^+ \wedge z) \vee (x^+ \wedge x^{++} \wedge z) \vee (y^{++} \wedge x^{++} \wedge z) \\
&= (x^+ \wedge y^+ \wedge z) \vee (y^{++} \wedge x^{++} \wedge z) \text{ as } x^+ \wedge x^{++} = 0, \forall x \in L.
\end{aligned}$$

Now, we use the fact $(x)^\nabla = (y)^\nabla \Leftrightarrow x^{++} = y^{++} \Leftrightarrow x^+ = y^+$, see Lemma 20(1).

It is easy to check that

$$\begin{aligned}
& \{\{x \wedge y^+ \wedge z^+\} \vee \{x \wedge z^{++} \wedge y^{++}\} \vee \{x^+ \wedge y \wedge z^+\} \vee \{x^+ \wedge z \wedge y^+\}\}^+ \\
&= \{\{x \wedge y^+ \wedge z^+\} \vee \{x^+ \wedge y \wedge z^+\} \vee \{x^{++} \wedge y^{++} \wedge z\} \vee \{x^+ \wedge y^+ \wedge z\}\}^+ \\
&= \{x^+ \vee y^{++} \vee z^{++}\} \wedge \{x^+ \vee z^+ \vee y^+\} \wedge \{x^{++} \vee y^+ \vee z^{++}\} \wedge \{x^{++} \vee z^+ \vee y^{++}\}.
\end{aligned}$$

368 Therefore, $(\{x \wedge y^+ \wedge z^+\} \vee \{x \wedge z^{++} \wedge y^{++}\} \vee \{x^+ \wedge y \wedge z^+\} \vee \{x^+ \wedge z \wedge y^+\})^\nabla = (\{x \wedge$
 369 $y^+ \wedge z^+\} \vee \{x^+ \wedge y \wedge z^+\} \vee \{x^{++} \wedge y^{++} \wedge z\} \vee \{x^+ \wedge y^+ \wedge z\})^\nabla$ implies $((x)^\nabla +$
 370 $(y)^\nabla) + (z)^\nabla = (x)^\nabla + ((y)^\nabla + (z)^\nabla)$.

371 (ii) Since $(x)^\nabla + (0)^\nabla = ((x \wedge 0^+) \vee (x^+ \wedge 0))^\nabla = (x \vee 0)^\nabla = (x)^\nabla$, then $(0)^\nabla$
 372 is the additive identity on $I_k^p(L)$.

373 (iii) Commutativity of $+$ and \bullet ,

$$\begin{aligned} (x)^\nabla + (y)^\nabla &= (x \wedge y^+) \vee (y \wedge x^+)^\nabla \\ &= (y \wedge x^+) \vee (y^+ \wedge x)^\nabla \\ &= (y)^\nabla + (x)^\nabla, \\ (x)^\nabla \bullet (y)^\nabla &= (x \wedge y)^\nabla \\ &= (y \wedge x)^\nabla \\ &= (y)^\nabla \bullet (x)^\nabla. \end{aligned}$$

374 (iv) It is clear that the additive inverse of $(x)^\nabla \in I_k^p(L)$ is $(x)^\nabla$ itself, that
 375 is, $-(x)^\nabla = (x)^\nabla$.

376 (v) The multiplicative identity of $I_k^p(L)$ is $(1)^\nabla$.

377 (vii) The distributive law on $I_k^p(L)$,

$$\begin{aligned} (x)^\nabla \bullet \{(y)^\nabla + (z)^\nabla\} &= (x)^\nabla \bullet ((y \wedge z^+) \vee (z \wedge y^+))^\nabla \\ &= (x \wedge \{(y \wedge z^+) \vee (z \wedge y^+)\})^\nabla \\ &= (\{x \wedge y \wedge z^+\} \vee \{x \wedge z \wedge y^+\})^\nabla, \end{aligned}$$

378 and

$$\begin{aligned} \{(x)^\nabla \bullet (y)^\nabla\} + \{(x)^\nabla \bullet (z)^\nabla\} &= (x \wedge y)^\nabla + (x \wedge z)^\nabla \\ &= (\{(x \wedge y) \wedge (x \wedge z)^+\} \vee \{(x \wedge y)^+ \wedge (x \wedge z)\})^\nabla \\ &= (\{(x \wedge y) \wedge (x^+ \vee z^+)\} \vee \{(x^+ \vee y^+) \wedge (x \wedge z)\})^\nabla \\ &= (\{x \wedge y \wedge x^+\} \vee \{x \wedge y \wedge z^+\} \vee \{x^+ \wedge x \wedge z\} \vee \{y^+ \wedge x \wedge z\})^\nabla. \end{aligned}$$

379 Then by Lemma 20(1), we get $(\{x \wedge y \wedge z^+\} \vee \{x \wedge z \wedge y^+\})^\nabla = (\{x \wedge y \wedge x^+\} \vee$
 380 $\{x \wedge y \wedge z^+\} \vee \{x^+ \wedge x \wedge z\} \vee \{y^+ \wedge x \wedge z\})^\nabla$.

381 Therefore, $(x)^\nabla \bullet \{(y)^\nabla + (z)^\nabla\} = \{(x)^\nabla \bullet (y)^\nabla\} + \{(x)^\nabla \bullet (z)^\nabla\}$.

382 (viii) $(x)^\nabla \bullet (x)^\nabla = (x \wedge x)^\nabla = (x)^\nabla$. Consequently $(I_k^p(L); +, \bullet, (0)^\nabla, (1)^\nabla)$
 383 is a Boolean ring. ■

384 It is known that there is a one-to-one correspondence between Boolean alge-
 385 bras and Boolean rings (see [17]). Then we can convert the Boolean ring $I_k^p(L)$
 386 into a Boolean algebra as follows.

387 **Corollary 24.** Let $(I_k^p(L); +, \bullet, (0)^\nabla, (1)^\nabla)$ be a Boolean ring of all principal k -
 388 ideals of a CRD-Stone algebra L . Then $(I_k^p(L); \vee, \wedge, ', (0)^\nabla, (1)^\nabla)$ is a Boolean
 389 algebra, where

$$390 \quad (x)^\nabla \vee (y)^\nabla = (x)^\nabla + (y)^\nabla + \{(x)^\nabla \bullet (y)^\nabla\} = (x \wedge y)^\nabla,$$

$$391 \quad (x)^\nabla \cap (y)^\nabla = (x)^\nabla \bullet (y)^\nabla = (x \wedge y)^\nabla,$$

$$392 \quad (x)^{\nabla'} = (x^+)^\nabla.$$

393 Now, we give an example to clarify the basic properties of the class of all
 394 principal k -ideals of a certain CRD-Stone algebra L .

