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FILTERS, IDEALS AND POWER OF DOUBLE BOOLEAN ALGEBRAS[1]

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Abstract

Double Boolean algebras (dBas) are algebras $\underline{D} = (D; \sqcap, \sqcup, \neg, \lrcorner, \bot, \top)$ of type $(2, 2, 1, 1, 0, 0)$, introduced by Rudolf Wille to capture the equational theory of the algebra of protoconcepts. Boolean algebras form a subclass of dBas. Our goal is an algebraic investigation of dBas, based on similar results on Boolean algebras. In this paper, we describe filters, ideals, homomorphisms and powers of dBas. We show that principal filters as well as

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principal ideals of dBas form (non necessary isomorphic) Boolean algebras. We also show that, a primary ideal (resp. primary filter) is exactly maximal ideal (resp. ultrafilter) in dBas and primary ideal (resp. filter) needs not be a prime ideal (resp. filter). For a finite dBa, a primary filters (resp. ideals) are principal filter (resp. ideals) generated by atom (resp. co-atom). Some properties of homomorphisms of dBas are investigated and the relationship between the homomorphism of dBas $\underline{D}$, $\underline{M}$ and the lattices of filters (resp. ideals) of these two dBas. Giving a dBa $\underline{D}$ and a non-emptyset X, we study some relationship between $\underline{D}$ and $\underline{L} = \underline{D}^X$ by showing that $\underline{D}$ is contextual, fully contextual (resp. trivial) if and only if $\underline{L}$ is contextual, fully contextual (resp. trivial). In addition, we show that $\underline{D}$ embeds into $\underline{L}$ and the lattice of filters $\mathcal{F}(\underline{D})$ (resp. of ideals $\mathcal{I}(\underline{D})$) is algebraic and embeds in the lattice $\mathcal{F}(\underline{L})$ (resp. $\mathcal{I}(\underline{L})$). We finish this paper by showing that some sets of polynomial functions of $\underline{D}$ form a Boolean algebra isomorphic to the set of principal filters (resp. principal ideals) of $\underline{D}$.

Keywords: double Boolean algebra, protoconcepts algebra, concept algebra, formal concept.

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1. INTRODUCTION

In order to extend Formal Concept Analysis (FCA) to Contextual Logic, a negation has to be formalized [10]. There are many options: We can require the negation of a concept to be concept [6, 10] or we want to preserve the correspondence between negation and set complementation. In the second option the notion of concept has to be generalized, and leads to the algebra of semi-concepts, protoconcepts and preconcepts [10]. To capture their equational theory, double Boolean algebras have been introduced by Rudolf Wille and coworkers. Wille proved that each double Boolean algebra "quasi-embeds" into an algebra of protoconcepts. Thus the equations defining a double Boolean algebra generate the equational theory of the algebra of protoconcepts [10] (Corollary 1).

To the best of our knowledge, the investigation of dBas has been so far concentrated on representation problem such as equational theory [10], contextual representation [9], and most recently topological representation [1, 5]. Of course the prime ideal theorem [7] plays a central role in such representation. For a better understanding of the structure of dBas, our goal is to start with purely algebraic notions such as filters, ideals, homomorphisms, powers,... of dBas. In particular, the study of the powers of dBas will help us better understand these algebras. They are the cornerstone in structure theory, representation, decomposition, construction as well as classification of algebraic structures.

In Boolean algebras, distributive lattices, and weakly dicomplemented lattices [2, 6] there are notions of ultrafilters, prime filters and primary filters which are closely related to the concepts of atoms, irreducible, prime and primary elements. We study some of these concepts on dBas. It is known (see [2]) that in a Boolean algebra, each prime filter is an ultrafilter and each prime ideal is a maximal ideal.

This work is an extended version of [8], and is organized as follows. In Section 2 we recall some basic notions and introduce protoconcept algebra which is a rich source of examples for dBas. Section 3 summarizes some results on filters and ideals of dBas: we describe filters (resp. ideals) generated by an arbitrary subset of dBas, and show that the set of principal filters (resp. principal ideals) of a dBa forms a bounded sublattice of the lattice of its filters (resp. ideals), and are (non necessary isomorphic) Boolean algebras. We show that primary filters (resp. primary ideals) are exactly ultrafilters (resp. maximal ideals) and primary filters (resp. primary ideals) need not be prime filters (resp. prime ideals) in dBa.

In Section 4, some properties of homomorphisms of dBas are investigated and the correspondence between the set of homomorphism of dBas $\underline{D}$, $\underline{M}$ and the lattices of filters (resp. ideals) of these two dBas.

We end this work with the study of the power of a dBa in Section 5. For a set X and a dBa $\underline{D}$, $\underline{L} := \underline{D^X}$ is a dBa as product of dBas. We give some properties of $\underline{L}$ following those of $\underline{D}$, we show that $\underline{D}$ is contextual, fully contextual (resp. trivial) if and only if $\underline{L}$ is contextual, fully contextual (resp. trivial) and if $\underline{D}$ is complete, then $\underline{L}$ is complete. In particular, we show that the lattice $\mathcal{F}(\underline{D})$ of filters (resp. $\mathcal{I}(\underline{D})$ of ideals) is algebraic and embeds in the lattice $\mathcal{F}(\underline{L})$ (resp. $\mathcal{I}(\underline{L})$). We end this section by showing that some sets of polynomial functions of $\underline{D}$ form a Boolean algebras isomorphic to $\underline{D_\sqcap}$ (resp. $\underline{D_\sqcup}$).

2. Concepts, protoconcepts and double Boolean algebras

In this section, we provide the reader with some basic notions and notations. For more details we refer to [3, 10]. A **formal context** is a triple $\mathbb{K} := (G, M, I)$ where G is a set of objects, M a set of attributes and $I \subseteq G \times M$, a binary relation to describe if an object of G has an attribute in M. We write gIm for $(g, m) \in I$. To extract clusters, the following derivation operators are defined on subsets $A \subseteq G$ and $B \subseteq M$ by

$$A' := \{m \in M \mid gIm \text{ for all } g \in A\} \quad \text{and} \quad B' := \{g \in G \mid \ gIm \text{ for all } m \in B\}.$$

The maps $A \mapsto A'$ and $B \mapsto B'$ form a Galois connection between the power set of G and that of M. The composition $''$ is a closure operator.

A **formal concept** is a pair (A, B) with $A' = B$ and $B' = A$. We call A the **extent** and B the **intent** of the formal concept (A, B). They are closed subsets with respect to $''$ (i.e., $X'' = X$). The set $\mathfrak{B}(\mathbb{K})$ of all formal concepts of the formal context $\mathbb{K}$ can be ordered by

$$(A_1, B_1) \leq (A_2, B_2) :\Longleftrightarrow A_1 \subseteq A_2 \quad \text{(or equivalently, } B_2 \subseteq B_1\text{)}.$$

The poset $\underline{\mathfrak{B}}(\mathbb{K}) := (\mathfrak{B}(\mathbb{K}), \leq)$ is a complete lattice, called the **concept lattice** of the context $\mathbb{K}$. Conversely each complete lattice is isomorphic to a concept lattice. This basic theorem on concept lattice ([3], Theorem 3) is a template for contextual representation problems. The lattice operations $\wedge$ (meet) and $\vee$ (join) can be interpreted as a logical conjunction and a logical disjunction for concepts, and are given by

$$\begin{aligned} \text{meet:}\quad & (A_1, B_1) \wedge (A_2, B_2) = \left(A_1 \cap A_2, (A_1 \cap A_2)'\right), \\ \text{join:}\quad & (A_1, B_1) \vee (A_2, B_2) = \left((B_1 \cap B_2)', B_1 \cap B_2\right). \end{aligned}$$

To extend FCA to contextual logic, we need to define the negation of a concept. Unfortunately, the complement of a closed subset is not always closed. To preserve the correspondence between set complementation and negation, the notion of concept is extended to that of protoconcept.

The pair (A, B) is called a **semi-concept** if $A' = B$ or $B' = A$, and a **protoconcept** if $A'' = B'$.

