STRUCTURES OF HALL SUBGROUPS OF FINITE METACYCLIC AND NILPOTENT GROUPS

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Abstract

In this paper, the structures of Hall subgroups of finite metacyclic and nilpotent groups are studied. It is proved that the collection of all Hall subgroups of a metacyclic group is a lattice and a group G is nilpotent if and only if its collection of Hall subgroups forms a distributive lattice. Also, lower semimodularity and complementation are studied in a collection of Hall subgroups of D_n for different values of n.

Keywords: group, Hall subgroup, lattice of subgroups, lower semimodular lattice, metacyclic group, nilpotent group.

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1. Introduction and notation

Throughout this article, G denotes a finite group. It is known that the set of all subgroups of a given finite group G forms a lattice denoted by $L(G)$ with $H \wedge K = H \cap K$ and $H \vee K = \langle H, K \rangle$ for subgroups H, K of G. The interrelations between the theory of lattices and the theory of groups have been studied by many researchers, see Pálfy [10], Schmidt [12], Suzuki [14]. For the group theoretic concepts and notations, we refer to Birkhoff [1], Luthar and Passi [8], Schmidt [12].

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There are a few types of subgroups such as Hall subgroups whose collections
may form lattices and these lattices can be used to study the properties of groups.
Accordingly, a study for collection of Hall subgroups of metacyclic and nilpotent
groups has been carried out.

The following notations are used throughout this article.

- $LH(G)$ – Collection of all Hall subgroups of G.

- $LN(G)$ – Collection of all normal subgroups of G, which is a sublattice of $L(G)$.

- $|G|$ – Order of G.

- $|L(G)|$ – Number of subgroups of G - Cardinality of $L(G)$.

- e – Neutral (Identity) element in G.

- $[m, r]$ – lcm of m and r.

- (m, r) – gcd of m and r.

- $\wedge_{LH}$ – g.l.b. in $LH(G)$.

- $\vee_{LH}$ – l.u.b. in $LH(G)$.

- $H \prec K$ – H is covered by K.

- D_n – Dihedral group of order $2n$: $\langle a,\ b \mid a^n = e = b^2,\ ba = a^{-1}b \rangle$.

The following definition of a Hall subgroup of a finite group is essentially due
to Hall [6].

Definition 1.1 [6]**.** A *Hall subgroup* of a finite group is a subgroup whose order
is coprime to its index.

Remark 1.2. Every Sylow p-subgroup of a finite group is a Hall subgroup.

The collection of Hall subgroups of a group is not necessarily a lattice, i.e.,
we have a group G in which $LH(G)$ does not form a lattice.

Consider $L(A_7)$ and its collection $LH(A_7)$ of all Hall subgroups of A_7. Note
that, the subgroups $H = \langle (1\,2\,3)\,(2\,3\,4\,5\,6) \rangle$ and $K = \langle (1\,2\,3),\ (2\,3\,4\,5\,7) \rangle$ are iso-
morphic to A_6 and so Hall subgroups of A_7. Moreover, $H \wedge K = \langle (1\,2\,3),\ (2\,3\,4\,5) \rangle$
is isomorphic to A_5. Note that, $\left(|H \wedge K|, \frac{|G|}{|H \wedge K|} \right) = (120, 42) = 6$ and so $H \wedge K$
is not a Hall subgroup.

Also, the subgroups $T = \langle (2\,3\,4\,5\,6) \rangle$ and $S = \langle (2\,4\,3\,5\,6) \rangle$ are Sylow
5-subgroups of A_7. Note that, $T \vee S = H \wedge K$ which is not a Hall subgroup of
A_7. Consequently, join of $T \vee_{LH} S$ as well as meet of $H \wedge_{LH} K$ does not exists
and therefore $LH(A_7)$ is not a lattice.

Next consider, the lattice depicted in Fig 1.1 which is the Hasse diagram of
$L(S_4)$. Note that, $LH(S_n) = L(S_n)$ for $n \leq 3$. The Hasse diagram of $LH(S_4)$

64 is depicted in Figure 1.2, and it is a lattice. Observe that for P_{28} and P_{27} in
65 $LH(S_n)$, we have $P_{28} \wedge P_{27} = M_{18}$ in $L(S_4)$, but $M_{18} \notin LH(S_4)$ and as such,
66 $LH(S_4)$ is not a sublattice of $L(S_4)$.
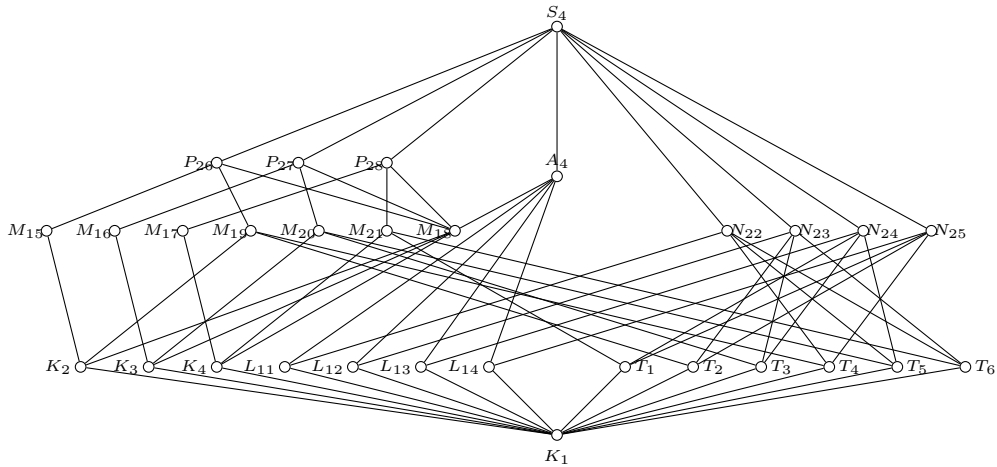
Figure 1.1. $L(S_4)$.
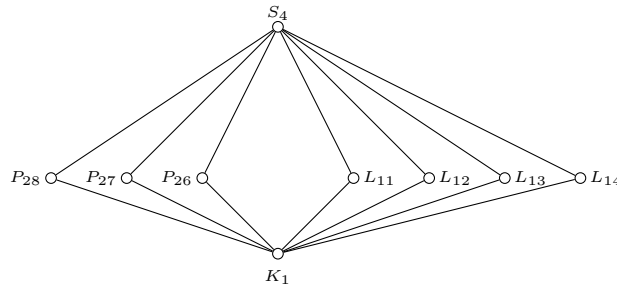
Figure 1.2. $LH(S_4)$.