395 **Example 25.** Consider the CRD-Stone algebra S_9 which is given in Example
 396 6(1) (see Figure 1). The principal k -ideals of S_9 are given as follows.

397 $(0)^\nabla = (c)^\nabla = (d)^\nabla = (k)^\nabla = (k]$, $(a)^\nabla = (x)^\nabla = (x]$, $(b)^\nabla = (y)^\nabla = (y]$
 398 and $(1)^\nabla = L = (1]$. We determine the algebras $(I_k^p(L), +)$ and $(I_k^p(L), \bullet)$ as in
 399 the following tables.

$+$	$(0)^\nabla$	$(a)^\nabla$	$(b)^\nabla$	$(1)^\nabla$
$(0)^\nabla$	$(0)^\nabla$	$(a)^\nabla$	$(b)^\nabla$	$(1)^\nabla$
$(a)^\nabla$	$(a)^\nabla$	$(0)^\nabla$	$(b)^\nabla$	$(1)^\nabla$
$(b)^\nabla$	$(b)^\nabla$	$(1)^\nabla$	$(0)^\nabla$	$(a)^\nabla$
$(1)^\nabla$	$(1)^\nabla$	$(b)^\nabla$	$(a)^\nabla$	$(0)^\nabla$

\bullet	$(0)^\nabla$	$(a)^\nabla$	$(b)^\nabla$	$(1)^\nabla$
$(0)^\nabla$	$(0)^\nabla$	$(0)^\nabla$	$(0)^\nabla$	$(0)^\nabla$
$(a)^\nabla$	$(0)^\nabla$	$(a)^\nabla$	$(0)^\nabla$	$(a)^\nabla$
$(b)^\nabla$	$(0)^\nabla$	$(0)^\nabla$	$(b)^\nabla$	$(b)^\nabla$
$(1)^\nabla$	$(0)^\nabla$	$(a)^\nabla$	$(b)^\nabla$	$(1)^\nabla$

400 From the above tables, we observe that $(I_k^p(L); +, \bullet)$ forms a Boolean ring.
 401 Also, Figure 3. Shows that $(I_k^p(L); \vee, \wedge, ', (0)^\nabla, (1)^\nabla)$ forms a Boolean algebra
 402 which is isomorphic to $B(L)$, where $'$ is given as, $(0)^{\nabla'} = (1)^\nabla$, $(a)^{\nabla'} = (b)^\nabla$,
 403 $(b)^{\nabla'} = (a)^\nabla$, $(1)^{\nabla'} = (0)^\nabla$.

404 **Theorem 26.** Let L be a CRD-Stone algebra. Then

- 405 (1) $(I_k(L); \vee, \wedge, \overline{D(L)}, L)$ is a $\{1\}$ -sublattice of $I(L)$,
 406 (2) $(I_k^p(L); \vee, \wedge, (0)^\nabla, (1)^\nabla)$ is a bounded sublattice of $I_k(L)$,
 407 (3) $B(L)$ is isomorphic to $I_k^p(L)$.

408 **Proof.** (1) Let $I, J \in I_k(L)$. Since $k \in I, J$, then $I \cap J$ and $\overline{I \vee J}$ are k -ideals.
 409 Since $k \in L = (1]$, then L is the greatest k -ideal of L , but $\overline{D(L)} = (k]$ is the
 410 smallest k -ideal of L . Then $I_k(L)$ is a $\{1\}$ -sublattice of the lattice $I(L)$.

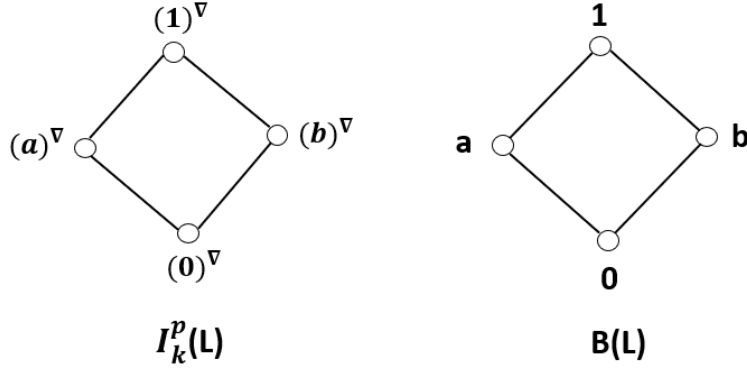


Figure 3. $I_k^p(L)$ and $B(L)$ are isomorphic Boolean algebras.

411 (2) We have $(x \vee y)^\nabla = (x)^\nabla \vee (y)^\nabla$ and $(x \wedge y)^\nabla = (x)^\nabla \wedge (y)^\nabla$ for all
 412 $(x)^\nabla, (y)^\nabla \in I_k^p(L)$. It is observed that $(0)^\nabla = \overline{D(L)}$, $(1)^\nabla = L$ are the smallest
 413 and the greatest members of $I_k^p(L)$, respectively. Therefore, $(I_k^p(L); \vee, \wedge, (0)^\nabla, (1)^\nabla)$
 414 is a bounded sublattice of the lattice $I_k(L)$.

415 (3) Define mapping: $f : B(L) \longrightarrow I_k^p(L)$ by $f(x) = (x)^\nabla$, for all $x \in B(L)$.
 416 To prove that f is a homomorphism, let $x, y \in B(L)$,

$$\begin{aligned} f(x \vee y) &= (x \vee y)^\nabla \\ &= (x)^\nabla \vee (y)^\nabla && \text{(by Lemma 19(3))} \\ &= f(x) \vee f(y) \end{aligned}$$

417 Thus $f(x \vee y) = f(x) \vee f(y)$. Similarly, we can get $f(x \wedge y) = f(x) \wedge f(y)$.
 418 Then f is homomorphism. Let $f(x) = f(y)$. Then $(x)^\nabla = (y)^\nabla$ and hence
 419 $x = x^{++} = y^{++} = y$. Then f is an injective map. For all $(x)^\nabla \in I_k^p(L)$, we have
 420 $(x)^\nabla = (x^{++})^\nabla = f(x^{++})$, $x^{++} \in B(L)$. Then f is a surjective map. Therefore
 421 f is an isomorphism and $B(L) \cong I_k^p(L)$. ■

422 5. k - $\{^+\}$ -CONGRUENCES ON A CRD -STONE ALGEBRA

423 In this section, we study the relationships between k -ideals and k - $\{^+\}$ -congruences
 424 of a CRD -Stone algebra L . Also, we describe the lattice $Con_k^+(L)$ of all k - $\{^+\}$ -
 425 congruences of L .

