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PRIMITIVE IDEALS AND JACOBSON'S STRUCTURE SPACES OF SEMIGROUPS

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Abstract

The purpose of this note is to introduce primitive ideals of semigroups and study some topological aspects of the corresponding structure spaces. We show that every structure space of a semigroup is T_0, quasi-compact, and every nonempty irreducible closed subset has a unique generic point. Moreover, such a structure space is Hausdorff if and only if every primitive ideal of the semirgroup is minimal. Finally, we define continuous maps between structure spaces of semigroups.

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1. Introduction

Since the introduction of primitive rings in [17], primitive ideals have shown their immense importance in understanding structural aspects of rings and modules [19, 27], Lie algebras [24], enveloping algebras [8, 21], PI-algebras [20], quantum groups [22], skew polynomial rings [16], and others. In [18], Jacobson has introduced a hull-kernel topology (also known as Jacobson topology) on the set of primitive ideals of a ring, and has obtained representations of biregular rings. This Jacobson topology also turns out to play a key role in representation of finite-dimensional Lie algebras (see [8]).

Compare to the above algebraic structures, after magmas (also known as groupoids), semigroups are the most basic ones. A detailed study of algebraic theory of semigroups can be found in one of the earliest textbooks [6] and [7] (see also [11, 13, 15]), whereas specific study of prime, semiprime, and maximal ideals of semigroups are done in [2, 4, 26, 28]. Furthermore, various notions of radicals of semigroups have been studied in [1, 10, 29]. Readers may consider [5] for a survey on ideal theory of semigroups.

The next question is of imposing topologies on various types of ideals of semigroups. To this end, hull-kernel topology on maximal ideals of (commutative) semigroups has been considered in [3], whereas the same on minimal prime ideals has been done in [23]. Using the notion of x-ideals introduced in [3], although in [14] a study of general notion of structure spaces for semigroups has been done, but having the assumption of commutativity restricts it to only certain types of ideals of semigroups, and hence did not have a scope for primitive ideals.

In [9], the spectrum of prime elements has been studied in the context of a multiplicative lattice which itself consists of a semigroup structure. One can further extend the theory developed there by defining ideals in a multiplicative lattice; and by considering modules over such lattices, it is not hard to see that the notion of primitive ideals can be studied over multiplicative lattices. All these and some other aspects of primitive ideals of quantales (a special type of multiplicative lattices) will be considered in the forthcoming paper [12].

The aim of this paper is to introduce primitive ideals of semigroups and endow Jacobson topology on primitive ideals to study some topological aspects of them. In order to have the notion of primitive ideals of semigroups, we furthermore need a notion of a module over a semigroup. We hope this notion of primitive ideals introduced here will in future shade some light on the structural aspects of semigroups.

2. Primitive ideals

A *semigroup* is a tuple $(S, \cdot)$ such that the binary operation $\cdot$ on the set S is associative. For all $a, b \in S$, we shall write ab to mean $a \cdot b$. Throughout this work, all semigroups are assumed to be noncommutative. If a semigroup S has an identity, we denote it by 1 satisfying the property: $s1 = s = 1s$ for all $s \in S$. If A and B are subsets of S, then by the *set product* AB of A and B we shall mean $AB = \{ab \mid a \in A, b \in B\}$. If $A = \{a\}$ we write AB as aB, and similarly for $B = \{b\}$. Thus

$$AB = \cup\{Ab \mid b \in B\} = \cup\{aB \mid a \in A\}.$$

A *left* (*right*) *ideal* of a semigroup S is a nonempty subset $\mathfrak{a}$ of S such that $S\mathfrak{a} \subseteq \mathfrak{a}$ ($\mathfrak{a}\, S \subseteq \mathfrak{a}$). A *two-sided ideal* or simply an *ideal* is a subset which is both a left and a right ideal of S. In this work the word "ideal" without modifiers will always mean two-sided ideal and we shall denote the set of all ideals of a semigroup S by Ideal(S). If X is a nonempty subset of a semigroup S, then the ideal $\langle X\rangle$ *generated by* X is the intersection of all ideals containing X. Therefore,

