

4 **PRIMITIVE IDEALS AND JACOBSON'S STRUCTURE**
5 **SPACES OF SEMIGROUPS**

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12 **Abstract**

13 The purpose of this note is to introduce primitive ideals of semigroups
14 and study some topological aspects of the corresponding structure spaces.
15 We show that every structure space of a semigroup is T_0 , quasi-compact,
16 and every nonempty irreducible closed subset has a unique generic point.
17 Moreover, such a structure space is Hausdorff if and only if every primitive
18 ideal of the semigroup is minimal. Finally, we define continuous maps
19 between structure spaces of semigroups.

20 **Keywords:** semigroup; primitive ideal; Jacobson topology.

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22 1. INTRODUCTION

23 Since the introduction of primitive rings in [17], primitive ideals have shown their
24 immense importance in understanding structural aspects of rings and modules
25 [19, 27], Lie algebras [24], enveloping algebras [8, 21], PI-algebras [20], quantum
26 groups [22], skew polynomial rings [16], and others. In [18], Jacobson has in-
27 troduced a hull-kernel topology (also known as Jacobson topology) on the set
28 of primitive ideals of a ring, and has obtained representations of biregular rings.
29 This Jacobson topology also turns out to play a key role in representation of
30 finite-dimensional Lie algebras (see [8]).

31 Compare to the above algebraic structures, after magmas (also known as
32 groupoids), semigroups are the most basic ones. A detailed study of algebraic
33 theory of semigroups can be found in one of the earliest textbooks [6] and [7] (see

34 also [11, 13, 15]), whereas specific study of prime, semiprime, and maximal ideals
 35 of semigroups are done in [2, 4, 26, 28]. Furthermore, various notions of radicals
 36 of semigroups have been studied in [1, 10, 29]. Readers may consider [5] for a
 37 survey on ideal theory of semigroups.

38 The next question is of imposing topologies on various types of ideals of
 39 semigroups. To this end, hull-kernel topology on maximal ideals of (commutative)
 40 semigroups has been considered in [3], whereas the same on minimal prime ideals
 41 has been done in [23]. Using the notion of x -ideals introduced in [3], although in
 42 [14] a study of general notion of structure spaces for semigroups has been done,
 43 but having the assumption of commutativity restricts it to only certain types of
 44 ideals of semigroups, and hence did not have a scope for primitive ideals.

45 In [9], the spectrum of prime elements has been studied in the context of
 46 a multiplicative lattice which itself consists of a semigroup structure. One can
 47 further extend the theory developed there by defining ideals in a multiplicative
 48 lattice; and by considering modules over such lattices, it is not hard to see that
 49 the notion of primitive ideals can be studied over multiplicative lattices. All
 50 these and some other aspects of primitive ideals of quantales (a special type of
 51 multiplicative lattices) will be considered in the forthcoming paper [12].

52 The aim of this paper is to introduce primitive ideals of semigroups and endow
 53 Jacobson topology on primitive ideals to study some topological aspects of them.
 54 In order to have the notion of primitive ideals of semigroups, we furthermore
 55 need a notion of a module over a semigroup. We hope this notion of primitive
 56 ideals introduced here will in future shade some light on the structural aspects
 57 of semigroups.

58 2. PRIMITIVE IDEALS

A *semigroup* is a tuple (S, \cdot) such that the binary operation \cdot on the set S is
 associative. For all $a, b \in S$, we shall write ab to mean $a \cdot b$. Throughout this
 work, all semigroups are assumed to be noncommutative. If a semigroup S has
 an identity, we denote it by 1 satisfying the property: $s1 = s = 1s$ for all $s \in S$.
 If A and B are subsets of S , then by the *set product* AB of A and B we shall
 mean $AB = \{ab \mid a \in A, b \in B\}$. If $A = \{a\}$ we write AB as aB , and similarly
 for $B = \{b\}$. Thus

$$AB = \cup\{Ab \mid b \in B\} = \cup\{aB \mid a \in A\}.$$

59 A *left (right) ideal* of a semigroup S is a nonempty subset \mathfrak{a} of S such that
 60 $S\mathfrak{a} \subseteq \mathfrak{a}$ ($\mathfrak{a}S \subseteq \mathfrak{a}$). A *two-sided ideal* or simply an *ideal* is a subset which is both
 61 a left and a right ideal of S . In this work the word “ideal” without modifiers
 62 will always mean two-sided ideal and we shall denote the set of all ideals of a