426 **Definition 18.** A $\{^+\}$ -congruence θ on a CRD -Stone algebra L is called a k -
 427 $\{^+\}$ -congruence if $k \in Ker \theta$, where $Ker \theta = \{x \in L : (x, 0) \in \theta\} = [0]_\theta$

428 **Proposition 27.** Define a binary relation θ on a core regular double Stone L as
429 follows:

$$430 \quad (x, y) \in \theta \Leftrightarrow (x)^\nabla = (y)^\nabla.$$

431 Then θ is a k - $\{^+\}$ -congruence on L . Moreover, $\theta = \psi^+$.

432 Let I be a k -ideal of CRD-Stone algebra L . Define a binary relation θ_I on
433 L as follows:

$$434 \quad \theta_I = \{(a, b) \in L \times L : a \vee i \vee k = b \vee i \vee k, \text{ for some } i \in I\}.$$

435 **Theorem 28.** Let I be a k -ideal of CRD-Stone algebra L . Then θ_I is a k - $\{^+\}$ -
436 congruence on L such that $\text{Ker } \theta_I = I$.

Proof. It is Clear that θ_I is an equivalent relation on L . Let $(a, b) \in \theta_I$. Then
 $a \vee i \vee k = b \vee i \vee k$ for some $i \in I$. Now for all $c \in L$, then by distributivity of
 L , we get

$$\begin{aligned} (a \wedge c) \vee i \vee k &= (b \wedge c) \vee i \vee k, \\ (a \vee c) \vee i \vee k &= (b \vee c) \vee i \vee k. \end{aligned}$$

437 Therefore $(a \wedge c, b \wedge c), (a \vee c, b \vee c) \in \theta_I$. So by **Theorem 8**, θ_I is a lattice
438 congruence on L . It remains to show that $(a, b) \in \theta_I$ implies $(a^+, b^+) \in \theta_I$.

$$\begin{aligned} (a, b) \in \theta_I &\Rightarrow a \vee i \vee k = b \vee i \vee k \\ &\Rightarrow a^+ \wedge i^+ \wedge k^+ = b^+ \wedge i^+ \wedge k^+ \\ &\Rightarrow a^+ \wedge i^+ = b^+ \wedge i^+ \text{ as } k^+ = 1 \\ &\Rightarrow (a^+ \wedge i^+) \vee i = (b^+ \wedge i^+) \vee i \\ &\Rightarrow (a^+ \vee i) \wedge (i^+ \vee i) = (b^+ \vee i) \wedge (i^+ \vee i) \quad (\text{by distributivity of } L) \\ &\Rightarrow (a^+ \vee i) \wedge 1 = (b^+ \vee i) \wedge 1 \quad (\text{by Theorem 1(2)}) \\ &\Rightarrow a^+ \vee i = b^+ \vee i \\ &\Rightarrow (a^+, b^+) \in \theta_I \end{aligned}$$

439 Then θ_I is a $\{^+\}$ -congruence on L .

440 Now, we prove that $\text{Ker } \theta_I = I$.

$$\begin{aligned} \text{Ker } \theta_I &= \{x \in L : (0, x) \in \theta_I\} \\ &= \{x \in L : 0 \vee i \vee k = x \vee i \vee k, i \in I\} \\ &= \{x \in L : i \vee k = x \vee i \vee k\} \\ &= \{x \in L : x \leq i \vee k\} \\ &= \{x \in L : x^{++} \leq i^{++} \leq i^{++} \vee k\} \\ &= \{x : x \in I^\nabla = I\} = I. \end{aligned}$$

441 Since $k \in I = \text{Ker } \theta_I$, then θ_I is a k - $\{^+\}$ -congruence on L . ■

442 **Theorem 29.** For any k -ideals I, J of a CRD-Stone algebra L , we have

- 443 (1) $I \subseteq J \Leftrightarrow \theta_I \subseteq \theta_J$,
 444 (2) $\psi^+ \subseteq \theta_I$, where ψ^+ is the dual Glivenko congruence on L ,
 445 (3) $\overline{\theta_{D(L)}} = \psi^+$,
 446 (4) $\theta_L = \nabla_L$,
 447 (5) the quotient lattice L/θ_I forms a Boolean algebra.

448 **Proof.** (1) Suppose $I \subseteq J$ and $(a, b) \in \theta_I$. Then there exists $i \in I$ such that
 449 $a \vee i \vee k = b \vee i \vee k$. Since $I \subseteq J$, then $(a, b) \in \theta_J$. Thus $\theta_I \subseteq \theta_J$. Conversely, let
 450 $\theta_I \subseteq \theta_J$. Then by the above **Theorem 28**, $I = \text{Ker } \theta_I \subseteq \text{Ker } \theta_J = J$.

451 (2) Let $(a, b) \in \psi^+$. Then $a^+ = b^+$ implies $a^{++} = b^{++}$. Now, we have

$$\begin{aligned} a \vee i \vee k &= (a^{++} \vee (a \wedge k)) \vee i \vee k && \text{(by Lemma 7(2))} \\ &= a^{++} \vee i \vee ((a \wedge k) \vee k) \\ &= a^{++} \vee i \vee k && \text{(by Definition 1(2))} \\ &= b^{++} \vee i \vee k \\ &= b^{++} \vee i \vee ((b \wedge k) \vee k) \\ &= (b^{++} \vee (b \wedge k)) \vee i \vee k \\ &= b \vee i \vee k. \end{aligned}$$

452 Thus $(a, b) \in \theta_I$ and hence $\psi^+ \subseteq \theta_I$.

(3) Since, $i^+ = 1$, for all $i \in \overline{D(L)}$, we get

$$\begin{aligned} \overline{\theta_{D(L)}} &= \{(a, b) \in L \times L : a \vee i \vee k = b \vee i \vee k, i \in \overline{D(L)}\} \\ &= \{(a, b) \in L \times L : a^+ \wedge i^+ \wedge k^+ = b^+ \wedge i^+ \wedge k^+\} \\ &= \{(a, b) \in L \times L : a^+ = b^+\} = \psi^+ \quad (\text{as } i^+ = k^+ = 1). \end{aligned}$$

453 (4) Since $a \vee 1 \vee k = b \vee 1 \vee k$ for all $a, b \in L$, then $(a, b) \in \theta_L$ and hence
 454 $\theta_L = \nabla_L$.