$$\langle X\rangle = X \cup XS \cup SX \cup XSX. \tag{1}$$

We say an ideal $\mathfrak{a} = \langle X\rangle$ is of *finite character* if X is equal to the set-theoretic union of all the ideals generated by finite subsets of X (*cf.* definition in [3, Chapter 1, p. 4]). Note that in our context, all ideals are of finite character. This follows from the fact that the property "being of finite character", in our context, should refers to the closure operator $\mathcal{C}(-)$ (see §3), and then equation (1) in [3, Chapter 1, p. 4] becomes: for any subset $X \subseteq S$, we have $\langle X\rangle = \cup\{\langle F\rangle \mid F \subseteq X, F \text{ finite }\}$. But this is always true, namely the x-system of "classical" ideals is of finite character, thanks to the fact that for any subset $X \subseteq S$, one has an expression (1).

To define primitive ideals of a semigroup S, we require the notion of a module over S, which we introduce now.

A (*left*) S*-module* is an abelian group $(M, +, 0)$ endowed with a map $S\times M \to M$ (denoted by $(s, m) \mapsto sm$) satisfying the identities:

1. $s(m + m') = sm + sm'$;
2. $(ss')m = s(s'm)$;
3. $s0 = 0$,

for all $s, s' \in S$ and for all $m, m' \in M$. Henceforth the term "S-module" without modifier will always mean left S-module. If M, M' are S-modules, then an S-*module homomorphism* from M into M' is a group homomorphism $f\colon M \to M'$ such that $f(sm) = sf(m)$ for all $s \in S$ and for all $m \in M$. A subset N of M is called an S*-submodule* of the module M if

1. $(N, +)$ is a subgroup of $(M, +)$;
2. for all $s \in S$ and for all $n \in N$, $sn \in N$.

If $\mathfrak{a}$ is an ideal of S, then the additive subgroup $\mathfrak{a}M$ of M generated by the elements of the form $\{am \mid a \in \mathfrak{a}, m \in M\}$ is an S-submodule. An S-module M is called *simple* (or *irreducible*) if

1. $SM = \{\sum s_i m_i \mid s_i \in S, m_i \in M\} \neq 0$.
2. There is no proper S-submodule of M other than 0.

A (*left*) *annihilator* of an S-module M is $\mathrm{Ann}_S(M) = \{s \in S \mid sm = 0 \text{ for all } m \in M\}$. When $M = \{m\}$, we write $\mathrm{Ann}_S(\{m\})$ as $\mathrm{Ann}_S(m)$.

Lemma 1. *An annihilator* $\mathrm{Ann}_S(M)$ *is an ideal of* S.

Proof. For all $s \in S$ and for all $x \in \mathrm{Ann}_S(M)$ we have $(sx)m = s(xm) = s0 = 0$. Similarly, we have $(xs)m = x(sm) = 0$ because $x \in \mathrm{Ann}_S(M)$ and $sm \in M$. ∎

Let S be a semigroup. A nonempty proper ideal $\mathfrak{p}$ of S is said to be *primitive* if $\mathfrak{p} = \mathrm{Ann}_S(M)$ for some simple S-module M. We denote the set of primitive ideals of a semigroup S by $\mathrm{Prim}(S)$. Let us provide some examples of primitive ideals of semigroups.

Example 2. Consider the semigroup S of 2×2 upper triangular matrices with real entries under matrix multiplication. An ideal

$$\mathfrak{p} := \left\{ \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \mid a \in \mathbb{R} \right\}$$

is a primitive ideal of S. The annihilator of the submodule consisting of scalar multiples of the identity matrix is $\mathfrak{p}$.

Example 3. Consider the semigroup $S = \mathbb{N}_0 \times \mathbb{N}_0$ (non-negative integer pairs) under componentwise addition. A primitive ideal of S is $\mathfrak{p} := \{(0, b) \mid b \in \mathbb{N}_0\}$. The annihilator of the submodule generated by the action of S on the set $\{(a, 0) \mid a \in \mathbb{N}_0\}$ is $\mathrm{Ann}_S(\{(a, 0) \mid a \in \mathbb{N}_0\}) = \mathfrak{p}$.