The set of all semi-concepts of $\mathbb{K}$ is denoted by $\mathfrak{h}(\mathbb{K})$, and that of all protoconcepts by $\mathfrak{P}(\mathbb{K})$. Note that $\mathfrak{h}(\mathbb{K}) \subseteq \mathfrak{P}(\mathbb{K})$. Meet and join of protoconcepts are then defined, similar as above for concepts. A negation (resp. opposition) is defined by taking the complement on objects (resp. attributes). More precisely, for protoconcepts (A_1, B_1), (A_2, B_2), (A, B) of $\mathbb{K}$ we define the operations:

$$\begin{aligned} \text{meet:}\quad & (A_1, B_1) \sqcap (A_2, B_2) := (A_1 \cap A_2, (A_1 \cap A_2)') \\ \text{join:}\quad & (A_1, B_1) \sqcup (A_2, B_2) := ((B_1 \cap B_2)', B_1 \cap B_2) \\ \text{negation:}\quad & \neg(A, B) := (G \setminus A, (G \setminus A)') \\ \text{opposition:}\quad & \lrcorner(A, B) := ((M \setminus B)', M \setminus B) \\ \text{nothing:}\quad & \bot := (\emptyset, M) \\ \text{all:}\quad & \top := (G, \emptyset). \end{aligned}$$

The algebra $\underline{\mathfrak{P}(\mathbb{K})} := (\mathfrak{P}(\mathbb{K}); \sqcap, \sqcup, \neg, \lrcorner, \bot, \top)$ is called the **algebra of protoconcepts** of $\mathbb{K}$. Note that applying any operation above on protoconcepts gives a semi-concept as result. Therefore $\underline{\mathfrak{H}(\mathbb{K})}$ is a sub-algebra of $\underline{\mathfrak{P}(\mathbb{K})}$. For the structural analysis of $\underline{\mathfrak{P}(\mathbb{K})}$, we split $\underline{\mathfrak{H}(\mathbb{K})}$ in $\sqcap$-**semi concepts** and $\sqcup$-**semi concepts**, $\mathfrak{P}(\mathbb{K})_\sqcap := \{(A, A') \mid A \subseteq G\}$ and $\mathfrak{P}(K)_\sqcup := \{(B', B) \mid B \subseteq M\}$, and set $x \vee y := \neg(\neg x \sqcap \neg y)$ and $x \wedge y = \lrcorner(\lrcorner x \sqcup \lrcorner y)$ for $x, y \in \mathfrak{P}(\mathbb{K})$.

$\underline{\mathfrak{P}(\mathbb{K})_\sqcap} := (\mathfrak{P}(\mathbb{K})_\sqcap; \sqcap, \vee, \neg, \bot, \neg\bot)$ (resp. $\underline{\mathfrak{P}(\mathbb{K})_\sqcup} := (\mathfrak{P}(\mathbb{K})_\sqcup; \wedge, \sqcup, \lrcorner, \lrcorner\top, \top)$)) is a Boolean algebra isomorphic (resp. anti-isomorphic) to the powerset algebra of G (resp. M) (see [10]).

Theorem 2.1 [10]**.** *The following equations hold in the algebra of protoconcepts* $\underline{\mathfrak{P}}(\mathbb{K})$*:*

(1a) $(x \sqcap x) \sqcap y = x \sqcap y$	(1b) $(x \sqcup x) \sqcup y = x \sqcup y$
(2a) $x \sqcap y = y \sqcap x$	(2b) $x \sqcup y = y \sqcup x$
(3a) $x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$	(3b) $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$
(4a) $\neg(x \sqcap x) = \neg x$	(4b) $\lrcorner(x \sqcup x) = \lrcorner x$
(5a) $x \sqcap (x \sqcup y) = x \sqcap x$	(5b) $x \sqcup (x \sqcap y) = x \sqcup x$
(6a) $x \sqcap (y \vee z) = (x \sqcap y) \vee (x \sqcap z)$	(6b) $x \sqcup (y \wedge z) = (x \sqcup y) \wedge (x \sqcup z)$
(7a) $x \sqcap (x \vee y) = x \sqcap x$	(7b) $x \sqcup (x \wedge y) = x \sqcup x.$
(8a) $\neg\neg(x \sqcap y) = x \sqcap y$	(8b) $\lrcorner\lrcorner(x \sqcup y) = x \sqcup y$
(9a) $x \sqcap \neg x = \bot$	(9b) $x \sqcup \lrcorner x = \top$
(10a) $\neg\bot = \top \sqcap \top$	(10b) $\lrcorner\top = \bot \sqcup \bot$
(11a) $\neg\top = \bot$	(11b) $\lrcorner\bot = \top$

$$(12)\ (x \sqcap x) \sqcup (x \sqcap x) = (x \sqcup x) \sqcap (x \sqcup x).$$

A **double Boolean algebra** (dBa) is an algebra $\underline{D} := (D; \sqcap, \sqcup, \neg, \lrcorner, \bot, \top)$ of type $(2, 2, 1, 1, 0, 0)$ that satisfies the equations in Theorem 2.1. Rudolf Wille showed that these identities generate the equational theory of protoconcept algebras [10]. In each dBa $\underline{D}$, a quasi-order $\sqsubseteq$ is defined by $x \sqsubseteq y :\iff x \sqcap y = x \sqcap x$ and $x \sqcup y = y \sqcup y$. It satisfies $x \sqsubseteq y$ iff $x \sqcap x \sqsubseteq y \sqcap y$ and $x \sqcup x \sqsubseteq y \sqcup y$, for all $x, y \in D$ [10]. We set $D_\sqcap := \{x \in D \mid x \sqcap x = x\}$ and $D_\sqcup := \{x \in D \mid x \sqcup x = x\}$. The algebra $\underline{D_\sqcap} := (D_\sqcap; \sqcap, \vee, \neg, \bot, \neg\bot)$ (resp. $\underline{D_\sqcup} = (D_\sqcup; \wedge, \sqcup, \lrcorner, \lrcorner\top, \top)$) is a Boolean algebras.

Definition 2.1 [7, 9, 10]**.** Let $\underline{D}$ be a double Boolean algebra. Then $\underline{D}$ is

- **contextual** if the quasi-order $\sqsubseteq$ is an order relation on $\underline{D}$.
- **fully contextual** if it is contextual and for all $x \in D_\sqcap$ and $y \in D_\sqcup$ such that $x \sqcup x = y \sqcap y$, there is a unique $z \in D$ such that $z \sqcap z = x$ and $z \sqcup z = y$.
- **complete** if its Boolean algebras $\underline{D_\sqcap}$ and $\underline{D_\sqcup}$ are complete.
- **trivial** if $\top \sqcap \top = \bot \sqcup \bot$.

- **pure** if for all $x \in D$, $x \sqcap x = x$ or $x \sqcup x = x$.

The next proposition collects some properties of dBas, that we will need later.

Proposition 2.2 [4, 5, 7, 10]**.** Let $\underline{D}$ be a double Boolean algebra and $x, y, a \in D$. Then

(1) $\bot \sqsubseteq x$ and $x \sqsubseteq \top$.

(2) $x \sqcap y \sqsubseteq x, y \sqsubseteq x \sqcup y$.

(3) $x \sqsubseteq y \implies \begin{cases} x \sqcap a \sqsubseteq y \sqcap a \\ x \sqcup a \sqsubseteq y \sqcup a. \end{cases}$

(4) $\neg(x \vee y) = \neg x \sqcap \neg y$.

(5) $\neg(x \sqcap y) = \neg x \vee \neg y$.

(6) $x \sqsubseteq \lrcorner y \iff y \sqsubseteq \lrcorner x$.

(7) $\neg\neg x = x \sqcap x$ and $\lrcorner\lrcorner x = x \sqcup x$.

(8) $\neg x, x \vee y \in D_{\sqcap}$ and $\lrcorner x, x \wedge y \in D_{\sqcup}$.

(9) $x \sqsubseteq y \iff \neg y \sqsubseteq \neg x$ and $\lrcorner y \sqsubseteq \lrcorner x$.