455 (5) The quotient set L/θ_I is $\{[a]\theta_I : a \in L\}$, where $[a]\theta_I$ is the congruence
 456 class of an element $a \in L$ modulo θ_I . It is known that $L/\theta_I = (L/\theta_I; \vee, \wedge, [1]\theta_I,$
 457 $[0]\theta_I)$ is a bounded distributive lattice, where $[0]\theta_I = I$, $[1]\theta_I$ are the bounds of
 458 L/θ_I and $[a]\theta_I \wedge [b]\theta_I = [a \wedge b]\theta_I$, $[a]\theta_I \vee [b]\theta_I = [a \vee b]\theta_I$. Define L/θ_I by $[a]'\theta_I =$
 459 $[a^+]\theta_I$, since $[a]\theta_I \wedge [a^+]\theta_I = [a \wedge a^+]\theta_I = [0]\theta_I$, $[a]\theta_I \vee [a^+]\theta_I = [a \vee a^+]\theta_I = [1]\theta_I$
 460 and $[a]''\theta_I = [a^{++}]\theta_I = [a]\theta_I$. Then $(L/\theta_I; \vee, \wedge, ', [0]\theta_I, [1]\theta_I)$ is a
 461 Boolean algebra. ■

462 Let $\text{Con}_k^+(L) = \{\theta_I : I \in I_k(L)\}$ be the set of all k - $\{^+\}$ -congruences on L
 463 which are induced by the k -ideals of L . Using Theorem 29. We can show the
 464 following results.

465 **Theorem 30.** For any θ_I and θ_J of $Con_k^+(L)$, we have the following:

- 466 (1) $\theta_I \cap \theta_J = \theta_{(I \cap J)}$,
 467 (2) $\theta_I \vee \theta_J = \theta_{(I \vee J)}$,
 468 (3) $(Con_k^+(L); \vee, \wedge, \theta_{\overline{D(L)}}, \theta_L)$ forms a bounded lattice and a sublattice of
 469 $Con^+(L)$.

Proof. (1) Since $I \cap J \subseteq I, J$, by Theorem 29 $\theta_{(I \cap J)} \subseteq \theta_I, \theta_J$ implies $\theta_{(I \cap J)} \subseteq \theta_I \cap \theta_J$. Conversely, let $(a, b) \in \theta_I \cap \theta_J$. We get

$$\begin{aligned} (a, b) \in \theta_I \cap \theta_J &\Rightarrow (a, b) \in \theta_I \text{ and } (a, b) \in \theta_J \\ &\Rightarrow a \vee i \vee k = b \vee i \vee k \text{ for some } i \in I \text{ and } a \vee j \vee k = b \vee j \vee k \\ &\quad k \text{ for some } j \in J \\ &\Rightarrow (a \vee i \vee k) \wedge (a \vee j \vee k) = (b \vee i \vee k) \wedge (a \vee j \vee k) \\ &\Rightarrow (a \vee k \vee i) \wedge (a \vee k \vee j) = (b \vee k \vee i) \wedge (a \vee k \vee j) \\ &\Rightarrow a \vee k \vee (i \wedge j) = b \vee k \vee (i \wedge j) \\ &\Rightarrow (a, b) \in \theta_{(I \cap J)} \text{ as } (i \wedge j) \in (I \cap J). \end{aligned}$$

470 Then $\theta_I \cap \theta_J \subseteq \theta_{(I \cap J)}$ and hence $\theta_I \cap \theta_J = \theta_{(I \cap J)}$.

471 (2) Since $I, J \subseteq I \vee J$, then by Theorem 29, $\theta_I, \theta_J \subseteq \theta_{(I \vee J)}$. Thus, $\theta_{(I \vee J)}$ is
 472 an upper bound of θ_I, θ_J . Conversely, let θ_k be an upper bound of θ_I and θ_J , for
 473 $k \in I_k(L)$. Then $\theta_I, \theta_J \subseteq \theta_k$. Hence $I, J \subseteq k$ as $I \vee J$ is the least upper bound of
 474 I, J on $I_k(L)$. By Theorem 29, $\theta_I, \theta_J \subseteq \theta_k$. Therefore $\theta_{(I \vee J)}$ is the least upper
 475 bound of θ_I, θ_J . This proves that $\theta_I \vee \theta_J = \theta_{(I \vee J)}$.

476 (3) From (1) and (2), it is clear that $(Con_k^+(L); \vee, \wedge)$ forms a sublattice of
 477 $Con^+(L)$. Since $\theta_{\overline{D(L)}}$ and θ_L are the smallest and the greatest members of
 478 $Con_k^+(L)$, respectively. Then $(Con_k^+(L); \vee, \wedge, \theta_{\overline{D(L)}}, \theta_L)$ is a bounded lattice. ■

479 Now, we introduce the following interesting results.

480 **Theorem 31.** For every k - $\{^+\}$ -congruence θ on a CRD-Stone algebra L , we
 481 have

- 482 (1) $[0]\theta$ is a k -ideal of L ,
 483 (2) θ can be expressed as θ_I for some k -ideal I of L .

484 **Proof.** (1) It is clear that $[0]\theta = \{x \in L : (x, 0) \in \theta\} = Ker \theta$. It is known
 485 that the $Ker \theta$ is an ideal of L . Since θ is a k - $\{^+\}$ -congruence, then $k \in Ker \theta$.
 486 Therefore $[0]\theta$ is a k -ideal of L .

487 (2) We claim that $\theta = \theta_{[0]\theta}$. Let $(x, y) \in \theta$. Since $(k, k) \in \theta$ hence
 488 $(x \wedge k, y \wedge k) \in \theta$. Since $[0]\theta$ is a k -ideal of L , then $x \wedge k, y \wedge k \in [0]\theta$. Hence

489 $(x \wedge k, y \wedge k) \in \theta_{[0]\theta}$. Now, we prove that $(x^{++}, y^{++}) \in \theta_{[0]\theta}$.

$$\begin{aligned}
 (x^+, y^+) \in \theta &\Rightarrow (x^+ \wedge x^{++}, y^+ \wedge x^{++}) \in \theta \text{ and } (x^+ \wedge y^{++}, y^+ \wedge y^{++}) \in \theta \\
 &\Rightarrow (0, y^+ \wedge x^{++}) \in \theta \text{ and } (x^+ \wedge y^{++}, 0) \in \theta \text{ (by Definition 8)} \\
 &\Rightarrow x^+ \wedge y^{++}, y^+ \wedge x^{++} \in [0]\theta \\
 &\Rightarrow (x^+ \wedge y^{++}, y^+ \wedge x^{++}) \in \theta_{[0]\theta} \\
 &\Rightarrow (x^+ \vee (x^+ \wedge y^{++}), x^+ \vee (y^+ \wedge x^{++})) = (x^+, x^+ \vee y^+) \theta_{[0]\theta} \\
 &\quad \text{(by Definition 1(2))} \\
 &\text{and } (y^+ \vee (x^+ \wedge y^{++}), y^+ \vee (y^+ \wedge x^{++})) = (x^+ \vee y^+, y^+) \in \theta_{[0]\theta} \\
 &\Rightarrow (x^+, y^+) \in \theta_{[0]\theta} \\
 &\Rightarrow (x^{++}, y^{++}) \in \theta_{[0]\theta}.
 \end{aligned}$$

490 Now, $(x^{++}, y^{++}) \in \theta_{[0]\theta}$ and $(x \wedge k, y \wedge k) \in \theta_{[0]\theta}$ imply that $(x, y) = (x^{++} \vee$
 491 $(x \wedge k), y^{++} \vee (y \wedge k)) = (x^{++}, y^{++}) \vee (x \wedge k, y \wedge k) \in \theta_{[0]\theta}$. Then $\theta \subseteq \theta_{[0]\theta}$. For
 492 the converse, let $(x, y) \in \theta_{[0]\theta}$. Then $(x \wedge k, y \wedge k) \in \theta_{[0]\theta}$. Since $x \wedge k, y \wedge k \in [0]\theta$,
 493 then $(x \wedge k, y \wedge k) \in \theta$.