67 So it is necessary to investigate the groups for which $LH(G)$ is a lattice and
68 similarly, $LH(G)$ is a sublattice of $L(G)$. It is also worth studying some properties
69 of $LH(G)$ in these situations.

70 Faigle, *et al.* (see [4, 11, 13]) studied strong lattices of finite length in which
71 the join-irreducible elements play a key role.

72 For the following definition and other relevant definitions in lattice theory we
73 refer to Birkhoff [1], Grätzer [5] and Stern [13].

74 **Definition 1.3** [13]**.** An element j of a lattice L is called *join-irreducible* if, for
75 all $x, y \in L$, $j = x \vee y$ implies $j = x$ or $j = y$.

For a lattice L of finite length $J(L)$ denotes the set of all non-zero join-irreducible elements.

We introduce the concept of join-irreducible subgroups as follows.

Definition 1.4. A subgroup of a group G is said to be *join-irreducible* if it is a join-irreducible element of $L(G)$.

We note that, every cyclic subgroup of prime power order of a finite group is a join-irreducible subgroup.

From this fact and Lemma 2 of [15], the following Lemma follows.

Lemma 1.5. *A subgroup of a finite group is a join-irreducible subgroup if and only if it is a cyclic subgroup of prime power order.*

The following concept of a strong element was coined by Faigle [4]; see also [13].

Definition 1.6 [4]**.** Let L be a lattice of finite length. A join-irreducible element $j \neq 0$ is called a *strong element* if the following condition holds for all $x \in L$:
(St) $j \leq x \vee j^- \implies j \leq x$, where j^- denotes the uniquely determined lower cover of j.

A lattice is said to be *strong* if every join-irreducible element of it is strong.

Remark 1.7. The condition (St) in the definition of a strong element is equivalent to the following; see [13] for more details.
(St') For every $q < j \in J(L)$, $x \in L$, $j \leq x \vee q$ implies $j \leq x$.

The following characterization of strong lattices is due to Richter and Stern [11].

Theorem 1.8 [11]**.** *A lattice L of finite length is strong if and only if it does not contain a special pentagon sublattice with $j \in J(L)$.*
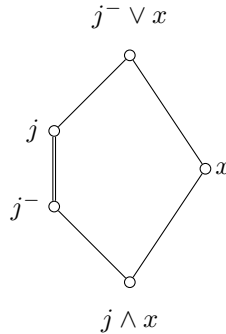

Figure 1.3. Special Pentagon.

100   Proof of the following Lemma follows from Theorem 1.8.

101 **Lemma 1.9.** *Let L be a finite lattice. If atoms are the only join-irreducible*
102 *elements in L, then L is strong.*

103 **Theorem 1.10.** *Let G be a group, if $LH(G)$ is a lattice, then $LH(G)$ is strong.*

104 ***Proof.*** In view of the Lemma 1.9, it is sufficient to prove that only atoms are
105 join-irreducible elements. Let $|G| = \prod_{i=1}^{m} p_i^{\alpha_i}$ and $J \in LH(G)$ a join-irreducible
106 Hall subgroup. Then $|J| = p_t^{\alpha_t}$ for some prime $t \in \{1, 2, \ldots, m\}$ and $|J^-| =$
107 $p_t^{\alpha_t - 1} \in L(G)$. Note that, $|J^-| = \{e\}$ in $LH(G)$. Consequently, if a subgroup J
108 is join-irreducible in $LH(G)$ then it is an atom. ∎

109   Note that, there exists a strong lattice which is not a Hall subgroup lattice
110 of any finite group. Figure 1.4 depicts a strong lattice, which is not a $LH(G)$ for
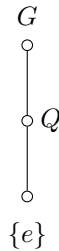111 any finite group G.

$$G$$
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$\{e\}$

Figure 1.4. C_3

112   2. HALL SUBGROUPS IN FINITE METACYCLIC GROUPS

113 In this section, the collection of Hall subgroups of metacyclic group is investigated.
114   Following is the definition of a metacyclic group, see [2].

115 **Definition 2.1** [2]**.** A finite group G is a *metacyclic* if it contains a cyclic normal
116 subgroup N such that $\frac{G}{N}$ is also cyclic.

117   It is observed that a metacyclic group can be written $G = SN$ with $S \leq G$
118 and $N \trianglelefteq G$ such that both S and N are cyclic. Such a product is a metacyclic
119 factorization of G.
120   Note that, Hall subgroups of a metacyclic group G are obtained with the
121 help of its metacyclic factorization. and so we have the following result which is
122 a Lemma 5.3 of [7].