(10) $\lrcorner(x \wedge y) = \lrcorner x \sqcup \lrcorner y$.

(11) $\lrcorner(x \sqcup y) = \lrcorner x \wedge \lrcorner y$.

(12) $\neg x \sqsubseteq y \iff \neg y \sqsubseteq x$.

(13) $x \sqcap y \sqsubseteq x \wedge y, x \vee y \sqsubseteq x \sqcup y$.

(14) $x \sqsubseteq y$ and $y \sqsubseteq x$ if and only if $x \sqcap x = y \sqcap y$ and $x \sqcup x = y \sqcup y$.

We close this section with some distributivity-like properties of dBas.

Proposition 2.3. Let $\underline{D}$ be a dBa and $a, b, c, d \in D$, we have

(i) $a \vee (b \sqcap c) = (a \vee b) \sqcap (a \vee c)$. (i)' $a \wedge (b \sqcup c) = (a \wedge b) \sqcup (a \wedge c)$.

(ii) $a \vee (a \sqcap b) = a \sqcap a$. (ii)' $a \wedge (a \sqcup b) = a \sqcup a$.

(iii) $(a \sqcap a) \vee (b \sqcap b) = a \vee b$. (iii)' $(a \sqcup a) \wedge (b \sqcup b) = a \wedge b$.

Proof. (i)', (ii)' and (iii)' are dual of (i), (ii) and (iii). Let $a, b, c, d \in D$.

(i) $a \vee (b \sqcap c) \overset{\text{def}}{=} \neg(\neg a \sqcap \neg(b \sqcap c)) = \neg(\neg a \sqcap (\neg b \vee \neg c))$ (by (5) of Proposition 2.2)
$= \neg((\neg a \sqcap \neg b) \vee (\neg a \sqcap \neg c))$ (by axiom (6a))
$= \neg(\neg(a \vee b) \vee \neg(a \vee c))$ (by (5) of proposition 2.2)
$= \neg[\neg[(a \vee b) \sqcap (a \vee c)]]$ (by (5) of Proposition 2.2)
$= \neg\neg[(a \vee b) \sqcap (a \vee c)] = (a \vee b) \sqcap (a \vee c)$ (by axiom (8a))

(ii) $a \vee (a \sqcap b) = \neg(\neg a \sqcap \neg(a \sqcap b)) = \neg(\neg a \sqcap (\neg a \vee \neg b))$ (by (5) of Proposition 2.2)
$= \neg(\neg a \sqcap \neg a)$ (by axiom (7a))
$= a \sqcap a$ (by (8) and (7) of Proposition 2.2)

(iii) $(a \sqcap a) \vee (b \sqcap b) = \neg(\neg(a \sqcap a) \sqcap \neg(b \sqcap b)) = \neg(\neg a \sqcap \neg b)$ (by axiom (4a)) $= a \vee b$. ■

3. Filters and ideals of a double Boolean algebra

The goal is to extent some results from Boolean algebras to dBas. Let $\underline{D}$ be a dBa. A nonempty subset F of D is called **filter** if for all $x, y \in D$, it holds:

$$x, y \in F \implies x \sqcap y \in F \text{ and } (x \in F,\, x \sqsubseteq y) \implies y \in F.$$

Ideal in dBa is defined dually. We denote by $F(\underline{D})$ (resp. $I(\underline{D})$) the set of filters (resp. ideals) of the dBa $\underline{D}$. Both sets are each closed under intersection [7]. Note that $F(\underline{D}) \cap I(\underline{D}) = \{D\}$. For $X \subseteq D$, the smallest filter (resp. ideal) containing X, denoted by $\text{Filter}\langle X\rangle$ (resp. $\text{Ideal}\langle X\rangle$), is the intersection of all filters (resp. ideals) containing X, and is called the **filter** (resp. **ideal**) **generated by** X. A **principal filter** (resp. **ideal**) is a filter (resp. ideal) generated by a singleton. In that case we omit the curly brackets and set $F(x) := \text{Filter}\langle\{x\}\rangle$, and $I(x) := \text{Ideal}\langle\{x\}\rangle$. Let F be a dBa filter. We call F **proper filter** if $F \neq D$, and **ultrafilter** if F is proper and is not contained in any other proper filter. We call F a **prime filter** if for all $a, b \in D$, we have $a \sqcup b \in F$ implies $a \in F$ or $b \in F$, dually **prime ideal** is defined. A set F_0 is a **base of a filter** F if $F_0 \subseteq F$ and $F = \{y \in D \mid x \sqsubseteq y \text{ for some } x \in F_0\}$. **Base of ideals** are defined dually.

Observation (Reviewer remark on base of filter)**.** *If F_0 is a base of a filter F, then $F = Filter\langle F_0\rangle$ and $F_0 = F$, if F_0 is a filter. Therefore a base of a filter F in general is supposed to be a proper subset of F that is not a filter.*

Lemma 3.1 [10]**.** *Let F be a filter and I be an ideal of a dBa $\underline{D}$.*

(1) *$F \cap D_\sqcap$ (resp. $F \cap D_\sqcup$) is a filter of the Boolean algebra $\underline{D_\sqcap}$ (resp. $\underline{D_\sqcup}$).*
(2) *$I \cap D_\sqcup$ (resp. $I \cap D_\sqcap$) is an ideal of $\underline{D_\sqcup}$ (resp. $\underline{D_\sqcap}$).*
(3) *Each filter of $\underline{D_\sqcap}$ is a base of some filter of $\underline{D}$.*
(4) *Each ideal of $\underline{D_\sqcup}$ is a base of some ideal of $\underline{D}$.*

To prove the prime ideal theorem for dBas, a description of filter (resp. ideal) generated by an element w and a filter F (resp. ideal J) was given in [7] as follows:

$$Filter\langle F \cup \{w\}\rangle = \{x \in D \mid w \sqcap b \sqsubseteq x \text{ for some } b \in F\},$$

$$Ideal\langle J \cup \{w\}\rangle) = \{x \in D \mid x \sqsubseteq w \sqcup b \text{ for some } b \in J\}.$$

To extend this description to arbitrary subsets, the following lemma is formulated.

Lemma 3.2. *Let $\underline{D}$ be a dBa and $a \in D$. Then*

(1) *The operations $\sqcap$ and $\sqcup$ are compatible with the quasi-order $\sqsubseteq$, i.e., for $a, b, c, d \in D$, if $a \sqsubseteq b$ and $c \sqsubseteq d$, then $a \sqcap c \sqsubseteq b \sqcap d$ and $a \sqcup c \sqsubseteq b \sqcup d$.*

(2) *The binary operations* $\vee$ *and* $\wedge$ *are compatible with* $\sqsubseteq$, *that is, if* $a \sqsubseteq b$ *and* $c \sqsubseteq d$, *then* $a \vee c \sqsubseteq b \vee d$ (2.1) *and* $a \wedge c \sqsubseteq b \wedge d$ (2.2) .

(3) $I(a) = \{x \in D \mid x \sqsubseteq a \sqcup a\}$ *and* $F(a) = \{x \in D \mid a \sqcap a \sqsubseteq x\}$.

(4) $\mathrm{Ideal}\langle\emptyset\rangle = I(\bot) = \{x \in D \mid x \sqsubseteq \bot \sqcup \bot\}$ *and* $\mathrm{Filter}\langle\emptyset\rangle = F(\top) = \{x \in D \mid \top \sqcap \top \sqsubseteq x\}$.

Proof. (1) It is a direct consequence of Proposition 2.2 (item 4).

(2) Since $\neg$ and $\lrcorner$ reverse the quasi-order $\sqsubseteq$ (Proposition 2.2, item 9) it follows from (1) that $\vee$ and $\wedge$ are also compatibe with $\sqsubseteq$.

(3) It is easy to show that the set $J := \{x \in D \mid x \sqsubseteq a \sqcup a\}$ is an ideal containing a by using (1) and Theorem 2.1(1b). If G is an ideal and $a \in G$, then $a \sqcup a \in G$. Each $x \in J$ satisfies $x \sqsubseteq a \sqcup a$, and is also in G. Thus $J = I(a)$. The description of $F(a)$ is obtained dually.