494 Now, we prove that $(x^{++}, y^{++}) \in \theta$ for all $(x, y) \in \theta_{[0]\theta}$

$$\begin{aligned}
 (x, y) \in \theta_{[0]\theta} &\Rightarrow (x^+, y^+) \in \theta_{[0]\theta} \\
 &\Rightarrow (x^+ \wedge x^{++}, y^+ \wedge x^{++}), (x^+ \wedge y^{++}, y^+ \wedge y^{++}) \in \theta_{[0]\theta} \\
 &\Rightarrow (0, y^+ \wedge x^{++}), (x^+ \wedge y^{++}, 0) \in \theta_{[0]\theta} \text{ as } x^+ \wedge x^{++} = 0, y^+ \wedge y^{++} = 0 \\
 &\Rightarrow x^+ \wedge y^{++}, y^+ \wedge x^{++} \in [0]\theta \\
 &\Rightarrow (x^+ \wedge y^{++}, y^+ \wedge x^{++}) \in [0]\theta \\
 &\Rightarrow (x^+ \vee (x^+ \wedge y^{++}), x^+ \vee (y^+ \wedge x^{++})), (y^+ \vee (x^+ \wedge y^{++}), y^+ \vee (y^+ \wedge x^{++})) \in \theta \\
 &\Rightarrow (x^+, (x^+ \vee y^+) \wedge (x^+ \vee x^{++})), ((y^+ \vee x^+) \wedge (y^+ \vee y^{++}), y^+) \in \theta \\
 &\quad \text{(by Definition 1(2))} \\
 &\Rightarrow (x^+, x^+ \vee y^+), (x^+ \vee y^+, y^+) \in \theta \text{ (by Definition 8)} \\
 &\Rightarrow (x^+, y^+) \in \theta \\
 &\Rightarrow (x^{++}, y^{++}) \in [0]\theta.
 \end{aligned}$$

495 Now, $(x^{++}, y^{++}) \in \theta$ and $(x \wedge k, y \wedge k) \in [0]\theta$ imply that $(x, y) = (x^{++}, y^{++})$
 496 $\vee (x \wedge k, y \wedge k) \in \theta$. Therefore $\theta_{[0]\theta} \subseteq \theta$ and $\theta = \theta_{[0]\theta}$. ■

497 According to Theorem 30 and Theorem 31, we observe that there is a one
 498 to one correspondence between the elements of the lattice $I_k(L)$ of all k -ideals of
 499 a CRD -Stone algebra L and the elements of the lattice $Con_k^+(L)$ of all k - $\{^+\}$ -
 500

501 Congruences of L . In fact, this deduces that the lattices $I_k(L)$ and $Con_k^+(L)$ are
 502 isomorphic and hence the lattice $Con_k^+(L)$ is a distributive lattice.

503 **Theorem 32.** *Let L be a CRD-Stone algebra. Then the lattices $I_k(L)$ and
 504 $Con_k^+(L)$ are isomorphic and hence $Con_k^+(L)$ is a distributive lattice.*

505 **Proof.** Define a map $h: I_k(L) \longrightarrow Con_k^+(L)$ by $h(I) = \theta_I$, for all $I \in I_k(L)$.
 506 From Theorem 30, for $I, J \in I_k(L)$, we have

$$\begin{aligned} 507 \quad h(I \vee J) &= \theta_I \vee \theta_J = \theta_{(I \vee J)} = h(I) \vee h(J), \\ 508 \quad h(I \cap J) &= \theta_I \cap \theta_J = \theta_{(I \cap J)} = h(I) \cap h(J), \\ 509 \quad h(\overline{D(L)}) &= \theta_{\overline{D(L)}} = \psi^+, \\ 510 \quad h(L) &= \theta_L = \nabla_L. \end{aligned}$$

511 Then h is $(0,1)$ -lattice homomorphism. Let $h(I) = h(J)$. Then $\theta_I = \theta_J$ implies
 512 $I = J$. Thus h is an injective map. For each $\theta \in Con_k^+(L)$, by Theorem 31(2),
 513 we have $\theta = \theta_I$ for some $I \in I_k(L)$. Then $h(I) = \theta_I = \theta$ implies that h is a
 514 surjective. Therefore, h is an isomorphism and hence $I_k(L)$ and $Con_k^+(L)$ are
 515 isomorphic lattices. Since $I_k(L)$ is a distributive lattice (see Theorem 16), then
 516 also, $Con_k^+(L)$ a distributive lattice. ■

517 6. PRINCIPAL k - $\{^+\}$ -CONGRUENCES ON A CRD-STONE ALGEBRA

518 In this section, we describe the principal k - $\{^+\}$ -Congruences on a CRD-Stone
 519 algebra L which are induced by the principal k -ideals of L . Also, we describe the
 520 algebraic structure of the class $Con_k^p(L)$ all principal k - $\{^+\}$ -ideals of L .