123 **Lemma 2.2** [7]. *Let G be a finite group with a metacyclic factorization $G = SN$,*
124 *to each set π of primes, the subgroup $H = S_\pi N_\pi$ is the unique Hall π-subgroup*
125 *of G such that $S_\pi = H \cap S$, $N_\pi = N \cap H$ and so $H = (H \cap S)(H \cap N)$.*

126   As observed, the collection of Hall subgroups of a finite group need not form a
127 lattice in general but in case of metacyclic group it forms a lattice as the following
128 result shows.

129 **Theorem 2.3.** *If G is a finite metacyclic group, then $LH(G)$ is a lattice. How-*
130 *ever, it is not necessarily a sublattice of $L(G)$.*

131 **Proof.** Let G be a finite metacyclic group, in order to show that $LH(G)$ is a
132 lattice, we prove that given two Hall subgroups H and K of G, $H \wedge_{LH} K$ and
133 $H \vee_{LH} K$ exist.

134   *Case* I. Let H and K be two distinct Hall π_1 and π_2-subgroups respectively
135 corresponding to metacyclic factorization SN of G.
136   In view of Lemma 2.2, the subgroups $H = S_{\pi_1} N_{\pi_1}$ and $K = S_{\pi_2} N_{\pi_2}$ are the
137 unique Hall π_1 and π_2-subgroups of G such that $S_{\pi_1} = H \cap S$, $N_{\pi_1} = H \cap N$, $S_{\pi_2} =$
138 $K \cap S$, $N_{\pi_2} = K \cap N$. Therefore, $H = (H \cap S)(H \cap N)$ and $K = (K \cap S)(K \cap N)$.
139 Now, for the set $\pi = \pi_1 \cap \pi_2$ of primes, there is the unique Hall π-subgroup say
140 $T = S_\pi N_\pi = (T \cap S)(T \cap N)$. Note that, T is the unique largest Hall subgroup of
141 G which is contained in both H and K. Consequently, $H \wedge_{LH} K = T$. Similarly,
142 for the set $\pi' = \pi_1 \cup \pi_2$ of primes there is the unique Hall π'-subgroup say
143 $R = S_{\pi'} N_{\pi'} = (R \cap S)(R \cap N)$. Note that, R is the unique smallest Hall subgroup
144 of G which contains both H and K. Therefore, $H \vee_{LH} K = R$.

145   *Case* II. Let H and K be two distinct Hall π_1 and π_2-subgroups respectively
146 corresponding to two different metacyclic factorizations SN and $S'N'$.
147   In view of the Lemma 2.2, $H = (H \cap S)(H \cap N) = S_{\pi_1} N_{\pi_1}$ and $K =$
148 $(K \cap S')(K \cap N') = S'_{\pi_2} N'_{\pi_2}$. Furthermore, each one of H and K is an unique
149 Hall π_1 and π_2-subgroups corresponding to two metacyclic factorizations SN and
150 $S'N'$ respectively. Now, corresponding to each prime $p_i \in \pi_1$ there is the unique
151 Sylow p_i-subgroup say P_i, corresponding to factorization SN of G and similarly,
152 corresponding to each prime $p_j \in \pi_2$ there is the unique Sylow p_j-subgroup say
153 Q_j, corresponding to factorization $S'N'$ of G.
154   Note that, the subgroup $H' = S_{\pi_1 \cap \pi_2} N_{\pi_1 \cap \pi_2}$ then H' is a subgroup of H.
155 If H' is also a subgroup of K then H' is the largest Hall subgroup of G which
156 is contained in both H and K. Consequently, $H \wedge_{LH} K = H'$. If H' is not a
157 subgroup of K, then choose the set π of primes of $p_i \in \pi_1 \cap \pi_2$ such that each
158 Sylow p_i-subgroup P_i of G contained in both H and K. Note that, if P is a Hall
159 π-subgroup of H then $P \supseteq \vee P_i$. Since every non-trivial Hall subgroup is join of
160 Sylow subgroups we have $P = \vee P_i$. And so, it is contained in both H and K.

As such P is the unique largest Hall subgroup of G corresponding to metacyclic factorization SN as well as $S'N'$ and so, $H \wedge_{LH} K = \vee_{p_i \in \pi} P_i$.

Similarly, choose the subgroup $H' = S_{\pi_1 \cup \pi_2} N_{\pi_1 \cup \pi_2}$ then H is the subgroup of H'. If K is also a subgroup of H' then H' is the smallest Hall subgroup of G which contains both H and K and therefore $H \vee_{LH} K = H'$. If K is not a subgroup of H', choose the least set π' of primes π' with $\pi_1 \cup \pi_2 \subseteq \pi'$ such that $H, \ K \subseteq \vee_{p_i \in \pi'} P_i$. Let R be a Hall subgroup of G such that $\vee_{p_i \in \pi'} P_i \subseteq R$ is the unique Hall π'-subgroup corresponding to metacyclic factorizations SN as well as $S'N'$. Note that, R is the least Hall subgroup which contains H and K and so, $H \vee_{LH} K = R$.

Hence $LH(G)$ is a lattice whenever G is metacyclic.

Consider a dihedral group D_n, which is metacyclic group. In [9] it is noted that $LH(D_n)$ is a lattice but not necessarily a sublattice of $L(D_n)$. ∎

Remark 2.4. Note that, a metacyclic group G may not have a unique metacyclic factorization, e.g., D_n. However, if G has unique meatcyclic factorization then $LH(G)$ is a sublattice of $L(G)$, e.g. $\mathbb{Z}_{pq}$. Also, for every finite group G whose order is square-free, $LH(G)$ is a sublattice of $L(G)$.