(4) From $\bot \sqsubseteq x$ for all $x \in D$ (Proposition 2.2(1)) and the fact that each ideal is non empty, it follows that $\bot$ is in each ideal. Thus $\mathrm{Ideal}\langle\emptyset\rangle = I(\bot) = \{x \in D \mid x \sqsubseteq \bot \sqcup \bot\}$. The last equality follows from (3). The equality $\mathrm{Filter}\langle\emptyset\rangle = F(\top) = \{x \in D \mid \top \sqcap \top \sqsubseteq x\}$ is proved similarly. ■

For $F \in F(\underline{D})$, $I \in I(\underline{D})$ and $a \in D$, we have $a \in F$ iff $a \sqcap a \in F$ and $a \in I$ iff $a \sqcup a \in I$. Therefore $F(a) = F(a \sqcap a)$ and $I(a) = I(a \sqcup a)$.

The following proposition describes filters and ideals generated by arbitrary non-empty subsets.

Proposition 3.1. Let $\underline{D}$ be a dBa, $\emptyset \neq X \subseteq D$, $F_1, F_2 \in F(\underline{D})$ and $I_1, I_2 \in I(\underline{D})$. Then

(a) $\mathrm{Ideal}\langle X\rangle = \{x \in D \mid x \sqsubseteq b_1 \sqcup \cdots \sqcup b_n \text{ for some } b_1, \ldots, b_n \in X, n \geq 1\}$.

(b) $\mathrm{Filter}\langle X\rangle = \{x \in D \mid x \sqsupseteq b_1 \sqcap \cdots \sqcap b_n \text{ for some } b_1, \ldots, b_n \in X, n \geq 1\}$.

(c) $\mathrm{Ideal}\langle I_1 \cup I_2\rangle = \{x \in D \mid x \sqsubseteq i_1 \sqcup i_2 \text{ for some } i_1 \in I_1 \text{ and } i_2 \in I_2\}$.

(d) $\mathrm{Filter}\langle F_1 \cup F_2\rangle = \{x \in D \mid f_1 \sqcap f_2 \sqsubseteq x \text{ for some } f_1 \in F_1 \text{ and } f_2 \in F_2\}$.

Proof. Let $\underline{D}$ be a dBa and $\emptyset \neq X \subseteq D$.

(a) We show that $J := \{x \in D \mid x \sqsubseteq b_1 \sqcup \cdots \sqcup b_n \text{ for some } b_1, \ldots, b_n \in X,\ n \geq 1\}$ is the smallest ideal that contains X. For $x \in X$, we have $x \sqsubseteq x \sqcup x$, and $x \in J$. If $y \in D$, $x \in J$ and $y \sqsubseteq x$, then $y \sqsubseteq x \sqsubseteq b_1 \sqcup \cdots \sqcup b_n$ for $n \geq 1$ and $b_1, \ldots, b_n \in X$. By transitivity of $\sqsubseteq$, we get $y \in J$. Now, let $a, b \in J$, then $a \sqsubseteq a_1 \sqcup \cdots \sqcup a_n$ and $b \sqsubseteq b_1 \sqcup \cdots \sqcup b_m$ for some $n, m \geq 1$ and $a_i, b_i \in X$, $1 \leq i \leq n, 1 \leq j \leq m$. Therefore $a \sqcup b \sqsubseteq a_1 \sqcup \cdots \sqcup a_n \sqcup b_1 \sqcup \cdots \sqcup b_m$ (by (1) of Lemma 3.2). Thus $a \sqcup b \in J$. It is easy to see that any ideal containing X also contains J.

(c) follows from (a), with the facts that $\sqcup$ is commutative, associative, and ideals are closed under $\sqcup$. (b) and (d) are dual to (a) and (c). ■

Note that $F(\bot) = D$ and $I(\top) = D$.

Remark 3.2. $F(\underline{D})$ and $I(\underline{D})$ are closure systems, and form two complete lattices

$$\mathcal{F}(\underline{D}) = (F(D); \wedge, \vee, F(\top), D) \text{ and } \mathcal{I}(\underline{D}) := (I(\underline{D}); \wedge, \vee, I(\bot), D).$$

For $\{F_\lambda : \lambda \in \Lambda\}$, on $\underline{D}$, we have $\bigwedge_{\lambda\in\Lambda} F_\lambda = \bigcap_{\lambda\in\Lambda} F_\lambda$ and $\bigvee_{\lambda\in\Lambda} F_\lambda = Filter\langle \bigcup_{\lambda\in\Lambda} F_\lambda\rangle$.

Let $F_p := \{F(a) \mid a \in D\} = \{F(a \sqcap a) \mid a \in D\} = \{F(a) \mid a \in D_\sqcap\}$ be the set of principal filters and $I_p := \{I(a) \mid a \in D\} = \{I(a) \mid a \in D_\sqcup\}$ be the set of principal ideals of $\underline{D}$.

It is known that, if $\underline{L}$ is a lattice, then the set $I(\underline{L})$ (resp. $F(\underline{L})$) of ideals (resp. filters) of $\underline{L}$ is a lattice and the set of principal ideals (resp. filters) of $\underline{L}$ forms a sublattice of $\mathcal{I}(\underline{L})$ (resp. $\mathcal{F}(\underline{L})$) isomorphic to $\underline{L}$. It is therefore natural to ask what happens in the case of double Boolean algebras.

Lemma 3.3. *Let $\underline{D}$ be a dBa and a, b in D. Then*

(i) $a \sqsubseteq b \implies F(a) \supseteq F(b)$. *The converse holds if $a, b \in D_\sqcap$.*

(ii) $a \sqsubseteq b \implies I(a) \subseteq I(b)$. *The converse holds if $a, b \in D_\sqcup$.*

(iii) $F(a \sqcup b) \subseteq F(a) \cap F(b) \subseteq F(a), F(b) \subseteq F(a \sqcap b)$.

(iv) $I(a \sqcap b) \subseteq I(a) \cap I(b) \subseteq I(a), I(b) \subseteq I(a \sqcup b)$.

Proof. (ii) and (iv) are dual of (i) and (iii), respectively. Let $a, b \in D$.

(i) We assume that $a \sqsubseteq b$. Then $a \sqcap a \sqsubseteq b \sqcap b \sqsubseteq b$. If $x \in F(b)$ then $b \sqcap b \sqsubseteq x$. Since $\sqsubseteq$ is transitive, we get $a \sqcap a \sqsubseteq x$, and $x \in F(a)$. Thus $F(b) \subseteq F(a)$. Conversely, if $F(b) \subseteq F(a)$ then $b \in F(a)$ and $a \sqcap a \sqsubseteq b$, which is equivalent to $a \sqsubseteq b$ if $a \in D_\sqcap$.

(iv) $a \sqcap b \sqsubseteq a, b \sqsubseteq a \sqcup b \implies F(a \sqcup b) \subseteq F(a) \cap F(b) \subseteq F(a), F(b) \subseteq F(a \sqcap b)$, by (i). ∎

The next result shows that the principal filters form a bounded sublattice of the lattice of all filters, that is a Boolean algebra.

Proposition 3.3. *Let $\underline{D}$ be a dBa. For any $a, b, c \in D$, the following hold.*

(1) $F(a) \vee F(b) = F(a \sqcap b) = Filter\langle\{a, b\}\rangle$.

(2) $I(a) \vee I(b) = I(a \sqcup b) = Ideal\langle\{a, b\}\rangle$.

(3) If $b \in F(\neg a)$, then $\neg b \in I(a)$.

(4) If $b \in I(\lrcorner a)$, then $\lrcorner b \in F(a)$.

(5) $F(a) \wedge (F(b) \vee F(c)) = (F(a) \wedge F(b)) \vee (F(a) \wedge F(c))$.

(6) $I(a) \vee (I(b) \wedge I(c)) = (I(a) \vee I(b)) \wedge (I(a) \vee I(c))$.

(7) $F(a) \cap F(b) = F(a \vee b)$.