521 **Proposition 33.** *Let L be a CRD-Stone algebra L and $I = (x)^\nabla$. Then $\theta_{(x)^\nabla}$
 522 is given as follows:*

$$523 \quad \theta_{(x)^\nabla} = \{(a, b) \in L \times L : a \vee x \vee k = b \vee x \vee k\} \text{ and } Ker \theta_{(x)^\nabla} = (x)^\nabla.$$

Proof. Let $I = (x)^\nabla$. Then

$$\theta_I = \theta_{(x)^\nabla} = \left\{ (a, b) \in L \times L : a \vee i \vee k = b \vee i \vee k, \text{ for some } i \in (x)^\nabla \right\}.$$

524 Let $(a, b) \in \theta_I$. Since $I = (x)^\nabla$, thus $a \vee i \vee k = b \vee i \vee k$, for some $i \in (x)^\nabla$ and
 525 hence $a^{++} \vee i^{++} = b^{++} \vee i^{++}$. Since $i \in (x)^\nabla$, then $i^{++} \leq x^{++} \vee k$ and we have
 526 $i^{++} \leq x^{++}$.

$$\begin{aligned}
 a \vee x \vee k &= (a^{++} \vee (a \wedge k)) \vee (x^{++} \vee (x \wedge k)) \vee k && \text{(by Lemma 7(2))} \\
 &= (a^{++} \vee (a \wedge k)) \vee x^{++} \vee ((x \wedge k) \vee k) \\
 &= (a^{++} \vee (a \wedge k)) \vee x^{++} \vee k && \text{(by Definition 1(2))} \\
 &= a^{++} \vee x^{++} \vee ((a \wedge k) \vee k) \\
 &= a^{++} \vee x^{++} \vee k && \text{(by Definition 1(2))} \\
 &= b^{++} \vee x^{++} \vee k \\
 &= b^{++} \vee x^{++} \vee (x \wedge k) \vee (b \wedge k) \vee k \\
 &= (b^{++} \vee (b \wedge k)) \vee (x^{++} \vee (x \wedge k)) \vee k \\
 &= b \vee x \vee k.
 \end{aligned}$$

527 Then, we have $(a, b) \in \theta_{(x)^\nabla}$ if and only if $a \vee x \vee k = b \vee x \vee k$ and hence $\theta_{(x)^\nabla} =$
 528 $\{(a, b) \in L \times L : a \vee x \vee k = b \vee x \vee k\}$. From Theorem 28, $Ker \theta_{(x)^\nabla} = (x)^\nabla$. ■

529 **Definition 19.** A k - $\{^+\}$ -congruence θ on a CRD -Stone algebra L is called a
 530 principal k - $\{^+\}$ -congruence if θ is a principal $\{^+\}$ -congruence on L .

531 **Proposition 34.** For any element x of a CRD -Stone algebra L , define $\theta(0, x^{++}$
 532 $\vee k)$ on L as follows

$$533 \quad \theta(0, x^{++} \vee k) = \{(a, b) \in L \times L : a \vee x^{++} \vee k = b \vee x^{++} \vee k\}.$$

534 Then $\theta(0, x^{++} \vee k)$ is a principal k - $\{^+\}$ -congruence on L and $Ker \theta(0, x^{++} \vee k) =$
 535 $(x^{++} \vee k) = (x)^\nabla$.

536 **Proof.** It is known that $\theta(0, x^{++} \vee k)$ is a principal lattice congruence on L (see
 537 Theorem 9(3)).

538 Let $(a, b) \in \theta(0, x^{++} \vee k)$. Then, we get

$$\begin{aligned}
 a \vee x^{++} \vee k &= b \vee x^{++} \vee k \\
 \Rightarrow a^+ \wedge x^+ \wedge k^+ &= b^+ \wedge x^+ \wedge k^+ \\
 \Rightarrow a^+ \wedge x^+ &= b^+ \wedge x^+ \text{ as } k^+ = 1 \\
 \Rightarrow (a^+ \wedge x^+) \vee (x^{++} \vee k) &= (b^+ \wedge x^+) \vee (x^{++} \vee k) \\
 \Rightarrow (a^+ \vee x^{++} \vee k) \wedge (x^+ \vee x^{++} \vee k) &= (b^+ \vee x^{++} \vee k) \wedge (x^+ \vee x^{++} \vee k) \\
 \Rightarrow a^+ \vee x^{++} \vee k &= b^+ \vee x^{++} \vee k \text{ as } x^+ \vee x^{++} = 1.
 \end{aligned}$$

539 Then $(a^+, b^+) \in \theta(0, x^{++} \vee k)$. Thus $\theta(0, x^{++} \vee k)$ a principal $\{^+\}$ -congruence
 540 on L . Since $0 \vee x^{++} \vee k = k \vee x^{++} \vee k$, then $(0, k) \in \theta(0, x^{++} \vee k)$. Then
 541 $k \in Ker \theta(0, x^{++} \vee k)$ and hence θ is a principal k - $\{^+\}$ -congruence on L .

Now, for every for all $x \in L$, we prove $Ker \theta(0, x^{++} \vee k) = (x^{++} \vee k]$.

$$\begin{aligned} Ker \theta(0, x^{++} \vee k) &= \{y \in L : (0, y) \in \theta(0, x^{++} \vee k)\} \\ &= \{y \in L : x^{++} \vee k = y \vee x^{++} \vee k\} \\ &= \{y \in L : y \leq x^{++} \vee k\} \\ &= (x^{++} \vee k] \\ &= (x)^\nabla. \end{aligned}$$

542

■

543 **Theorem 35.** *Let x be an element of a CRD-Stone algebra L . Then*

544

$$\theta(0, x^{++} \vee k) = \theta_{(x)^\nabla}.$$

Proof. Let $(a, b) \in \theta(0, x^{++} \vee k)$. Then

$$\begin{aligned} a \vee x^{++} \vee k = b \vee x^{++} \vee k &\Rightarrow a \vee x^{++} \vee x \vee k = b \vee x^{++} \vee x \vee k \\ &\Rightarrow a \vee x \vee k = b \vee x \vee k \\ &\Rightarrow (a, b) \in \theta_{(x)^\nabla}. \end{aligned}$$

545 Thus $\theta(0, x^{++} \vee k) \subseteq \theta_{(x)^\nabla}$. Conversely, let $(a, b) \in \theta_{(x)^\nabla}$. Then we get

$$\begin{aligned} a \vee x \vee k &= b \vee x \vee k \\ \Rightarrow a \vee (x^{++} \vee (x \wedge k)) \vee x \vee k &= b \vee (x^{++} \vee (x \wedge k)) \vee x \vee k \text{ (by Lemma 7(2))} \\ \Rightarrow a \vee x^{++} \vee ((x \wedge k) \vee k) &= b \vee x^{++} \vee ((x \wedge k) \vee k) \text{ (by Definition 1(2))} \\ \Rightarrow a \vee x^{++} \vee k &= b \vee x^{++} \vee k \\ \Rightarrow (a, b) &\in \theta(0, x^{++} \vee k). \end{aligned}$$

546 Then $\theta_{(x)^\nabla} \subseteq \theta(0, x^{++} \vee k)$ and hence $\theta_{(x)^\nabla} = \theta(0, x^{++} \vee k)$. ■

547 **Corollary 36.** *Let L be a CRD-Stone algebra. Then*

548

$$Ker \theta_{(x)^\nabla} = Ker \theta(0, x^{++} \vee k) = (x^{++} \vee k] = (x)^\nabla.$$

549 A characterization of a principle k - $\{^+\}$ -congruence on a CRD-Stone algebra
550 L is given in the following two theorems.