We note that,dihedral groups are metacyclic and so $LH(D_n)$ is a lattice. However, $LH(D_n)$ is a lattice is proved independently in [9] using the classification of the subgroups given in [3] as follows;

Theorem 2.5 [3]. *Every subgroup of D_n is cyclic or dihedral. A complete listing of the subgroups is as follows:*

(1) $\langle a^d \rangle$*, where $d|n$, with index $2d$,*

(2) $\langle a^k, a^i b \rangle$*, where $k|n$ and $0 \leq i \leq k-1$, with index k.*

Every subgroup of D_n occurs exactly once in this listing.

Remark 2.6. 1. A subgroup of D_n is said to be of *Type* (1) if it is cyclic subgroup as stated in (1) of Theorem 2.5.

2. A subgroup of D_n is said to be of *Type* (2) if it is dihedral subgroup as stated in (2) of Theorem 2.5.

A study of collection of Hall subgroups of D_n namely $LH(D_n)$ is carried out by Mitkari et. al. in [9], where the binary operations $\wedge_{LH}$ and $\vee_{LH}$ in $LH(D_n)$ are defined as per the classification of subgroups of D_n as follows.

Let $n = 2^\alpha \prod_{i=1}^m p_i^{\alpha_i}$.

1. If $T = \langle a^t \rangle$ for some $s, \ t \in \mathbb{N}$ and $S = \langle a^s \rangle$ are Hall subgroups of Type (1), then $T \vee_{LH} S = \langle a^g \rangle$ where $g = (s,t)$ and $T \wedge_{LH} S = \langle a^l \rangle$, where $l = [s,t]$.

2. If $T = \langle a^t \rangle$ is a Hall subgroup of Type (1) and $S = \langle a^s, a^i b \rangle$ is a Hall subgroups of Type (2) for some $s, \ t \in \mathbb{N}$, then $T \vee_{LH} S = \langle a^g, a^i b \rangle$ where $g = (s,t)$ and $T \wedge_{LH} S = \langle a^l \rangle$, where $l = [s,t]$.

199   3. If $T = \langle a^t, a^i b \rangle$ and $S = \langle a^s, a^j b \rangle$ are Hall subgroups of Type (2) for some
200   s, $t \in \mathbb{N}$, then $T \vee_{LH} S = \langle a^g, a^i b \rangle$ where $g = \frac{g_1}{r}$ and $g_1 = (t, s, i - j)$, $r = \left(\frac{2n}{g_1}, g_1 \right)$
201   and

$$
202 \quad T \wedge_{LH} S = \begin{cases} \langle a^s \rangle, & \text{if } tx + sy = k - j \text{ has no integer solution} \\ \quad \text{where } s = \frac{2^{\alpha+1}n}{(|T|,|S|)} & \\ \langle a^d, a^{k - n_1 x_0} b \rangle, & \text{if } tx + sy = k - j \text{ has an integer solution} \\ \quad \text{where } d = \frac{2n}{(|T|,|S|)} & \end{cases}
$$

203   where (x_0, y_0) is an integer solution of an equation $tx + sy = k - j$.

204   Now, we establish some lattice theoretic property such as lower semimodular-
205   ity, complementation, atomic covering condition and Mac-lanes exchange prop-
206   erty in the subgroup lattice $LH(D_n)$.

207   **Definition 2.7** [13]. A lattice L is said to be *lower semimodular*, for every
208   T, $S \in L$, if $T \prec T \vee S$, then $T \wedge S \prec S$.

209   **Theorem 2.8.** *The lattice $LH(D_n)$ is lower semimodular.*

210   **Proof.** Let T and $S \in LH(D_n)$ be such that $T \prec T \vee S$.

211   **Claim.** $T \wedge S \prec S$.

212   Consider $n = 2^\alpha \prod_{i=1}^m p_i^{\alpha_i}$ where each p_i is an odd prime. Note that, if a
213   Type (1) subgroup H of D_n generated by a^h is also a Hall subgroup, then it is
214   necessary that $h = 2^\alpha \prod_{x \in M} p_x^{\alpha_x}$ for some subset $M \subseteq \{1, 2, \ldots, m\}$. Moreover,
215   if a Type (2) subgroup H of D_n generated by $\{a^h, a^i b\}$ is also a Hall subgroup,
216   then it is necessary that $h = \prod_{x \in N} p_x^{\alpha_x}$ for some subset $N \subseteq \{1, 2, \ldots, m\}$.

217   *Case* I. Let $T = \langle a^t \rangle$, where $t = 2^\alpha \prod_{x \in U \subseteq \{1,2,\ldots,m\}} p_x^{\alpha_x}$.

218   *Subcase* I(i). If $S = \langle a^s \rangle$ where $s = 2^\alpha \prod_{y \in V \subseteq \{1,2,\ldots,m\}} p_y^{\alpha_y}$ then $T \vee S = \langle a^g \rangle$
219   where $g = (s, t)$. In view of $T \prec T \vee S$, Note that, $\langle a^t \rangle \prec \langle a^g \rangle$ if and only if
220    $g = \frac{t}{p_*^{\alpha_*}} = \frac{2^\alpha \prod_{x \in U} p_x^{\alpha_x}}{p_*^{\alpha_*}}$ and p_* is an odd prime dividing n with largest power α_*.
221   We have $g | s$ (say $gk = s$ where $k \in \mathbb{Z}$) and $p_*^{\alpha_*} \nmid s$ since $T \nsubseteq S$.
222   Now $S \wedge T = \langle a^l \rangle$, where $l = [s, t] = [gk, gp_*^{\alpha_*}] = gkp_*^{\alpha_*} = sp_*^{\alpha_*}$ $(p_* \nmid s)$.
223   Consequently, $T \wedge S = \langle a^{sp_*^{\alpha_*}} \rangle \prec \langle a^s \rangle$.