(8) $I(a) \cap I(b) = I(a \wedge b)$.

(9) $F(a) \wedge F(\neg a) = F(\top)$.

(10) $I(a) \vee I(\lrcorner a) = I(\top)$.

(11) $F(a) \vee F(\neg a) = F(\bot)$.

(12) $I(a) \wedge I(\lrcorner a) = I(\bot)$.

Proof. The even numbered items are dual of the odd ones. Let $a, b, c, x \in D$.

(1) From $a \sqcap b \sqsubseteq a, b$ we get $F(a), F(b) \subseteq F(a \sqcap b)$ and $F(a) \vee F(b) \subseteq F(a \sqcap b)$ (Lemma 3.3(i)). If $x \in F(a \sqcap b)$, then $a \sqcap b \sqsubseteq x$. But $a \in F(a)$ and $b \in F(b)$ imply $a \sqcap b \in F(a) \vee F(b)$, and yields $x \in F(a) \vee F(b)$. Thus $F(a \sqcap b) = F(a) \vee F(b)$.

(3) $b \in F(\neg a) \iff \neg a \sqcap \neg a \sqsubseteq b \iff \neg a \sqsubseteq b$ (by (8) of Proposition 2.2)
$\iff \neg b \sqsubseteq a$ (by (12) of Proposition 2.2) $\implies \neg b \in I(a)$.

(7) $x \in F(a \vee b) \iff (a \vee b) \sqcap (a \vee b) \sqsubseteq x \iff a \vee b \sqsubseteq x$ (due to $a \vee b \in D_{\sqcap}$)
$\iff (a \sqcap a) \vee (b \sqcap b) \sqsubseteq x$ (by (iii) of Proposition 2.3)
$\iff a \sqcap a \sqsubseteq x$ and $b \sqcap b \sqsubseteq x$ (since $a \sqcap a, b \sqcap b \sqsubseteq a \vee b$)
$\iff x \in F(a)$ and $x \in F(b) \iff x \in F(a) \cap F(b)$.

(5) $F(a) \vee (F(b) \wedge F(c)) = F(a) \vee (F(b \vee c))$ (by (7)) $= F(a \sqcap (b \vee c))$ (by (1))
$= F((a \sqcap b) \vee (a \sqcap c))$ (axiom (6a))
$= F(a \sqcap b) \wedge F(a \sqcap c)$ (by (7))
$= (F(a) \vee F(b)) \wedge (F(a) \vee F(c))$ (by (1)).

(9) $F(a) \wedge F(\neg a) = F(a \vee \neg a) = F(\neg(\neg a \sqcap \neg\neg a) = F(\neg\bot) = F(\top \sqcap \top) = F(\top)$.

(11) $F(a) \vee F(\neg a) = F(a \sqcap \neg a) = F(\bot)$. ■

Using (9) and (11) (resp. (10) and (12)) of Proposition 3.3 we observe that any principal filter $F(a)$ (resp. principal ideal $I(a)$) of $\underline{D}$ has a complement, namely the principal filter $F(a)^c =: F(\neg a)$ (resp. ideal $I(a)^c := I(\lrcorner a)$). From (5) and (6) we get the distributivity. Thus; the algebras

$$\underline{\mathcal{F}_p}(\underline{D}) := (F_p(\underline{D}); \wedge, \vee, ^c, F(\top), F(\bot)) \text{ and } \underline{\mathcal{I}_p}(\underline{D}) := (I_p(\underline{D}); \wedge, \vee, ^c, I(\bot), I(\top))),$$

are Boolean algebras. The following holds.

Theorem 3.4. *Let $\underline{D}$ be a double Boolean algebra. The set of principal filters (resp. ideals) of $\underline{D}$ forms a Boolean algebra dual-isomorphic (resp. isomorphic) to $\underline{D_\sqcap}$ (resp. $\underline{D_\sqcup}$).*

Proof. Let $\underline{D}$ be a dBa define $\varphi : D_\sqcap \to F_p(\underline{D}), a \mapsto F(a)$.

- For $a, b \in D_\sqcap$, we have
  $\varphi(a \sqcap b) = F(a \sqcap b) = F(a) \vee F(b) = \varphi(a) \vee \varphi(b)$,
  $\varphi(a \vee b) = F(a \vee b) = F(a) \wedge F(b) = \varphi(a) \wedge \varphi(b)$ and
  $\varphi(\neg a) = F(\neg a) = F(a)^c = \varphi(a)^c$.

  We deduce that φ is a dual homomorphism of Boolean algebras.
- φ is onto, since any principal filter $F(a)$ is equal to $F(a \sqcap a)$, with $a \sqcap a \in D_\sqcap$.
- It remains to show that ϕ is one-to-one. Let $a, b \in D_\sqcap$ with $\varphi(a) = \varphi(b)$. Then $F(a) = F(b) = F(a) \vee F(b) = F(a \sqcap b)$. Henceforth, $a \sqcap b \in F(a) = F(b)$, that is $a \sqcap a \sqsubseteq a \sqcap b \sqsubseteq a$ and $b \sqcap b \sqsubseteq a \sqcap b \sqsubseteq b$. Since $a, b \in D_\sqcap$, we deduce that $a = a \sqcap b = b$. Thus $\mathcal{F}_p(\underline{D})$ is anti-isomorphic to $\underline{D_\sqcap}$.

The proof for the set of principal ideals is similar to that principal of filters, using the mapping $\psi : D_\sqcup \to I_p(\underline{D}), a \mapsto I(a)$. ■

Corollary 3.4. *If $\underline{D}$ is a complete dBa, then the lattices $\mathcal{F}(\underline{D})$ of filters and $\mathcal{I}(\underline{D})$ of ideals of $\underline{D}$ are Boolean algebras.*

Proof. For $\emptyset \neq X \subseteq D$, set $\sqcap X := \underset{a \in X}{\sqcap} a$, $\sqcup X := \underset{a \in X}{\sqcup} a$. We assume that $\underline{D}$ is complete, then for all $X \subseteq D$, $\sqcap X \in D_\sqcap$ and $\sqcup X \in D_\sqcup$ and all filters (resp. ideals) of $\underline{D}$ are principal. Thus $\mathcal{F}(\underline{D}) = \mathcal{F}_p(\underline{D})$, which is a Boolean algebra, by Theorem 3.4. ■

Let $\underline{L}$ be a lattice. An element $a \in L$ is said **compact** if whenever $\bigvee A$ exists and $a \leq \bigvee A$ for $A \subseteq L$, then $a \leq \bigvee B$ for some finite $B \subseteq A$. $\underline{L}$ is said **compactly generated** if every element in L is a supremum of compact elements. $\underline{L}$ is said **algebraic** if $\underline{L}$ is complete and compactly generated [2].

Remark 3.5. Let $\underline{D}$ be a dBa. Let J be an arbitrary nonempty set, $\{F_j : j \in J\} \subseteq \mathcal{F}(\underline{D})$ and $\{I_j : j \in J\} \subseteq \mathcal{I}(\underline{D})$. Set $\tilde{J} := \{(i_1, \ldots, i_n) \in J^n \mid n \geq 1, n \in \mathbb{N}\}$. The following hold.

$$(*)\ \underset{j \in J}{\vee} F_j = \bigcup_{(i_1,\ldots,i_n) \in \tilde{J}} (F_{i_1} \vee \cdots \vee F_{i_n}), \quad (**)\ \underset{j \in J}{\vee} I_j = \bigcup_{(i_1,\ldots,i_n) \in \tilde{J}} (I_{i_1} \vee \cdots \vee I_{i_n}).$$

Lemma 3.5. *Let $\underline{D}$ be a dBa. A filter or an ideal of $\underline{D}$ is compact if and only if it is principal.*