63 semigroup S by $\text{Ideal}(S)$. If X is a nonempty subset of a semigroup S , then the
 64 ideal $\langle X \rangle$ generated by X is the intersection of all ideals containing X . Therefore,

$$\langle X \rangle = X \cup XS \cup SX \cup XSX. \quad (1)$$

65 We say an ideal $\mathfrak{a} = \langle X \rangle$ is of *finite character* if X is equal to the set-theoretic
 66 union of all the ideals generated by finite subsets of X (*cf.* definition in [3, Chapter
 67 1, p. 4]). Note that in our context, all ideals are of finite character. This follows
 68 from the fact that the property “being of finite character”, in our context, should
 69 refer to the closure operator $\mathcal{C}(-)$ (see §3), and then equation (1) in [3, Chapter 1,
 70 p. 4] becomes: for any subset $X \subseteq S$, we have $\langle X \rangle = \cup\{\langle F \rangle \mid F \subseteq X, F \text{ finite}\}$.
 71 But this is always true, namely the x -system of “classical” ideals is of finite
 72 character, thanks to the fact that for any subset $X \subseteq S$, one has an expression
 73 (1).

74 To define primitive ideals of a semigroup S , we require the notion of a module
 75 over S , which we introduce now.

76 A (*left*) S -module is an abelian group $(M, +, 0)$ endowed with a map $S \times M \rightarrow$
 77 M (denoted by $(s, m) \mapsto sm$) satisfying the identities:

- 78 1. $s(m + m') = sm + sm'$;
- 79 2. $(ss')m = s(s'm)$;
- 80 3. $s0 = 0$,

81 for all $s, s' \in S$ and for all $m, m' \in M$. Henceforth the term “ S -module” without
 82 modifier will always mean left S -module. If M, M' are S -modules, then an S -
 83 module homomorphism from M into M' is a group homomorphism $f: M \rightarrow M'$
 84 such that $f(sm) = sf(m)$ for all $s \in S$ and for all $m \in M$. A subset N of M is
 85 called an S -submodule of the module M if

- 86 1. $(N, +)$ is a subgroup of $(M, +)$;
- 87 2. for all $s \in S$ and for all $n \in N$, $sn \in N$.

88 If \mathfrak{a} is an ideal of S , then the additive subgroup $\mathfrak{a}M$ of M generated by the
 89 elements of the form $\{am \mid a \in \mathfrak{a}, m \in M\}$ is an S -submodule. An S -module M
 90 is called *simple* (or *irreducible*) if

- 91 1. $SM = \{\sum s_i m_i \mid s_i \in S, m_i \in M\} \neq 0$.
- 92 2. There is no proper S -submodule of M other than 0.

93 A (*left*) annihilator of an S -module M is $\text{Ann}_S(M) = \{s \in S \mid sm = 0 \text{ for all } m \in$
 94 $M\}$. When $M = \{m\}$, we write $\text{Ann}_S(\{m\})$ as $\text{Ann}_S(m)$.

95 **Lemma 1.** *An annihilator $\text{Ann}_S(M)$ is an ideal of S .*

96 **Proof.** For all $s \in S$ and for all $x \in \text{Ann}_S(M)$ we have $(sx)m = s(xm) = s0 = 0$.
 97 Similarly, we have $(xs)m = x(sm) = 0$ because $x \in \text{Ann}_S(M)$ and $sm \in M$. ■

98 Let S be a semigroup. A nonempty proper ideal \mathfrak{p} of S is said to be *primitive*
 99 if $\mathfrak{p} = \text{Ann}_S(M)$ for some simple S -module M . We denote the set of primitive
 100 ideals of a semigroup S by $\text{Prim}(S)$. Let us provide some examples of primitive
 101 ideals of semigroups.

102 **Example 2.** Consider the semigroup S of 2×2 upper triangular matrices with
 103 real entries under matrix multiplication. An ideal

$$\mathfrak{p} := \left\{ \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \mid a \in \mathbb{R} \right\}$$

104 is a primitive ideal of S . The annihilator of the submodule consisting of scalar
 105 multiples of the identity matrix is \mathfrak{p} .