224   *Subcase* II(ii). Let $S = \langle a^{s'}, a^i b \rangle$ for some subset $M \subseteq \{1, 2, \ldots, m\}$ where
225    $s' = \prod_{y \in W \subseteq \{1,2,\ldots,m\}} p_y^{\alpha_y}$ such that $T \prec T \vee S$. Note that, $T \vee S = \langle a^g, a^i b \rangle$
226   where $g = (s', t)$. Since $T \prec T \vee S$ we have $\langle a^t \rangle \prec \langle a^g, a^i b \rangle$ if and only if
227    $g = \frac{t}{2^\alpha} = \prod_{x \in U} p_x^{\alpha_x}$. As $g | s'$ ((say $gk = s'$ where $k \in \mathbb{Z}$), i.e., $\prod_{x \in U} p_x^{\alpha_x} | \prod_{y \in W} p_y^{\alpha_y}$
228   and so $\prod_{x \in U} p_x^{\alpha_x} \prod_{q \in X \subseteq W} p_q^{\alpha_q} = \prod_{y \in W} p_y^{\alpha_y}$. Now consider $T \wedge S = \langle a^l \rangle$ where

229 $l = [s', t] = [gk, 2^\alpha g] = 2^\alpha gk = 2^\alpha s'$ $(2 \nmid s')$. Consequently, $T \wedge S = \langle a^{2^\alpha s'} \rangle \prec$
230 $\langle a^{s'}, a^i b \rangle = S$, as $\frac{|S|}{|S \wedge T|} = 2^{\alpha+1}$.

231   *Case* II. Let $T = \langle a^t, a^i b \rangle$ where $t = \prod_{x \in U} p_x^{\alpha_x}$.

232   *Subcase* II(i). Let $S = \langle a^s \rangle$ where $s = 2^\alpha \prod_{y \in V} p_y^{\alpha_y}$ such that $T \prec T \vee S$. We
233 have $T \vee S = \langle a^g, a^i b \rangle$ where $g = (s, t)$. Since $T \prec T \vee S$, we have $\langle a^t, a^i b \rangle \prec$
234 $\langle a^g, a^i b \rangle$ if and only if $g = \frac{t}{p_*^{\alpha_*}} = \frac{\prod_{x \in U} p_x^{\alpha_x}}{p_*^{\alpha_*}}$. Note that, $g | s$ ((say $gk = s$ where
235 $k \in \mathbb{Z}$) and $T \not\subset S$ which implies $p_*^{\alpha_*} \nmid s$.
236   Now consider $S \wedge T = \langle a^l \rangle$ where $l = [s, t] = [gq, gp_*^{\alpha_*}] = gqp_*^{\alpha_*} = sp_*^{\alpha_*}$
237 $(p_*^{\alpha_*} \nmid s)$. Consequently, $T \wedge S = \langle a^{sp_*^{\alpha_*}} \rangle \prec \langle a^s \rangle = S$.