Proof. (1) Let $a \in D$, to show that $F(a)$ is compact, we take $\{F_j : j \in J\} \subseteq \mathcal{F}(\underline{D})$ with J an arbitrary nonempty set such that $F(a) \subseteq \bigvee_{j \in J} F_j$. Since $a \in F(a)$, by $(*)$ of Remark 3.5 there are $n \geq 1$ and $i_1, \ldots, i_n \in J$ such that $a \in F_{i_1} \vee \cdots \vee F_{i_n}$. Therefore $F(a) \subseteq F_{i_1} \vee \cdots \vee F_{i_n}$. Hence it follows that $F(a)$ is a compact filter. Conversely, let F be a compact filter. Since $F \subseteq \bigvee_{a \in F} F(a)$ and F compact, there are $a_1, \ldots, a_n \in F$ such that $F \subseteq F(a_1) \vee \cdots \vee F(a_n) \subseteq F$, by (1) of Proposition 3.3 we get $F(a_1) \vee \cdots \vee (F_{a_n}) = F(a_1 \sqcap \cdots \sqcap a_n)$ and F is a principal filter. A similar argument shows the result for ideals. ■

Theorem 3.6. *Let $\underline{D}$ be a dBa. $\mathcal{F}(\underline{D})$ and $\mathcal{I}(\underline{D})$ are algebraic lattices.*

Proof. By Remark 3.2, we know that $\mathcal{F}(\underline{D})$ and $\mathcal{I}(\underline{D})$ are complete lattices. By Lemma 3.5 all principal filters $F(a)$ and principal ideals $I(a)$ are compact. Since $F = \bigvee_{a \in F} F(a)$ for any filter F of $\underline{D}$, and $I = \bigvee_{a \in I} I(a)$ for any ideal I of $\underline{D}$, we see that $\mathcal{F}(\underline{D})$ and $\mathcal{I}(\underline{D})$ are compactly generated. Therefore $\mathcal{F}(\underline{D})$ and $\mathcal{I}(\underline{D})$ are algebraic lattices. ■

In Boolean algebras there are several equivalent definitions of prime filters. These definitions can be carried over to dBas. To solve the equational theory problem for protoconcepts algebra, Rudolf Wille introduced in [10] the set $\mathfrak{F}_P(\underline{D})$ of filters F of $\underline{D}$ such that $F \cap D_\sqcap$ are prime filters of the Boolean algebra $\underline{D_\sqcap}$, and the set $\mathfrak{I}_P(\underline{D})$ of ideals I of $\underline{D}$ such that $I \cap D_\sqcup$ are prime ideals of the Boolean algebra $\underline{D_\sqcup}$. To prove the prime ideal theorem for double Boolean algebras, Léonard Kwuida introduced in [7] **primary filters** as proper filters F for which $x \in F$ or $\neg x \in F$, for each $x \in D$. Dually, a **primary ideal** is a proper ideal I for which $x \in I$ or $\lrcorner x \in I$, for each $x \in D$. We denote by $\mathfrak{F}_{pr}(\underline{D})$ the set of primary filters of $\underline{D}$, and by $\mathfrak{I}_{pr}(\underline{D})$ its set of primary ideals. In [4] Prosenjit Howlader and Mohua Banerjee showed that $\mathfrak{F}_P(\underline{D}) = \mathfrak{F}_{pr}(\underline{D})$ and $\mathfrak{I}_P(\underline{D}) = \mathfrak{I}_{pr}(\underline{D})$. Recall that the set of principal filters of $\underline{D}$ is $F_p(\underline{D}) = \{F(a) \mid a \in D_\sqcap\}$ and that of principals ideals $I_p(\underline{D}) = \{I(a) \mid a \in D_\sqcup\}$, and that $\underline{D_\sqcap}$ and $\underline{D_\sqcup}$ are Boolean algebras whose order relation is the restriction of $\sqsubseteq$ on $D_\sqcap$ and $D_\sqcup$, respectively. Thus, by Theorem 3.4, the lattice of principal filters of a dBa $\underline{D}$ is isomorphic to the lattice of principal filters of the Boolean algebra $\underline{D_\sqcap}$. Similarly, the lattice of principal ideals of a dBa $\underline{D}$ is isomorphic to the lattice of principal ideals,of the Boolean algebra $\underline{D_\sqcup}$.

On the basis of these results, an interesting question is how filters and ideals of $\underline{D}$ and those of $\underline{D_\sqcap}$ or $\underline{D_\sqcup}$ are related.

Theorem 3.7. *Let $\underline{D}$ be a double Boolean algebra. Then*

(1) *$\mathcal{F}(\underline{D_\sqcap})$ and $\mathcal{F}(\underline{D})$ are isomorphic lattices.*

(2) *$\mathcal{I}(\underline{D_\sqcup})$ and $\mathcal{I}(\underline{D})$ are isomorphic lattices.*

Proof. To prove (1), let $\underline{D}$ be a double Boolean algebra and E be a filter of $\underline{D_\sqcap}$. We set $\Phi(E) = \{x \in D \mid \exists u \in E, u \sqsubseteq x\}$.

- $\Phi(E)$ is a filter of $\underline{D}$ containing E. In fact, for any $x \in E$, $x \sqsubseteq x$, and thus $x \in \Phi(E)$. For $x, y \in \Phi(E)$, there are $x_0, y_0 \in E$ such that $x_0 \sqsubseteq x$ and $y_0 \sqsubseteq y$. By (1) of Lemma 3.2 we deduce that $x_0 \sqcap y_0 \sqsubseteq x \sqcap y \in \Phi(E)$. For $x, y \in D$ with $x \in \Phi(E)$ and $x \sqsubseteq y$; there is then $x_0 \in E$ such that $x_0 \sqsubseteq x \sqsubseteq y$; thus $y \in \Phi(E)$. Actually, E is a base of the filter $\Phi(E)$, and Φ defines a map from $F(\underline{D_\sqcap})$ to $F(\underline{D})$.
- $\Phi(E)$ is the only filter F of $\underline{D}$ such that $F \cap D_\sqcap = E$. In fact, $E \subseteq \Phi(E) \cap D_\sqcap$, and for any $x \in \Phi(E) \cap D_\sqcap$ there is $u \in E$ such that $u \sqsubseteq x$, implying $x \in E$, since E is a filter of $\underline{D_\sqcap}$. Thus $E = \Phi(E) \cap D_\sqcap$. If F_1, F_2 are in $F(\underline{D})$ such that $F_1 \cap D_\sqcap = F_2 \cap D_\sqcap$, then $F_1 = F_2$ since for any $x \in D$, we have

$$x \in F_1 \Leftrightarrow x \sqcap x \in F_1 \cap D_\sqcap \Leftrightarrow x\sqcap \in F_2 \cap D_\sqcap \Leftrightarrow x \in F_2.$$

- Φ is bijective. In fact, if if $E_1; E_2 \in F(\underline{D_\sqcap})$ such that $\Phi(E_1) = \Phi(E_2)$, then $E_1 = \Phi(E_1) \cap D_\sqcap = \Phi(E_2) \cap D_\sqcap = E_2$, and Φ is injective. Furthermore, for any filter F of $\underline{D}$, $F \cap D_\sqcap$ is a filter of $\underline{D_\sqcap}$ by (1) of Lemma 3.1 and $\Phi(F \cap D_\sqcap) = F$.
- Φ preserves and reflects the order relation. In fact, let $E_1, E_2 \in F(\underline{D_\sqcap})$. If $E_1 \subseteq E_2$, then for any $x \in \Phi(E_1)$, then there exists $u \in E_1 \subseteq E_2$ such that $u \sqsubseteq x$. Thus $u \in E_2$ and $u \sqsubseteq x$; that is $x \in \Phi(E_2)$. Thus $\Phi(E_1) \subseteq \Phi(E_2)$. Conversely, $\Phi(E_1) \subseteq \Phi(E_2) \Rightarrow \Phi(E_1) \cap D_\sqcap \subseteq \Phi(E_2) \cap D_\sqcap \Rightarrow E_1 \subseteq E_2$. Therefore, Φ is a lattice isomorphism, and $\mathcal{F}(\underline{D_\sqcap})$ and $\mathcal{F}(\underline{D})$ are isomorphic.