106 **Example 3.** Consider the semigroup $S = \mathbb{N}_0 \times \mathbb{N}_0$ (non-negative integer pairs)
 107 under componentwise addition. A primitive ideal of S is $\mathfrak{p} := \{(0, b) \mid b \in \mathbb{N}_0\}$.
 108 The annihilator of the submodule generated by the action of S on the set $\{(a, 0) \mid$
 109 $a \in \mathbb{N}_0\}$ is $\text{Ann}_S(\{(a, 0) \mid a \in \mathbb{N}_0\}) = \mathfrak{p}$.

110 **Example 4.** Consider the semigroup $S = (\mathbb{N}, +)$, where \mathbb{N} is the set of natural
 111 numbers. Let $M = (\mathbb{Z}, +, 0)$ be the additive group of integers. Define the action
 112 of S on M as $n \cdot m = nm$ for all $n \in \mathbb{N}$ and $m \in \mathbb{Z}$. The trivial ideal 0 is a
 113 primitive ideal of S .

114 **Example 5.** Let S be the semigroup of $n \times n$ non-negative integer matrices under
 115 matrix multiplication. For $M = (\mathbb{R}^n, +, 0)$, where 0 is the zero vector, define the
 116 action of S on M as $A \cdot v = Av$ for all $A \in S$ and $v \in \mathbb{R}^n$. The annihilator of M
 117 is the set of matrices with a row of zeros, denoted as

$$\text{Ann}_S(M) = \{A \in S \mid \exists v \neq 0, Av = 0\}.$$

118 A primitive ideal of S is $\mathfrak{p} := \{A \in S \mid \text{some row of } A \text{ is } 0\}$.

119 **Example 6.** Consider the free semigroup S generated by two elements a and b
 120 with the operation being string concatenation. Let $M = (\mathbb{Z}, +, 0)$ be the additive
 121 group of integers. Define the action of S on M by the concatenation of strings
 122 followed by addition, i.e., $s \cdot m = sm$, for all $s \in S$ and $m \in \mathbb{Z}$. A primitive ideal
 123 of S is $\mathfrak{p} := \{s \in S \mid b \text{ does not appear in } s\}$.

124 A nonempty proper ideal \mathfrak{q} of a semigroup S is said to be *prime* if for any
 125 two ideals $\mathfrak{a}, \mathfrak{b}$ of S and $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{q}$ implies $\mathfrak{a} \subseteq \mathfrak{q}$ or $\mathfrak{b} \subseteq \mathfrak{q}$, where the product $\mathfrak{a}\mathfrak{b}$ of
 126 ideals \mathfrak{a} and \mathfrak{b} is defined to be the set of all finite sums $\sum i_\alpha j_\alpha$ (where $i_\alpha \in \mathfrak{a}$,
 127 $j_\alpha \in \mathfrak{b}$).

128 The proof of the following result is easy to verify.

129 **Lemma 7.** *If \mathfrak{a} and \mathfrak{b} are any two ideals of a semigroup, then $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b}$.*

130 The following proposition gives an alternative formulation of prime ideals of
 131 semigroups. For a proof, see [26, Lemma 2.2].

132 **Proposition 8.** Suppose S is a semigroup. Then the following conditions are
 133 equivalent:

- 134 1. \mathfrak{q} is a prime ideal of S .
- 135 2. $aSb \subseteq \mathfrak{q}$ implies $a \in \mathfrak{q}$ or $b \in \mathfrak{q}$ for all $a, b \in S$.

136 Primitive ideals and prime ideals of a semigroup are related as follows.

137 **Proposition 9.** Every primitive ideal of a semigroup is a prime ideal.

Proof. Suppose \mathfrak{p} is a primitive ideal and $\mathfrak{p} = \text{Ann}_S(M)$ for some simple S -
 module M . Let $a, b \notin \text{Ann}_S(M)$. Then $am \neq 0$ and $bm' \neq 0$ for some $m, m' \in M$.
 Since M is simple, there exists an $s \in S$ such that $s(bm') = m$. Then

$$(asb)m' = a(s(bm')) = am \neq 0,$$

138 and hence $asb \notin \text{Ann}_S(M)$. Therefore, $\text{Ann}_S(M)$ is a prime ideal by Lemma 8.

139 ■

140 In the next section we talk about Jacobson topology on the set of primitive
 141 ideals of a semigroup and discuss about some of the topological properties of the
 142 corresponding structure spaces.