238   *Subcase* II(ii). Let S be a dihedral subgroup with $|S| = |T|$ and $T \prec T \vee S$.
239 Then $S = \langle a^t, a^j b \rangle$. Note that, $S \vee T = \langle a^g, a^i b \rangle = \langle a^g, a^j b \rangle$. Since $T \prec T \vee S$, we
240 have $\langle a^t, a^i b \rangle \prec \langle a^g, a^i b \rangle$ if and only if $g = \frac{t}{p_*^{\alpha_*}} = \frac{\prod_{x \in U} p_x^{\alpha_x}}{p_*^{\alpha_*}}$. Note that, i, $j \leq t$
241 and so $i - j \leq t$. Consider the equation $tx_1 + tx_2 = i - j$ for x_1, $x_2 \in \mathbb{Z}$ and this
242 equation does not have a solution as $i - j \leq t$, $t \nmid i - j$. Therefore, $T \wedge S$ is a
243 cyclic subgroup, suppose that $T \wedge S = \langle a^l \rangle$ where $l = \frac{2^{\alpha+1}n}{(|T|, |S|)} = \frac{2^{\alpha+1}n}{(\frac{2n}{t}, \frac{2n}{t})} = t2^\alpha$.
244 Therefore, $S \wedge T = \langle a^{t2^\alpha} \rangle$. Note that, $\frac{|S|}{|S \wedge T|} = 2^{\alpha+1}$ and hence $T \wedge S \prec S$ for such
245 choice of S and T.
246   Now suppose S be a dihedral subgroup such that $|T| \neq |S|$ and $T \prec T \vee S$,
247 say $S = \langle a^{s'}, a^j b \rangle$ where $s' = \prod_{y \in V} p_y^{\alpha_y}$ for some $y \in V \subseteq \{1, 2, \ldots, m\}$. Note
248 that, $S \vee T = \langle a^g, a^i b \rangle$ where $g = \frac{g_1}{r}$ and $g_1 = (t, s, i - j), r = \left(\frac{2n}{g_1}, g_1 \right)$. Since
249 $T \prec T \vee S$ we have $\langle a^t, a^i b \rangle \prec \langle a^g, a^i b \rangle$ if and only if $g = \frac{t}{p_*^{\alpha_*}} = \frac{\prod_{x \in U} p_x^{\alpha_x}}{p_*^{\alpha_*}}$. Now as
250 $g | s'$ and $g | i - j$ there exists $\alpha, \beta \in \mathbb{Z}$ we have $\alpha g = i - j$ and $\beta g = s'$. Consider the
251 equation $tx_1 + sx_2 = i - j$, i.e., $g(p_*^{\alpha_*})x_1 + g(\beta)x_2 = g\alpha$, i.e., $(p_*^{\alpha_*})x_1 + (\beta)x_2 = \alpha$.
252   We have two cases: $p_*^{\alpha_*} \nmid \beta$ and $p_*^{\alpha_*} | \beta$ and we contend that in each case
253 $T \wedge S \prec S$.
254   Suppose that, $p_*^{\alpha_*} \nmid \beta$, then $(p_*^{\alpha_*}, \beta) = 1$. Therefore, the equation $(p_*^{\alpha_*})x_1 +$
255 $\beta x_2 = \alpha$ will always have a solution. In this case $T \wedge S = \langle a^d, a^z b \rangle$, where
256 $d = \frac{2n}{\left(\frac{2n}{\prod_{x \in U} p_x^{\alpha_x}}, \frac{2n \cdot p_*^{\alpha_*}}{\prod_{x \in U} p_x^{\alpha_x} \beta} \right)} = \beta \prod_{x \in U} p_x^{\alpha_x}$. Note that, $\frac{|S|}{|S \wedge T|} = p_*^{\alpha_*}$. Consequently,
257 $T \wedge S \prec S$.
258   Now suppose that $p_*^{\alpha_*} | \beta$. If the equation $(p_*^{\alpha_*})x_1 + \beta x_2 = \alpha$ for $x_1, x_2 \in \mathbb{Z}$
259 has a solution, then $p_*^{\alpha_*} | \alpha$. Now as $\left(\frac{\prod_{x \in U} p_x^{\alpha_x}}{p_*^{\alpha_*}}, p_*^{\alpha_*} \right) = 1$ implies $\prod_{x \in U} p_x^{\alpha_x} | i - j$
260 and also $\prod_{x \in U} p_x^{\alpha_x} | s'$. Consequently, $T \vee S = \langle a^g, a^i b \rangle = \langle a^t, a^i b \rangle = T$ (as $g_1 =$
261 $(t, s', i - j) = t$ and $r = \left(\frac{2n}{g_1}, g_1 \right) = 1$ then $g = \frac{g_1}{r} = g_1 = t$) which is not
262 true since $T \prec T \vee S$. Therefore $p_*^{\alpha_*} \nmid \alpha$ and so the equation does not have
263 a solution. As such $S \wedge T$ is not a Type (2) subgroup of D_n and we must

have $S \wedge T = \langle a^l \rangle$, for $l = \dfrac{2^{\alpha+1}n}{\left(\frac{2n}{\Pi_{x \in U} p_x^{\alpha_x}}, \frac{2n.p_*^{\alpha_*}}{\Pi_{x \in U} p_x^{\alpha_x} \Pi p_q^{\alpha_q}} \right)} = \dfrac{2^\alpha . \Pi_{x \in U} p_x^{\alpha_x} \Pi p_q^{\alpha_q}}{p_*^{\alpha_*}} = 2^\alpha s'$.

Therefore, $\langle a^l \rangle = \langle a^{2^\alpha s'} \rangle \prec \langle a^{s'}, a^j b \rangle = S$. Note that, $\dfrac{|S|}{|T \wedge S|} = 2^{\alpha+1}$ and hence $T \wedge S \prec S$ for such choice of S and T. ∎

A lattice is said to be complemented if every element has a complement. In what follows, we have a Theorem about $LH(D_n)$.

Theorem 2.9. *Let D_n be the dihedral group with $2n$ elements where $n = 2^\alpha \prod_{i=1}^m p_i^{\alpha_i}$. Then, the lattice $LH(D_n)$ is complemented.*

Proof. In order to show that $LH(D_n)$ is complemented, it is sufficient to show that every cyclic Hall subgroup has a complement in $LH(D_n)$.

Note that, if a cyclic subgroup $\langle a^h \rangle$ is also a Hall subgroup, then it is necessary that $h = 2^\alpha \prod_M p_x^{\alpha_x}$ such that $x \in M \subseteq \{1, 2, \ldots, m\}$. Moreover, if a dihedral subgroup $\langle a^h, a^i b \rangle$ is also a Hall subgroup, then it is necessary that $h = \prod_N p_x^{\alpha_x}$ such that $x \in N \subseteq \{1, 2, \ldots, m\}$.

Let $A = \langle a^k \rangle$ be a cyclic Hall subgroup, then $k = 2^\alpha \prod_U p_x^{\alpha_x}$ such that $x \in U \subseteq \{1, 2, \ldots, m\}$. Choose the subgroup $B = \langle a^t, a^i b \rangle$ where $t = \frac{n}{k}$. But then $g = (k, t) = 1$ and so $A \vee B = \langle a^g, a^i b \rangle = D_n$. Moreover $l = [k, t] = n$ this implies $A \wedge B = \langle a^l \rangle = \langle a^n \rangle = I$. Therefore, every cyclic Hall subgroup has complement and so every dihedral Hall subgroup has a complement. ∎

It is known that the number of subgroups of D_n for $n \geq 3$ is $|L(D_n)| =$ Number of divisors of n + Sum of divisors of n. Along the same line, we have the following formula for the number of Hall subgroups of D_n, i.e., $|LH(D_n)|$.