For (2), we can check using a similar arguments as above, that the mapping $\Psi : I(\underline{D_\sqcup}) \to I(\underline{D})$, $I \mapsto \Psi(I) = \{x \in D \mid \exists x_0 \in I, x \sqsubseteq x_0\}$ is a lattice isomorphism. ■

An **atom** in a dBa is an element x such that $x \neq \bot$ and if $y \sqsubseteq x, x \neq y$ then $y = \bot$. **co-atom** are defined dually. In Boolean algebras, distributive lattices [2] and weakly dicomplemented lattices [6] there are notions of ultrafilters, prime filters and primary filters which are closely related to the concept of atom, irreducible, prime and primary elements. These concepts can be transfered to dBas.

Definition 3.6. Let $\underline{D}$ be a dBa and F be a filter of $\underline{D}$. F is a **maximal filter** or an **ultrafilter** if F is a maximal element of $(F(\underline{D}) \setminus \{D\}, \subseteq)$; i.e., F is a proper filter not contained in any other proper filter. F is a **prime filter** if for all $a, b \in D$, $a \sqcup b \in F$ implies $a \in F$ or $b \in F$. Maximal and prime ideals are defined dually.

It is known (see [2]) that in Boolean algebras, each prime filter is an ultrafilter and vice versa. Dually each prime ideal is a maximal ideal and vice versa. The following theorem presents the corresponding results for dBas.

Theorem 3.8. *Let $\underline{D}$ be a dBa, $a \in D$, F a filter and I an ideal of $\underline{D}$.*

(1) *The principal filter $F(a \sqcap a)$ is an ultrafilter if and only if $a \sqcap a$ is an atom.*

(2) *The principal ideal $I(a \sqcup a)$ is maximal if and only if $a \sqcup a$ is a co-atom.*

(3) *F is an ultrafilter if and only if F is a primary filter.*

(4) *I is a maximal ideal if and only if I is a primary ideal.*

Proof. The even numbered items are dual of the odd ones. Let $a \in D$. Note that $F(a) = D \Leftrightarrow a \sqcap a = \bot$. Let F be a filter of $\underline{D}$.

(1) Assume that $a \sqcap a$ is an atom of $\underline{D}$ and $F(a \sqcap a) \subsetneq F$. Let $b \in F \setminus F(a \sqcap a)$. Since $a \sqcap a$ is an atom of $\underline{D}$, $(a \sqcap a) \sqcap b = \bot$ or $(a \sqcap a) \sqcap b = a \sqcap a \neq \bot$, because $(a \sqcap a) \sqcap b \sqsubseteq a \sqcap a$. If $(a \sqcap a) \sqcap b = a \sqcap a$, then $a \sqcap a \sqsubseteq b$, and $b \in F(a)$, which is a contradiction. Thus $\bot = (a \sqcap a) \sqcap b = a \sqcap b \in F$, since $a, b \in F$ and $F = D$. This prove that $F(a)$ is an ultrafilter.

Conversely, assume that $F(a \sqcap a)$ is an ultrafilter. Let $b \in D_{\sqcap}$ such that $b \sqsubseteq a \sqcap a$. Then $F(a) = F(a \sqcap a) \subseteq F(b)$. Since $F(a \sqcap a)$ is an ultrafilter, we get $F(b) = F(a \sqcap a)$ or $F(b) = D$. If $F(a) = F(b)$ or $F(b) = D$, then $b \in F(a)$ and $a \sqcap a \sqsubseteq b$, which is a contradiction to $b \neq a \sqcap a$. It follows that $F(b) = D$, and $\bot = b \sqcap b$. Therefore $a \sqcap a$ is an atom of $\underline{D}$.

(3) We assume that F is an ultrafilter of $\underline{D}$. To show that F is primary, we start by proving that $F \cap D_{\sqcap}$ is an ultrafilter of $\underline{D_{\sqcap}}$. By (1) of Lemma 3.1, $F \cap D_{\sqcap}$ is a filter of $\underline{D_{\sqcap}}$. Let G_0 be a proper filter of $\underline{D_{\sqcap}}$ such that $F \cap D_{\sqcap} \subseteq G_0$. G_0 is a base of a dBa filter G containing F. In fact, for any $y \in D$ we have

$$y \in F \Rightarrow y \sqcap y \in F \cap D_{\sqcap} \subseteq G_0 \Rightarrow y \in G.$$

Since F is a filter, $F = G$ or $G = D$. But $\bot \notin G$, so $F = G$ and $G_0 = G \cap D_{\sqcap} = F \cap D_{\sqcap}$. Thus $F \cap D_{\sqcap}$ is an ultrafilter of the Boolean algebra $\underline{D_{\sqcap}}$. Now let $x \in D$. Then $x \sqcap x \in F \cap D_{\sqcap}$ or $\neg(x \sqcap x) = \neg x \in F \cap D_{\sqcap}$, which is equivalent to $x \in F$ or $\neg x \in F$, since $\neg(x \sqcap x) = \neg x$. Thus, F is primary.

Conversely, we assume that F is primary. Then $F \cap D_{\sqcap}$ is a prime filter of the Boolean algebra $\underline{D_{\sqcap}}$, and therefore a maximal filter of $\underline{D_{\sqcap}}$.

Let G be a filter of $\underline{D}$ such that $F \subseteq G$, then $F \cap D_{\sqcap} \subseteq G \cap D_{\sqcap}$, as $F \cap D_{\sqcap}$ is maximal, we have $F \cap D_{\sqcap} = G \cap D_{\sqcap}$ or $G \cap D_{\sqcap} = D_{\sqcap}$. If $G \cap D_{\sqcap} = D_{\sqcap}$, then $D_{\sqcap} \subseteq G$. For any $x \in D$, $x \sqcap x \in D_{\sqcap} \subseteq G$, since G is a filter. We deduce that $x \in G$, and $G = D$. If $F \cap D_{\sqcap} = G \cap D_{\sqcap}$, then $F = G$ by the proof of Theorem 3.7. Thus F is a ultrafilter of $\underline{D}$. ∎

Remark 3.7. Let $\underline{D_3}$ be the trivial dBa with $D_3 = \{\bot, a, \top\}$, $\bot \sqcup \bot = a = \top \sqcap \top$ ([7]). Then $F = \{a, \top\}$ is a primary filter, $\bot \sqcup \bot = a \in F$, but $\bot \notin F$, so F is not a prime filter. $I(a) = \{\bot, a\}$ is a primary ideal. But $\top \sqcap \top = a \in I(a)$ and $\top \notin I(a)$. Therefore primary filters and primary ideals need not be prime.

4. HOMOMORPHISMS OF dBAS

Let $\underline{D}$ and $\underline{M}$ be two dBas and h be a map from D to M.

Definition 4.1 [4, 10]**.** h is a **homomorphism** from $\underline{D}$ to $\underline{M}$, if h preserves dBa operations, i.e., $\forall a, b \in D$, $h(a \sqcap b) = h(a) \sqcap h(b), h(a \sqcup b) = h(a) \sqcup h(b)$, $h(\neg a) = \neg h(a)$, $h(\lrcorner a) = \lrcorner h(a)$, and $h(\top) = \top$, $h(\bot) = \bot$. An **embedding** is a one-to-one homomorphism. An **isomorphism** is a bijective homomorphism.

If h is an embedding, then we say that $\underline{D}$ embeds into $\underline{M}$. If h is an isomorphism, we say that the dBas $\underline{D}$ and $\underline{M}$ are isomorphic.

Definition 4.2 [10]**.** h is **quasi-injective**, if its preserves and reflect the quasi-order, i.e., $\forall a, b \in D$, $a \sqsubseteq b$ iff $h(a) \sqsubseteq h(b)$. If h is quasi-injective and surjective, then h is called **quasi-isomorphism**.

If h is quasi-injective, then we say that $\underline{D}$ quasi-embeds into $\underline{M}$. If h is an quasi-isomorphism, we say that the dBas $\underline{D}$ and $\underline{M}$ are quasi-isomorphic. Note that composition of two dBa homomorphisms is also a dBas homomorphism. For a dBa $\underline{D}$ we denote by D_P the set $D_\sqcap \cup D_\sqcup$.