551 **Theorem 37.** *Let θ be a principle $\{^+\}$ -congruence of L . Then $\theta(0, a)$ is principle
552 k - $\{^+\}$ -congruence if and only if $k \leq a$.*

553 **Proof.** If θ is a principle k - $\{^+\}$ -congruence, then $k \in Ker \theta(0, a)$ implies $(k, 0) \in$
554 $\theta(0, a)$ and hence $k \vee a = 0 \vee a = a$. Thus $k \leq a$. Conversely, let $k \leq a$ and $\theta(0, a)$
555 is a principal k - $\{^+\}$ -congruence. Then $(k, 0) \in \theta(0, a)$. Since $k \in Ker \theta(0, a)$,
556 thus $\theta(0, a)$ is a k - $\{^+\}$ -congruence on L . ■

557 **Theorem 38.** Let $\theta(0, a)$ be principle k - $\{^+\}$ -congruence on L . Then $\theta(0, a) =$
 558 $\theta_{(a)\nabla}$ if and only if $k \leq a$.

559 **Proof.** Let $\theta(a, b)$ be a k - $\{^+\}$ -congruence on L and $\theta(0, a) = \theta_{(a)}$

$$\begin{aligned} \theta(0, a) = \theta_{(a)\nabla} &\Rightarrow k \in Ker \theta(0, a) = Ker \theta_{(a)\nabla} \\ &\Rightarrow (k, 0) = \theta(0, a) \\ &\Rightarrow k \vee a = 0 \vee a = a \\ &\Rightarrow k \leq a. \end{aligned}$$

560 Conversely, let $k \leq a$ and $(x, y) \in \theta(0, a)$.

$$\begin{aligned} (x, y) \in \theta(0, a) &\Rightarrow x \vee a = y \vee a \\ &\Rightarrow x \vee a \vee k = y \vee a \vee k \\ &\Rightarrow (x, y) \in \theta_{(a)\nabla}. \end{aligned}$$

561 Then $\theta(0, a) \subseteq \theta_{(a)\nabla}$. Let $(x, y) \in \theta_{(a)\nabla}$. Then we have

$$\begin{aligned} (x, y) \in \theta_{(a)\nabla} &\Rightarrow x \vee a \vee k = y \vee a \vee k \\ &\Rightarrow x \vee a = y \vee a \\ &\Rightarrow (x, y) \in \theta(0, a). \end{aligned}$$

562 Then $\theta_{(a)\nabla} \subseteq \theta(0, a)$ and hence $\theta_{(a)\nabla} = \theta(0, a)$. ■

563 **Corollary 39.** Every principle k - $\{^+\}$ -congruence $\theta(0, a)$ on CRD-Stone algebra
 564 L can be expressed as $\theta(0, a^{++} \vee k)$.

565 Let $Con_k^p(L) = \{\theta_{(x)\nabla} : x \in L\}$ be the class of all principal k - $\{^+\}$ -congruences
 566 which are induced by the principal k -ideals of L . Theorem 40 shows that the class
 567 $Con_k^p(L)$ forms a Boolean ring which is isomorphic to the Boolean ring $I_k^p(L)$.

Theorem 40. Let L be a CRD-Stone algebra. Then $(Con_k^p(L); \oplus, \odot, \theta_{(1)\nabla}, \theta_{(0)\nabla})$
 forms a Boolean ring, where

$$\begin{aligned} \theta_{(x)\nabla} \oplus \theta_{(y)\nabla} &= \theta_{(x)\nabla + (y)\nabla}, \\ \theta_{(x)\nabla} \odot \theta_{(y)\nabla} &= \theta_{(x)\nabla \bullet (y)\nabla}. \end{aligned}$$

568 Moreover, $Con_k^p(L)$ and $I_k^p(L)$ are isomorphic Boolean rings.

569 **Proof.** According to Theorem 23, $(I_k^p(L); +, \bullet, (0)\nabla, (1)\nabla)$ is a Boolean ring.
 570 Consequently, for any $\theta_{(x)\nabla}, \theta_{(y)\nabla}, \theta_{(z)\nabla} \in Con_k^\nabla(L)$, we use the properties of the
 571 ring $(I_k^p(L), +, \bullet)$ to show the following properties.

(i) The associativity of \oplus and \odot .

$$\begin{aligned}
\theta_{(x)\nabla} \oplus \left\{ \theta_{(y)\nabla} \oplus \theta_{(z)\nabla} \right\} &= \theta_{(x)\nabla} \oplus \theta_{(y)\nabla+(z)\nabla} \\
&= \theta_{(x)\nabla+\{(y)\nabla+(z)\nabla\}} \\
&= \theta_{\{(x)\nabla+(y)\nabla\}+(z)\nabla} \text{ by associativity of } + \\
&= \theta_{(x)\nabla+(y)\nabla} \oplus \theta_{(z)\nabla} \\
&= \left\{ \theta_{(x)\nabla} \oplus \theta_{(y)\nabla} \right\} \oplus \theta_{(z)\nabla},
\end{aligned}$$

and

$$\begin{aligned}
\theta_{(x)\nabla} \odot \left\{ \theta_{(y)\nabla} \odot \theta_{(z)\nabla} \right\} &= \theta_{(x)\nabla} \odot \theta_{(y)\nabla \bullet (z)\nabla} \\
&= \theta_{(x)\nabla \bullet \{(y)\nabla \bullet (z)\nabla\}} \\
&= \theta_{\{(x)\nabla \bullet (y)\nabla\} \bullet (z)\nabla} \text{ by associativity of } \bullet \\
&= \theta_{(x)\nabla \bullet (y)\nabla} \odot \theta_{(z)\nabla} \\
&= \left\{ \theta_{(x)\nabla} \odot \theta_{(y)\nabla} \right\} \odot \theta_{(z)\nabla}.
\end{aligned}$$

572 (ii) The additive identity and the multiplicative identity in $\text{Con}_k^p(L)$ are $\theta_{(1)\nabla}$
573 and $\theta_{(0)\nabla}$, respectively.

574 (iii) The commutativity of \oplus and \odot .

$$\begin{aligned}
\theta_{(x)\nabla} \oplus \theta_{(y)\nabla} &= \theta_{(x)\nabla+(y)\nabla} \\
&= \theta_{(y)\nabla+(x)\nabla} \text{ as } + \text{ is commutative in } I_k^p(L) \\
&= \theta_{(y)\nabla} \oplus \theta_{(x)\nabla}, \\
\theta_{(x)\nabla} \odot \theta_{(y)\nabla} &= \theta_{(x)\nabla \bullet (y)\nabla} \\
&= \theta_{(y)\nabla \bullet (x)\nabla} \text{ as } \bullet \text{ is commutative in } I_k^p(L) \\
&= \theta_{(y)\nabla} \odot \theta_{(x)\nabla}.
\end{aligned}$$

575 (iv) The additive inverse of $\theta_{(x)\nabla}$ is $\theta_{(x)\nabla}$ itself.