143 3. JACOBSON TOPOLOGY

144 We shall introduce Jacobson topology in $\text{Prim}(S)$ by defining a closure operator
 145 for the subsets of $\text{Prim}(S)$. Once we have a closure operator, closed sets are
 146 defined as sets which are invariant under this closure operator¹. Suppose X is a
 147 subset of $\text{Ideal}(S)$. Set $\mathcal{D}_X = \bigcap_{\mathfrak{q} \in X} \mathfrak{q}$. We define the closure of the set X as

$$\mathcal{C}(X) = \{\mathfrak{p} \in \text{Prim}(S) \mid \mathfrak{p} \supseteq \mathcal{D}_X\}. \quad (2)$$

¹The origin of Kuratowski's closure operator on the set of primitive ideals of a ring can be traced back to [18].

148 If $X = \{x\}$, we will write $\mathcal{C}(\{x\})$ as $\mathcal{C}(x)$. We wish to verify that the closure
 149 operation defined in (2) satisfies Kuratowski's closure conditions and that is done
 150 in the following

151 **Proposition 10.** *The sets $\{\mathcal{C}(X)\}_{X \subseteq \text{Ideal}(S)}$ satisfy the following conditions:*

- 152 1. $\mathcal{C}(\emptyset) = \emptyset$,
- 153 2. $\mathcal{C}(X) \supseteq X$,
- 154 3. $\mathcal{C}(\mathcal{C}(X)) = \mathcal{C}(X)$,
- 155 4. $\mathcal{C}(X \cup Y) = \mathcal{C}(X) \cup \mathcal{C}(Y)$.

Proof. The proofs of (1)-(3) are straightforward, whereas for (4), it is easy to see
 that $\mathcal{C}(X \cup Y) \supseteq \mathcal{C}(X) \cup \mathcal{C}(Y)$. To obtain the the other inclusion, let $\mathfrak{p} \in \mathcal{C}(X \cup Y)$.
 Then

$$\mathfrak{p} \supseteq \mathcal{D}_{X \cup Y} = \mathcal{D}_X \cap \mathcal{D}_Y.$$

Since \mathcal{D}_X and \mathcal{D}_Y are ideals of S , by Lemma 7, it follows that

$$\mathcal{D}_X \mathcal{D}_Y \subseteq \mathcal{D}_X \cap \mathcal{D}_Y \subseteq \mathfrak{p}.$$

156 Since by Proposition 9, \mathfrak{p} is prime, either $\mathcal{D}_X \subseteq \mathfrak{p}$ or $\mathcal{D}_Y \subseteq \mathfrak{p}$. This means either
 157 $\mathfrak{p} \in \mathcal{C}(X)$ or $\mathfrak{p} \in \mathcal{C}(Y)$. Thus $\mathcal{C}(X \cup Y) \subseteq \mathcal{C}(X) \cup \mathcal{C}(Y)$. ■

158 The set $\text{Prim}(S)$ of primitive ideals of a semigroup S topologized (the Jacob-
 159 son topology) by the closure operator defined in (2) is called the *structure space* of
 160 the semigroup S . It is evident from (2) that if $\mathfrak{p} \neq \mathfrak{p}'$ for any two $\mathfrak{p}, \mathfrak{p}' \in \text{Prim}(S)$,
 161 then $\mathcal{C}(\mathfrak{p}) \neq \mathcal{C}(\mathfrak{p}')$. Thus

162 **Proposition 11.** *Every structure space $\text{Prim}(S)$ is a T_0 -space.*

163 **Theorem 12.** *If S is a semigroup with identity then the structure space $\text{Prim}(S)$
 164 is quasi-compact.*

165 **Proof.** Suppose that $\{K_\lambda\}_{\lambda \in \Lambda}$ is a family of closed sets of the structure space
 166 $\text{Prim}(S)$ such that $\bigcap_{\lambda \in \Lambda} K_\lambda = \emptyset$. This implies that the ideal $\bigvee_{\lambda \in \Lambda} \mathcal{D}_{K_\lambda}$ generated
 167 by $\{\mathcal{D}_{K_\lambda}\}_{\lambda \in \Lambda}$ must be equal to S . Indeed: $\bigvee_{\lambda \in \Lambda} \mathcal{D}_{K_\lambda} \neq S$ implies there exists a
 168 maximal ideal \mathfrak{m} in S such that $\mathcal{D}_{K_\lambda} \subseteq \mathfrak{m}$ for all $\lambda \in \Lambda$, whence $\mathfrak{m} \in \bigcap_{\lambda \in \Lambda} K_\lambda$, a
 169 contradiction. Therefore, in particular, $1 = x_1 \cdots x_n$, where $x_i \in \mathcal{D}_{K_{\lambda_i}}$ ($1 \leq i \leq$
 170 n). Hence, $\bigvee_{i=1}^n \mathcal{D}_{K_{\lambda_i}} = S$. This subsequently implies $\bigcap_{i=1}^n K_{\lambda_i} = \emptyset$. By finite
 171 intersection property, we then have the desired quasi-compactness. ■