Example 4. Consider the semigroup $S = (\mathbb{N}, +)$, where $\mathbb{N}$ is the set of natural numbers. Let $M = (\mathbb{Z}, +, 0)$ be the additive group of integers. Define the action of S on M as $n \cdot m = nm$ for all $n \in \mathbb{N}$ and $m \in \mathbb{Z}$. The trivial ideal 0 is a primitive ideal of S.

Example 5. Let S be the semigroup of $n \times n$ non-negative integer matrices under matrix multiplication. For $M = (\mathbb{R}^n, +, 0)$, where 0 is the zero vector, define the action of S on M as $A \cdot v = Av$ for all $A \in S$ and $v \in \mathbb{R}^n$. The annihilator of M is the set of matrices with a row of zeros, denoted as

$$\mathrm{Ann}_S(M) = \{A \in S \mid \exists v \neq 0, Av = 0\}.$$

A primitive ideal of S is $\mathfrak{p} := \{A \in S \mid \text{some row of } A \text{ is } 0\}$.

Example 6. Consider the free semigroup S generated by two elements a and b with the operation being string concatenation. Let $M = (\mathbb{Z}, +, 0)$ be the additive group of integers. Define the action of S on M by the concatenation of strings followed by addition, i.e., $s \cdot m = sm$, for all $s \in S$ and $m \in \mathbb{Z}$. A primitive ideal of S is $\mathfrak{p} := \{s \in S \mid b \text{ does not appear in } s\}$.

A nonempty proper ideal $\mathfrak{q}$ of a semigroup S is said to be *prime* if for any two ideals $\mathfrak{a}$, $\mathfrak{b}$ of S and $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{q}$ implies $\mathfrak{a} \subseteq \mathfrak{q}$ or $\mathfrak{b} \subseteq \mathfrak{q}$, where the product $\mathfrak{a}\mathfrak{b}$ of ideals $\mathfrak{a}$ and $\mathfrak{b}$ is defined to be the set of all finite sums $\sum i_\alpha j_\alpha$ (where $i_\alpha \in \mathfrak{a}$, $j_\alpha \in \mathfrak{b}$).

The proof of the following result is easy to verify.

Lemma 7. *If $\mathfrak{a}$ and $\mathfrak{b}$ are any two ideals of a semigroup, then $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b}$.*

The following proposition gives an alternative formulation of prime ideals of semigroups. For a proof, see [26, Lemma 2.2].

Proposition 8. Suppose S is a semigroup. Then the following conditions are equivalent:

1. $\mathfrak{q}$ is a prime ideal of S.
2. $aSb \subseteq \mathfrak{q}$ implies $a \in \mathfrak{q}$ or $b \in \mathfrak{q}$ for all $a, b \in S$.

Primitive ideals and prime ideals of a semigroup are related as follows.

Proposition 9. Every primitive ideal of a semigroup is a prime ideal.

Proof. Suppose $\mathfrak{p}$ is a primitive ideal and $\mathfrak{p} = \mathrm{Ann}_S(M)$ for some simple S-module M. Let $a, b \notin \mathrm{Ann}_S(M)$. Then $am \neq 0$ and $bm' \neq 0$ for some $m, m' \in M$. Since M is simple, there exists an $s \in S$ such that $s(bm') = m$. Then

$$(asb)m' = a(s(bm')) = am \neq 0,$$

and hence $asb \notin \mathrm{Ann}_S(M)$. Therefore, $\mathrm{Ann}_S(M)$ is a prime ideal by Lemma 8. ■

In the next section we talk about Jacobson topology on the set of primitive ideals of a semigroup and discuss about some of the topological properties of the corresponding structure spaces.

3. JACOBSON TOPOLOGY

We shall introduce Jacobson topology in $\mathrm{Prim}(S)$ by defining a closure operator for the subsets of $\mathrm{Prim}(S)$. Once we have a closure operator, closed sets are defined as sets which are invariant under this closure operator[1]. Suppose X is a subset of $\mathrm{Ideal}(S)$. Set $\mathcal{D}_X = \bigcap_{\mathfrak{q} \in X} \mathfrak{q}$. We define the closure of the set X as

$$\mathcal{C}(X) = \{\mathfrak{p} \in \mathrm{Prim}(S) \mid \mathfrak{p} \supseteq \mathcal{D}_X\}. \tag{2}$$

[1]The origin of Kuratowski's closure operator on the set of primitive ideals of a ring can be traced back to [18].