Theorem 2.10. *For any $n \geq 3$, $|LH(D_n)| = 2^z + \prod_{m=1}^z (1 + p_m^{\alpha_m})$ where $n = 2^\alpha \prod_{m=1}^z p_m^{\alpha_m}$, where p is prime and z is the number of odd primes dividing n.*

Proof. Let $n = 2^\alpha \prod_{m=1}^z p_m^{\alpha_m}$, p being prime. If H is a cyclic Hall subgroup of D_n, then $|H| = \prod_{x \in S \subseteq \{1,2,\ldots,z\}} p_x^{\alpha_x}$ and $|H|$ is not a multiple of 2. Note that, number of subgroups whose order is divisible by single odd prime is given by $\binom{z}{1}$. Similarly, number of subgroups whose order contains exactly two odd prime factors is given by $\binom{z}{2}$. Consequently, number of cyclic Hall subgroups$=$ $\binom{z}{0} + \binom{z}{1} + \binom{z}{2} + \binom{z}{3} + \cdots + \binom{z}{z} = 2^z$.

Now consider a dihedral Hall subgroup H then $|H| = 2^{\alpha+1} \prod_{x \in S \subseteq \{1,2,\ldots,z\}} p_x^{\alpha_x}$. If H_1 be a dihedral Hall subgroup whose order is divisible by single odd prime say p_1, then $H_1 = \left\langle a^{\prod_{m=2}^z p_m^{\alpha_m}}, a^i b \right\rangle$ and number of subgroups whose order is equal to order of H_1 is $\prod_{m=2}^z p_m^{\alpha_m}$. Consequently, the number of all such subgroups whose order is divisible by exactly single odd prime is equal to $\sum_{x \in S \subset \{1,2,\ldots,z\}} \prod p_x^{\alpha_x}$ such

298 that $|S| = z-1$. Similarly, if H_2 is a dihedral Hall subgroup whose order is divis-
299 ible by exactly two odd prime factors, say p_1 and p_2, then $H_2 = \left\langle a^{\prod_{m=3}^{z} p_m^{\alpha_m}}, a^i b \right\rangle$
300 and the number of subgroups whose order is equal to order of H_2 is $\prod_{m=3}^{z} p_m^{\alpha_m}$.
301 Consequently, number of all such subgroups whose order contains exactly two
302 odd primes is equal to $\sum_{x \in S \subset \{1,2,\ldots,z\}} \prod p_x^{\alpha_x}$ such that $|S| = z-2$. As such, num-
303 ber of all such dihedral Hall subgroups considering the number of prime divisors
304 involved is given by $\sum_{m=1}^{z} p_m^{\alpha_m} + \sum_{x \in S_1 \subset \{1,2,\ldots,z\}} \prod p_x^{\alpha_x} + \sum_{x \in S_2 \subset \{1,2,\ldots,z\}} \prod p_x^{\alpha_x} +$
305 $\sum_{x \in S \subset \{1,2,\ldots,z\}} \prod p_x^{\alpha_x} + \cdots + \sum_{x \in S_{z-1} \subset \{1,2,\ldots,z\}} \prod p_x^{\alpha_x} + 1 = \prod_{m=1}^{z} \left(1 + p_m^{\alpha_m}\right)$, where
306 $|S_i| = z - i$ for $i = 1, 2, \ldots, z-1$.
307 Therefore, number of Hall subgroups of $D_n = |LH(D_n)| = 2^z + \prod_{m=1}^{z} (1 + p_m^{\alpha_m})$,
308 whenever $n = 2^\alpha \prod_{m=1}^{z} p_m^{\alpha_m}$. ∎

309 ## 3.   Hall subgroups of finite nilpotent groups

310 In this section, properties of collection of Hall subgroups of finite nilpotent groups
311 are investigated.

312 We recall the following characterization, see Grätzer [5].

313 **Theorem 3.1.** *A modular lattice is distributive if and only if it does not a sub-*
314 *lattice isomorphic to diamond ($\mathcal{M}_3$).*

315 **Remark.** For every Hall subgroup K of G, $LH(K)$ is a sublattice of $LH(G)$
316 whenever $LH(G)$ is a lattice.

317 **Theorem 3.2.** *Let G be a finite group. Then $LH(G)$ is a distributive lattice if*
318 *and only if G is a nilpotent group.*

319 *Proof.* Let G be a finite nilpotent group, we first show that $LH(G)$ is a sublattice
320 of $L(G)$. Let $|G| = \prod_{i=1}^{m} p_i^{\alpha_i}$ and the subgroups H, K are Hall subgroups of G.
321 Note that, G is nilpotent if and only if it is direct product of its Sylow p-subgroups,
322 i.e., $G = G_1 \times G_2 \times \cdots \times G_m = \prod_{i=1}^{m} G_i$, where each G_i is the Sylow p_i-subgroup of
323 G. Also, Note that, each G_i is unique being part of direct product and so normal
324 in G.

325 **Claim I.** $H \wedge K$ is a Hall subgroup.

326 Let $H = \prod_{i \in S_1} G_i$ and $K = \prod_{i \in S_2} G_i$ such that S_1, $S_2 \subseteq \{1, 2, \ldots, m\}$ are
327 unique of its order being normal in G. But then the subgroup $H \cap K = T =$
328 $\prod_{i \in S_1 \cap S_2} G_i$ is the Hall subgroup of G and so $H \cap K$ is a Hall subgroup.

329 **Claim II.** $H \vee K$ is a Hall subgroup.