172 Recall that a nonempty closed subset K of a topological space X is *irreducible*
 173 if $K \neq K_1 \cup K_2$ for any two proper closed subsets K_1, K_2 of K . A maximal
 174 irreducible subset of a topological space X is called an *irreducible component* of
 175 X . A point x in a closed subset K is called a *generic point* of K if $K = \mathcal{C}(x)$.

176 **Lemma 13.** *The irreducible closed subsets of a structure space $\text{Prim}(S)$ are of*
 177 *the form: $\{\mathcal{C}(\mathfrak{p})\}_{\mathfrak{p} \in \text{Prim}(S)}$.*

Proof. Since $\{\mathfrak{p}\}$ is irreducible, so is $\mathcal{C}(\mathfrak{p})$. Suppose $\mathcal{C}(\{\mathfrak{a}\})$ is an irreducible closed subset of $\text{Prim}(S)$ and $\mathfrak{a} \notin \text{Prim}(S)$. Here, by $\mathcal{C}(\{\mathfrak{a}\})$, we mean $\mathcal{C}(X)$ with $\mathcal{D}_X = \{\mathfrak{a}\}$. This implies there exist ideals \mathfrak{b} and \mathfrak{c} of S such that $\mathfrak{b} \not\subseteq \mathfrak{a}$ and $\mathfrak{c} \not\subseteq \mathfrak{a}$, but $\mathfrak{bc} \subseteq \mathfrak{a}$. Then

$$\mathcal{C}(\langle \mathfrak{a}, \mathfrak{b} \rangle) \cup \mathcal{C}(\langle \mathfrak{a}, \mathfrak{c} \rangle) = \mathcal{C}(\langle \mathfrak{a}, \mathfrak{bc} \rangle) = \mathcal{C}(\mathfrak{a}).$$

178 But $\mathcal{C}(\langle \mathfrak{a}, \mathfrak{b} \rangle) \neq \mathcal{C}(\mathfrak{a})$ and $\mathcal{C}(\langle \mathfrak{a}, \mathfrak{c} \rangle) \neq \mathcal{C}(\mathfrak{a})$, and hence $\mathcal{C}(\mathfrak{a})$ is not irreducible. ■

179 **Proposition 14.** Every irreducible closed subset of $\text{Prim}(S)$ has a unique generic
 180 point.

181 **Proof.** The existence of generic point follows from Lemma 13, and the uniqueness
 182 of such a point follows from Proposition 11. ■

183 In the following proposition, we will find examples of irreducible components
 184 of a structure space.

185 **Proposition 15.** If \mathfrak{p} is a minimal primitive ideal of S , then $\mathcal{C}(\mathfrak{p})$ is an irreducible
 186 component of a structure space $\text{Prim}(S)$. The converse also holds.

187 **Proof.** If $\mathcal{C}(\mathfrak{p})$ is not a maximal irreducible subset of $\text{Prim}(S)$, then there exists a
 188 maximal irreducible subset $\mathcal{C}(\mathfrak{p}')$ with $\mathfrak{p}' \in \text{Prim}(S)$ such that $\mathcal{C}(\mathfrak{p}) \subsetneq \mathcal{C}(\mathfrak{p}')$. This
 189 implies that $\mathfrak{p} \in \mathcal{C}(\mathfrak{p}')$ and hence $\mathfrak{p}' \subsetneq \mathfrak{p}$, contradicting the minimality property
 190 of \mathfrak{p} . To show the converse, let K be an irreducible component. By Lemma 13,
 191 $K = \mathcal{C}(\mathfrak{p})$ for some primitive ideal \mathfrak{p} . If \mathfrak{p} is not minimal, then there is a primitive
 192 ideal \mathfrak{q} properly contained in \mathfrak{p} . Then, $K \subsetneq \mathcal{C}(\mathfrak{q})$, contradicting the maximality
 193 of K . ■

194 While the next corollary provides a characterization of Hausdorff structure
 195 spaces of semigroups, the author, however, hasn't encountered any examples of
 196 semigroups where $\text{Prim}(S)$ is not Hausdorff.