If $X = \{x\}$, we will write $\mathcal{C}(\{x\})$ as $\mathcal{C}(x)$. We wish to verify that the closure operation defined in (2) satisfies Kuratowski's closure conditions and that is done in the following

Proposition 10. *The sets* $\{\mathcal{C}(X)\}_{X\subseteq \mathrm{Ideal}(S)}$ *satisfy the following conditions:*

1. $\mathcal{C}(\emptyset) = \emptyset$,
2. $\mathcal{C}(X) \supseteq X$,
3. $\mathcal{C}(\mathcal{C}(X)) = \mathcal{C}(X)$,
4. $\mathcal{C}(X \cup Y) = \mathcal{C}(X) \cup \mathcal{C}(Y)$.

Proof. The proofs of (1)-(3) are straightforward, whereas for (4), it is easy to see that $\mathcal{C}(X\cup Y) \supseteq \mathcal{C}(X)\cup\mathcal{C}(Y)$. To obtain the the other inclusion, let $\mathfrak{p} \in \mathcal{C}(X\cup Y)$. Then

$$\mathfrak{p} \supseteq \mathcal{D}_{X\cup Y} = \mathcal{D}_X \cap \mathcal{D}_Y.$$

Since $\mathcal{D}_X$ and $\mathcal{D}_Y$ are ideals of S, by Lemma 7, it follows that

$$\mathcal{D}_X\mathcal{D}_Y \subseteq \mathcal{D}_X \cap \mathcal{D}_Y \subseteq \mathfrak{p}.$$

Since by Proposition 9, $\mathfrak{p}$ is prime, either $\mathcal{D}_X \subseteq \mathfrak{p}$ or $\mathcal{D}_Y \subseteq \mathfrak{p}$ This means either $\mathfrak{p} \in \mathcal{C}(X)$ or $\mathfrak{p} \in \mathcal{C}(Y)$. Thus $\mathcal{C}(X \cup Y) \subseteq \mathcal{C}(X) \cup \mathcal{C}(Y)$. ∎

The set $\mathrm{Prim}(S)$ of primitive ideals of a semigroup S topologized (the Jacobson topology) by the closure operator defined in (2) is called the *structure space* of the semigroup S. It is evident from (2) that if $\mathfrak{p} \neq \mathfrak{p}'$ for any two $\mathfrak{p}, \mathfrak{p}' \in \mathrm{Prim}(S)$, then $\mathcal{C}(\mathfrak{p}) \neq \mathcal{C}(\mathfrak{p}')$. Thus

Proposition 11. *Every structure space* $\mathrm{Prim}(S)$ *is a* T_0*-space.*

Theorem 12. *If* S *is a semigroup with identity then the structure space* $\mathrm{Prim}(S)$ *is quasi-compact.*

Proof. Suppose that $\{K_\lambda\}_{\lambda\in\Lambda}$ is a family of closed sets of the structure space $\mathrm{Prim}(S)$ such that $\bigcap_{\lambda\in\Lambda} K_\lambda = \emptyset$. This implies that the ideal $\bigvee_{\lambda\in\Lambda} \mathcal{D}_{K_\lambda}$ generated by $\{\mathcal{D}_{K_\lambda}\}_{\lambda\in\Lambda}$ must be equal to S. Indeed: $\bigvee_{\lambda\in\Lambda} \mathcal{D}_{K_\lambda} \neq S$ implies there exists a maximal ideal $\mathfrak{m}$ in S such that $\mathcal{D}_{K_\lambda} \subseteq \mathfrak{m}$ for all $\lambda \in \Lambda$, whence $\mathfrak{m} \in \bigcap_{\lambda\in\Lambda} K_\lambda$, a contradiction. Therefore, in particular, $1 = x_1 \cdots x_n$, where $x_i \in \mathcal{D}_{K_{\lambda_i}}$ $(1 \leqslant i \leqslant n)$. Hence, $\bigvee_{i=1}^n \mathcal{D}_{K_{\lambda_i}} = S$. This subsequently implies $\bigcap_{i=1}^n K_{\lambda_i} = \emptyset$. By finite intersection property, we then have the desired quasi-compactness. ∎

Recall that a nonempty closed subset K of a topological space X is *irreducible* if $K \neq K_1 \cup K_2$ for any two proper closed subsets K_1, K_2 of K. A maximal irreducible subset of a topological space X is called an *irreducible component* of X. A point x in a closed subset K is called a *generic point* of K if $K = \mathcal{C}(x)$.