576 (v) The distributive law holds as

$$\begin{aligned}
\theta_{(x)\nabla} \odot \left\{ \theta_{(y)\nabla} \oplus \theta_{(z)\nabla} \right\} &= \theta_{(x)\nabla} \odot \theta_{\{(y)\nabla+(z)\nabla\}} \\
&= \theta_{(x)\nabla \bullet \{(y)\nabla+(z)\nabla\}} \\
&= \theta_{\{(x)\nabla \bullet (y)\nabla\} + \{(x)\nabla \bullet (z)\nabla\}} \text{ by distributivity of } I_k^p(L) \\
&= \theta_{\{(x)\nabla \bullet (y)\nabla\}} \oplus \theta_{\{(x)\nabla \bullet (z)\nabla\}} \\
&= \left\{ \theta_{(x)\nabla} \odot \theta_{(y)\nabla} \right\} \oplus \left\{ \theta_{(x)\nabla} \odot \theta_{(z)\nabla} \right\}.
\end{aligned}$$

577 (vii) $[\theta_{(x)\nabla}]^2 = \theta_{(x)\nabla} \odot \theta_{(x)\nabla} = \theta_{(x)\nabla} \bullet_{(x)\nabla} = \theta_{(x)\nabla}$.

578 Therefore $(\text{Con}_k^p(L); \oplus, \odot, \theta_{(1)\nabla}, \theta_{(0)\nabla})$ is a Boolean ring. It is observed that the
 579 two rings $I_k^p(L)$ and $\text{Con}_k^p(L)$ are isomorphic under the isomorphism $(x)^\nabla \mapsto$
 580 $\theta_{(x)\nabla}$. ■

581 Combining the above Theorem 40 and Corollary 24, we will investigate the
 582 following interesting result.

Corollary 41. *Let $(\text{Con}_k^p(L); \oplus, \odot, \theta_{(1)\nabla}, \theta_{(0)\nabla})$ be the Boolean ring of all principal k - $\{^+\}$ -congruences on a CRD-Stone algebra L . Then $(\text{Con}_k^p(L); \vee, \cap, ', \theta_{(1)\nabla}, \theta_{(0)\nabla})$ is a Boolean algebra, where*

$$\begin{aligned} \theta_{(x)\nabla} \vee \theta_{(y)\nabla} &= \theta_{(x \vee y)\nabla}, \\ \theta_{(x)\nabla} \cap \theta_{(y)\nabla} &= \theta_{(x \wedge y)\nabla}, \\ \theta'_{(x)\nabla} &= \theta_{(x^+)\nabla}. \end{aligned}$$

583 **Example 42.** Consider the CRD-Stone algebra S_9 as in Figure 1. The principal
 584 k - $\{^+\}$ -congruences of S_9 are gives as follows:

$$\begin{aligned} \theta(0, 0) &= \theta(0, c) = \theta(0, d) = \theta(0, k) = \triangle_L, \\ \theta(0, a) &= \theta(0, x) = \{\{0, d, c, k, a, x\}, \{b, y, 1\}\}, \\ \theta(0, b) &= \theta(0, y) = \{\{0, d, c, k, b, y\}, \{a, x, 1\}\}, \\ \theta(0, 1) &= \nabla_L. \end{aligned}$$

585 Then the following two tables show that $(\text{Con}_k^p(L); \oplus, \odot)$ is a Boolean ring, where
 586 $\text{Con}_k^p(L) = \{\theta(0, 0), \theta(0, a), \theta(0, b), \theta(0, 1)\} = \{\theta_{(0)\nabla}, \theta_{(a)\nabla}, \theta_{(b)\nabla}, \theta_{(1)\nabla}\}$.

587

\oplus	$\theta(0, 0)$	$\theta(0, a)$	$\theta(0, b)$	$\theta(0, 1)$
$\theta(0, 0)$	$\theta(0, 0)$	$\theta(0, a)$	$\theta(0, b)$	$\theta(0, 1)$
$\theta(0, a)$	$\theta(0, a)$	$\theta(0, 0)$	$\theta(0, 1)$	$\theta(0, b)$
$\theta(0, b)$	$\theta(0, b)$	$\theta(0, 1)$	$\theta(0, 0)$	$\theta(0, a)$
$\theta(0, 1)$	$\theta(0, 1)$	$\theta(0, b)$	$\theta(0, a)$	$\theta(0, 0)$

\odot	$\theta(0, 0)$	$\theta(0, a)$	$\theta(0, b)$	$\theta(0, 1)$
$\theta(0, 0)$	$\theta(0, 0)$	$\theta(0, 0)$	$\theta(0, 0)$	$\theta(0, 0)$
$\theta(0, a)$	$\theta(0, 0)$	$\theta(0, a)$	$\theta(0, 0)$	$\theta(0, a)$
$\theta(0, b)$	$\theta(0, 0)$	$\theta(0, 0)$	$\theta(0, b)$	$\theta(0, b)$
$\theta(0, 1)$	$\theta(0, 0)$	$\theta(0, a)$	$\theta(0, b)$	$\theta(0, 1)$

588 Figure 4. Shows that $(\text{Con}_k^p(L); \oplus, \odot, \theta_{(1)\nabla}, \theta_{(0)\nabla})$ forms a Boolean algebra which is
 589 isomorphic to the Boolean algebra $I_k^p(L)$.

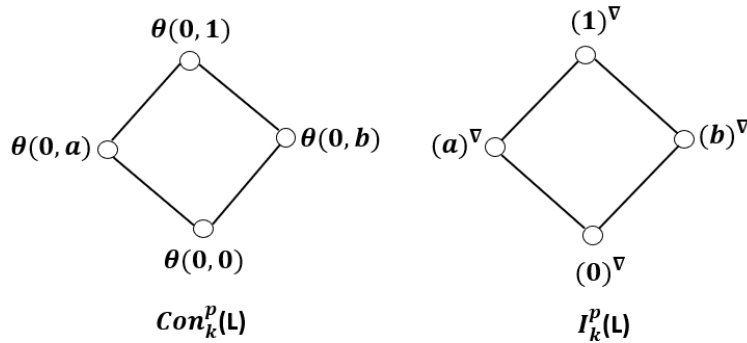


Figure 4. $\text{Con}_k^p(L)$ and $I_k^p(L)$ are isomorphic Boolean algebras.

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