197 **Corollary 16.** A structure space $\text{Prim}(S)$ is Hausdorff if and only if every prim-
 198 itive ideal of S is minimal.

199 Recall that a semigroup is called *Noetherian* if it satisfies the ascending chain
 200 condition on its ideals, whereas a topological space X is called *Noetherian* if the
 201 descending chain condition holds for closed subsets of X . A relation between these
 202 two notions is shown in the following

203 **Proposition 17.** If a semigroup S is Noetherian, then $\text{Prim}(S)$ is a Noetherian
 204 space.

205 **Proof.** It suffices to show that a collection of closed sets in $\text{Prim}(S)$ satisfies the
 206 descending chain condition. Let $\mathcal{C}(\mathfrak{a}_1) \supseteq \mathcal{C}(\mathfrak{a}_2) \supseteq \cdots$ be a descending chain of
 207 closed sets in $\text{Prim}(S)$. Once again, by $\mathcal{C}(\{\mathfrak{a}\})$, we mean $\mathcal{C}(X)$ with $\mathcal{D}_X = \{\mathfrak{a}\}$.
 208 Then, $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \cdots$ is an ascending chain of ideals in S . Since S is Noetherian,
 209 the chain stabilizes at some $n \in \mathbb{N}$. Hence, $\mathcal{C}(\mathfrak{a}_n) = \mathcal{C}(\mathfrak{a}_{n+k})$ for any k . Thus
 210 $\text{Prim}(S)$ is Noetherian. ■

211 **Corollary 18.** The set of minimal primitive ideals in a Noetherian semigroup is
 212 finite.

213 **Proof.** By Proposition 17, $\text{Prim}(S)$ is Noetherian, thus $\text{Prim}(S)$ has a finitely
 214 many irreducible components. By Proposition 15, every irreducible closed subset
 215 of $\text{Prim}(S)$ is of form $\mathcal{C}(\mathfrak{p})$, where \mathfrak{p} is a minimal primitive ideal. Thus $\mathcal{C}(\mathfrak{p})$ is
 216 irreducible components if and only if \mathfrak{p} is minimal primitive. Hence, S has only
 217 finitely many minimal primitive ideals. ■

218 **Proposition 19.** Suppose $\phi: S \rightarrow T$ is a semigroup homomorphism and define
 219 the map $\phi_*: \text{Prim}(T) \rightarrow \text{Prim}(S)$ by $\phi_*(\mathfrak{p}) = \phi^{-1}(\mathfrak{p})$, where $\mathfrak{p} \in \text{Prim}(T)$. Then
 220 ϕ_* is a continuous map.

221 **Proof.** To show ϕ_* is continuous, we first show that $f^{-1}(\mathfrak{p}) \in \text{Prim}(S)$, whenever
 222 $\mathfrak{p} \in \text{Prim}(T)$. Note that $\phi^{-1}(\mathfrak{p})$ is an ideal of S and a union of $\ker\phi$ -classes (see [11,
 223 Proposition 3.4]). Suppose $\mathfrak{p} = \text{Ann}_T(M)$ for some simple T -module. Then $\phi^{-1}(\mathfrak{p})$
 224 is the annihilator of the simple T -module M obtained by defining $sm := \phi(s)m$.
 225 Therefore $f^{-1}(\mathfrak{p}) \in \text{Prim}(S)$. Now consider a closed subset $\mathcal{C}(\mathfrak{a})$ of $\text{Prim}(S)$,
 226 where by $\mathcal{C}(\mathfrak{a})$, we mean $\mathcal{C}(X)$ with $\mathcal{D}_X = \{\mathfrak{a}\}$. Then for any $\mathfrak{q} \in \text{Prim}(T)$, we
 227 have:

$$\mathfrak{q} \in \phi_*^{-1}(\mathcal{C}(\mathfrak{a})) \Leftrightarrow \phi^{-1}(\mathfrak{q}) \in \mathcal{C}(\mathfrak{a}) \Leftrightarrow \mathfrak{a} \subseteq \phi^{-1}(\mathfrak{q}) \Leftrightarrow \mathfrak{q} \in \mathcal{C}(\langle \phi(\mathfrak{a}) \rangle),$$

228 and this proves the desired continuity of ϕ_* . ■

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