Lemma 13. *The irreducible closed subsets of a structure space* $\mathrm{Prim}(S)$ *are of the form:* $\{\mathcal{C}(\mathfrak{p})\}_{\mathfrak{p}\in\mathrm{Prim}(S)}$.

Proof. Since $\{\mathfrak{p}\}$ is irreducible, so is $\mathcal{C}(\mathfrak{p})$. Suppose $\mathcal{C}(\{\mathfrak{a}\})$ is an irreducible closed subset of $\mathrm{Prim}(S)$ and $\mathfrak{a} \notin \mathrm{Prim}(S)$. Here, by $\mathcal{C}(\{\mathfrak{a}\})$, we mean $\mathcal{C}(X)$ with $\mathcal{D}_X = \{\mathfrak{a}\}$. This implies there exist ideals $\mathfrak{b}$ and $\mathfrak{c}$ of S such that $\mathfrak{b} \nsubseteq \mathfrak{a}$ and $\mathfrak{c} \nsubseteq \mathfrak{a}$, but $\mathfrak{b}\mathfrak{c} \subseteq \mathfrak{a}$. Then

$$\mathcal{C}(\langle \mathfrak{a}, \mathfrak{b}\rangle) \cup \mathcal{C}(\langle \mathfrak{a}, \mathfrak{c}\rangle) = \mathcal{C}(\langle \mathfrak{a}, \mathfrak{b}\mathfrak{c}\rangle) = \mathcal{C}(\mathfrak{a}).$$

But $\mathcal{C}(\langle \mathfrak{a}, \mathfrak{b}\rangle) \neq \mathcal{C}(\mathfrak{a})$ and $\mathcal{C}(\langle \mathfrak{a}, \mathfrak{c}\rangle) \neq \mathcal{C}(\mathfrak{a})$, and hence $\mathcal{C}(\mathfrak{a})$ is not irreducible. ■

Proposition 14. Every irreducible closed subset of $\mathrm{Prim}(S)$ has a unique generic point.

Proof. The existence of generic point follows from Lemma 13, and the uniqueness of such a point follows from Proposition 11. ■

In the following proposition, we will find examples of irreducible components of a structure space.

Proposition 15. If $\mathfrak{p}$ is a minimal primitive ideal of S, then $\mathcal{C}(\mathfrak{p})$ is an irreducible component of a structure space $\mathrm{Prim}(S)$. The converse also holds.

Proof. If $\mathcal{C}(\mathfrak{p})$ is not a maximal irreducible subset of $\mathrm{Prim}(S)$, then there exists a maximal irreducible subset $\mathcal{C}(\mathfrak{p}')$ with $\mathfrak{p}' \in \mathrm{Prim}(S)$ such that $\mathcal{C}(\mathfrak{p}) \subsetneq \mathcal{C}(\mathfrak{p}')$. This implies that $\mathfrak{p} \in \mathcal{C}(\mathfrak{p}')$ and hence $\mathfrak{p}' \subsetneq \mathfrak{p}$, contradicting the minimality property of $\mathfrak{p}$. To show the converse, let K be an irreducible component. By Lemma 13, $K = \mathcal{C}(\mathfrak{p})$ for some primitive ideal $\mathfrak{p}$. If $\mathfrak{p}$ is not minimal, then there is a primitive ideal $\mathfrak{q}$ properly contained in $\mathfrak{p}$. Then, $K \subsetneq \mathcal{C}(\mathfrak{q})$, contradicting the maximality of K. ■

While the next corollary provides a characterization of Hausdorff structure spaces of semigroups, the author, however, hasn't encountered any examples of semigroups where $\mathrm{Prim}(S)$ is not Hausdorff.