330 Let $H = \prod_{i \in S_1} G_i$ and $K = \prod_{i \in S_2} G_i$ such that $S_1, S_2 \subseteq \{1, 2, \ldots, m\}$ are
331 unique of its order being normal in G. But then the subgroup $\langle H, K \rangle = T =$

332 $\prod_{i \in S_1 \cup S_2} G_i$ is the Hall subgroup of G and so $\langle H, K \rangle$ is a Hall subgroup. This
333 proves that $LH(G)$ is a sublattice of $L(G)$.

334   Note that, each Hall subgroup is normal as it is join of Sylow p-subgroups
335 and every Sylow p-subgroup is unique as G is direct product of its Sylow p-
336 subgroups being nilpotent. Consequently, $LH(G)$ is a sublattice of $LN(G)$ which
337 implies that $LH(G)$ is modular since $LN(G)$ is a modular lattice and sublattice
338 of modular lattice is modular. We show that $LH(G)$ does not contain diamond
339 ($\mathcal{M}_3$) as its sublattice.

340   Suppose $LH(G)$ contains a diamond as its sublattice. Note that, the five
341 subgroups H_i, $i \in \{1, 2, \ldots, 5\}$ in M_3 as depicted in Figure 3.1. The each one of
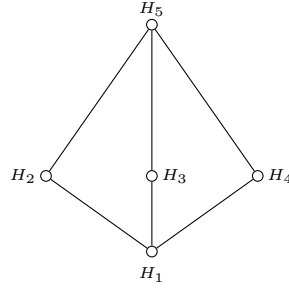342 the five subgroups are of different orders these are of different orders.

Figure 3.1. Figure $\mathcal{M}_3$.

343   Now $H_2 \vee H_3 = H_2 H_3 = H_4 \vee H_3 = H_4 H_3 = H_2 \vee H_4 = H_4 H_2$. Consequently,
344 $|H_4 H_3| = |H_4 H_2| = |H_2 H_3| = |H_5|$, but then $|H_4 H_3| = \frac{|H_4||H_3|}{|H_4 \cap H_3|} = |H_4 H_2| =$
345 $\frac{|H_4||H_2|}{|H_4 \cap H_2|}$ which implies $|H_2| = |H_3|$, a contradiction.

346   Conversely, suppose that $LH(G)$ is a distributive lattice. We contend that,
347 G is direct product of its Sylow p-subgroups. If not, then there exists a prime p
348 such that $p||G|$ and a Sylow p-subgroup of G is not normal. Let P_1 and P_2 be
349 two Sylow p-subgroups of G, then these are also Hall subgroups.

350   Note that, $|G|$ is divisible by at least two primes since every finite group with
351 prime power order is nilpotent.

352   *Case* I. Let $|G| = p^\alpha q^\beta$ where p, q are distinct primes. Choose a subgroup
353 Q of G such that Q is a Sylow q-subgroup, which is also a Hall subgroup. Note
354 that, $P_1 \wedge_{LH} Q = P_2 \wedge_{LH} Q = P_1 \wedge_{LH} P_2 = \{e\}$ and $P_1 \vee_{LH} Q = P_2 \vee_{LH}$
355 $Q = P_1 \vee_{LH} P_2 = G$. Moreover P_1, P_2, Q Hall subgroup. Consequently, $LH(G)$
356 contains sublattice $S = \{\{e\}, P_1, P_2, Q, G\}$ isomorphic to M_3, a contradiction to
357 the fact that $LH(G)$ is distributive.

358   *Case* II. Let $|G| = p^\alpha q_1^{\beta_1} \cdots q_m^{\beta_m}$ where p, q_i's are distinct primes. Since
359 $LH(G)$ is a lattice, $P_1 \vee_{LH} P_2 = T$ is a Hall subgroup of G, let $|T| = p^\alpha \prod_{i \in X} q_i^{\beta_i}$

for a subset $X \subseteq \{1, 2, \ldots, m\}$. Note that, if there exists a Hall subgroup Q of order $\prod_{i \in X} q_i^{\beta_i}$ then this subgroup is such that $p \nmid |Q|$ is a co-atom in $LH(T)$. If not, then consider a subgroup Q which is Hall subgroup with order $\prod_{i \in Y \subset X} q_i^{\beta_i}$. Such Q exists, since at least we have a Sylow q_i-subgroup which is a Hall subgroup. Also, such Q is co-atom in $LH(T)$ and $p \nmid |Q|$.

Now, consider the subset $\{\{e\}, P_1, P_2, Q, T\}$ with $P_1 \wedge_{LH} Q = P_2 \wedge_{LH} Q = P_1 \wedge_{LH} P_2 = \{e\}$ and $P_1 \vee_{LH} Q = P_2 \vee_{LH} Q = P_1 \vee_{LH} P_2 = T$, which forms a sublattice isomorphic to M_3 of $LH(T)$ and so, $LH(T)$ is not distributive. Consequently, $LH(G)$ is not distributive, a contradiction.

Therefore, G is direct product of its Sylow p-subgroups and so nilpotent. ∎

In the next Lemma the number of Hall subgroups of finite nilpotent groups is obtained.

Lemma 3.3. *Let G be a finite nilpotent group and $|G| = \prod_{i=1}^{m} p_i^{\alpha_i}$, then $|LH(G)| = 2^m$.*

Proof. Note that, if G is a finite nilpotent group and π is any set of primes, then G has a Hall π-subgroup. Moreover, by Theorem 3.2, we have the unique Hall π-subgroup for each set π of primes. Consequently, the number of distinct Hall subgroups of G is $\binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \binom{m}{3} + \cdots + \binom{m}{m} = 2^m$. ∎

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