Corollary 16. A structure space $\mathrm{Prim}(S)$ is Hausdorff if and only if every primitive ideal of S is minimal.

Recall that a semigroup is called *Noetherian* if it satisfies the ascending chain condition on its ideals, whereas a topological space X is called *Noetherian* if the descending chain condition holds for closed subsets of X. A relation between these two notions is shown in the following

Proposition 17. If a semigroup S is Noetherian, then $\mathrm{Prim}(S)$ is a Noetherian space.

Proof. It suffices to show that a collection of closed sets in $\mathrm{Prim}(S)$ satisfies the descending chain condition. Let $\mathcal{C}(\mathfrak{a}_1) \supseteq \mathcal{C}(\mathfrak{a}_2) \supseteq \cdots$ be a descending chain of closed sets in $\mathrm{Prim}(S)$. Once again, by $\mathcal{C}(\{\mathfrak{a}\})$, we mean $\mathcal{C}(X)$ with $\mathcal{D}_X = \{\mathfrak{a}\}$. Then, $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \cdots$ is an ascending chain of ideals in S. Since S is Noetherian, the chain stabilizes at some $n \in \mathbb{N}$. Hence, $\mathcal{C}(\mathfrak{a}_n) = \mathcal{C}(\mathfrak{a}_{n+k})$ for any k. Thus $\mathrm{Prim}(S)$ is Noetherian. ∎

Corollary 18. The set of minimal primitive ideals in a Noetherian semigroup is finite.

Proof. By Proposition 17, $\mathrm{Prim}(S)$ is Noetherian, thus $\mathrm{Prim}(S)$ has a finitely many irreducible components. By Proposition 15, every irreducible closed subset of $\mathrm{Prim}(S)$ is of form $\mathcal{C}(\mathfrak{p})$, where $\mathfrak{p}$ is a minimal primitive ideal. Thus $\mathcal{C}(\mathfrak{p})$ is irreducible components if and only if $\mathfrak{p}$ is minimal primitive. Hence, S has only finitely many minimal primitive ideals. ∎

Proposition 19. Suppose $\phi\colon S \to T$ is a semigroup homomorphism and define the map $\phi_*\colon \mathrm{Prim}(T) \to \mathrm{Prim}(S)$ by $\phi_*(\mathfrak{p}) = \phi^{-1}(\mathfrak{p})$, where $\mathfrak{p} \in \mathrm{Prim}(T)$. Then ϕ_* is a continuous map.

Proof. To show ϕ_* is continuous, we first show that $f^{-1}(\mathfrak{p}) \in \mathrm{Prim}(S)$, whenever $\mathfrak{p} \in \mathrm{Prim}(T)$. Note that $\phi^{-1}(\mathfrak{p})$ is an ideal of S and a union of $\ker\phi$-classes (see [11, Proposition 3.4]. Suppose $\mathfrak{p} = \mathrm{Ann}_T(M)$ for some simple T-module. Then $\phi^{-1}(\mathfrak{p})$ is the annihilator of the simple T-module M obtained by defining $sm := \phi(s)m$. Therefore $f^{-1}(\mathfrak{p}) \in \mathrm{Prim}(S)$. Now consider a closed subset $\mathcal{C}(\mathfrak{a})$ of $\mathrm{Prim}(S)$, where by $\mathcal{C}(\mathfrak{a})$, we mean $\mathcal{C}(X)$ with $\mathcal{D}_X = \{\mathfrak{a}\}$. Then for any $\mathfrak{q} \in \mathrm{Prim}(T)$, we have:

$$\mathfrak{q} \in \phi_*^{-1}(\mathcal{C}(\mathfrak{a})) \Leftrightarrow \phi^{-1}(\mathfrak{q}) \in \mathcal{C}(\mathfrak{a}) \Leftrightarrow \mathfrak{a} \subseteq \phi^{-1}(\mathfrak{q}) \Leftrightarrow \mathfrak{q} \in \mathcal{C}(\langle \phi(\mathfrak{a}) \rangle),$$

and this proves the desired continuity of ϕ_*. ∎

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