Let $\underline{L} = (L, \leq)$ be a quasi-ordered set and $F \subseteq L$. F is **quasi-order filter** on $\underline{L}$ if for all $x \in F, y \in L$, if $x \leq y$ then $y \in F$. A **quasi-order ideal** is defined dually. If the quasi-order $\leq$ is an order relation, then we get the well-known notions of order filter and order ideal.

Definition 4.3 [10]**.** Let $h : D \to M$ be a homomorphism of dBas. The kernel of h is defined by

$$\ker(h) := \{(a, b) \in D^2 \mid h(a) = h(b)\}.$$

It is known that if the map $h : D \to M$ is a homomorphism of Boolean algebras then, $\ker(h)$ is a congruence on $\underline{D}$ and the class $[\bot]_{\ker(h)}$ (resp. $[\top]_{\ker(h)}$) is an ideal (resp. a filter) of $\underline{D}$. In the sequel we will see that for a homomorphism h of dBas the class $[\bot]_{\ker(h)}$ (resp. $[\top]_{\ker(h)}$) is not always an ideal (resp. a filter) of $\underline{D}$. It was shown in [5] that if $h : D \to M$ is a homomorphism of dBas, then for all $a, b \in D$, if $a \sqsubseteq b$, then $h(a) \sqsubseteq h(b)$. We say that h is isotone. We give some properties of the sets $A = [\bot]_{\ker(h)}$, $B = [\bot \sqcup \bot]_{\ker(h)}$, $C = [\top \sqcap \top]_{\ker(h)}$ and $E = [\top]_{\ker(h)}$ in the following remark. One can see that $B = \{x \in D \mid h(x) = \bot \sqcup \bot\}$ and $C = \{x \in D \mid h(x) = \top \sqcap \top\}$.

Remark 4.4. Let $h : D \to M$ be a homomorphism of dBas, from Theorem 2.1 we obtain the following statements.

(1) A is an ideal (resp. E is a filter) iff $\bot \sqcup \bot \in A$ (resp. $\top \sqcap \top \in E$).

(2) A is a quasi-order (resp. order) ideal if $\sqsubseteq$ is a quasi-order (resp. order) relation on D.

(3) E is a quasi-order (resp. order) filter if $\sqsubseteq$ is a quasi-order (resp. order) relation on D.

(4) A is preserved by $\sqcap, \sqcup$ and $h(\lrcorner x) = \top$ for any $x \in A$.

(5) C is preserved by $\sqcup$ and $\sqcap$ and $h(\neg x) = \bot$ for any $x \in C$.

(6) If h is injective, then h is quasi-injective.

Now we investigate how filters of $\underline{D}$ and $\underline{M}$ are related.

Remark 4.5. Let $h : \underline{D} \to \underline{M}$ be a homomorphism of dBas and F be a filter of $\underline{D}$. Then the filter generated by $h(F)$ is equal to $\{y \in M \mid \exists x \in F, h(x) \sqsubseteq y\}$, and $h(F)$ is a base of $Filter\langle h(F)\rangle$. Dually, if I is an ideal of $\underline{D}$, then $h(I)$ is a base of $Ideal\langle h(I)\rangle = \{y \in M \mid \exists x \in I, h(x) \sqsubseteq y\}$.

We consider the maps:

$$\tau : F(\underline{D}) \to F(\underline{M}),\ F \mapsto \tau(F) := Filter\langle h(F)\rangle$$

and

$$\nu : I(\underline{D}) \to I(\underline{M}),\ I \mapsto \nu(I) := Ideal\langle h(I)\rangle.$$

Clearly the maps τ and ν are well defined.

Lemma 4.1. *Let $h : D \to M$ be a homomorphism, $I_1, I_2 \in I(\underline{D}), F_1, F_2 \in F(\underline{D})$ and $a \in D$. The following hold.*

(1) $\tau(F(a)) = F(h(a))$ *and* $\nu(I(a)) = I(h(a))$.

(2) τ *and* ν *preserve the order, and also reflect the order if h is quasi-injective.*

(3) (ii) $\tau(F_1 \cap F_2) = \tau(F_1) \cap \tau(F_2)$ *and* $\tau(F_1 \vee F_2) = \tau(F_1) \vee \tau(F_2)$. *Similarly* $\nu(I_1 \cap I_2) = \nu(I_1) \cap \nu(I_2)$ *and* $\nu(I_1 \vee I_2) = \nu(I_1) \vee \nu(I_2)$.

(4) *If h is quasi-injective, then $h(F)$ is a filter and $h(I)$ an ideal of $h(\underline{D})$, and $h(F_1 \vee F_2) = h(F_1) \vee^h h(F_2)$, $h(F_1 \cap F_2) = h(F_1) \cap h(F_2)$, $h(I_1 \vee I_2) = h(I_1) \vee^h h(I_2)$ and $h(I_1 \cap I_2) = h(I_1) \cap h(I_2)$, where $\vee^h$ is the join in the lattice of filters/ideals of $h(\underline{D})$.*

Proof. Let $y \in M$.

(1) If $y \in F(h(a))$, then $h(a \sqcap a) = h(a) \sqcap h(a) \sqsubseteq y$, and $y \in \tau(F(a))$. If $y \in \tau(F(a))$, then there exists $x \in F(a)$ such that $h(x) \sqsubseteq y$. Since $x \in F(a)$; we get $a \sqcap a \sqsubseteq x$ and $h(a \sqcap a) = h(a) \sqcap h(a) \sqsubseteq h(x) \sqsubseteq y$, so $y \in F(h(a))$. Thus $F(h(a)) = \tau(F(a))$. The rest is proved similarly.

(2) If $F_1 \subseteq F_2$, then obviously $\tau(F_1) \subseteq \tau(F_2)$. Conversely, if h is quasi-injective and $\tau(F_1) \subseteq \tau(F_2)$, then for any $x_1 \in F_1, h(x_1) \in \tau(F_1) \subseteq h(F_2)$, and there exists $x_2 \in F_2$ such that $h(x_2) \sqsubseteq h(x_1)$. Therefore $x_2 \sqsubseteq x_1$, since h is quasi-injective. Since F_2 is a filter and $x_2 \in F_2$, we deduce that $x_1 \in F_2$ and $F_1 \subseteq F_2$. The proof for ν is obtained similarly.

(3) $\tau(F_1 \cap F_2) \subseteq \tau(F_1) \cap \tau(F_2)$ and $\tau(F_1) \vee \tau(F_2) \subseteq \tau(F_1 \vee F_2)$ follow from (2). If $y \in \tau(F_1) \cap \tau(F_2)$, then there are $x_1 \in F_1$ and $x_2 \in F_2$ such that $h(x_1) \sqsubseteq y$ and $h(x_2) \sqsubseteq y$. We get $h(x_1 \vee x_2) = h(x_1) \vee h(x_2) \sqsubseteq y \vee y = y \sqcap y \sqsubseteq y$ and $x_1 \vee x_2 \in F_1 \cap F_2$, since $F(x_1 \vee x_2) = F(x_1) \cap F(x_2) \subseteq F_1 \cap F_2$ (by (7) of Proposition 3.3). Thus $\tau(F_1 \cap F_2) = \tau(F_1) \cap \tau(F_2)$.

If $y \in \tau(F_1 \vee F_2)$, then there is $x \in F_1 \vee F_2$ such that $h(x) \sqsubseteq y$. Thus there are $x_1 \in F_1$ and $x_2 \in F_2$ such that $x_1 \sqcap x_2 \sqsubseteq x$. Since h is a homomorphism we get $h(x_1) \sqcap h(x_2) = h(x_1 \sqcap x_2) \sqsubseteq h(x) \sqsubseteq y$ and $y \in \tau(F_1) \vee \tau(F_2)$. Thus $\tau(F_1) \vee \tau(F_2) = \tau(F_1 \vee F_2)$.

The proof for ν is obtained similarly.

(4) We assume that h is quasi-injective. Note that $h(\underline{D})$ is a dBa.

If $a_1, a_2 \in h(F)$, then there are $x_1, x_2 \in F$ such that $a_1 = h(x_1)$ and $a_2 = h(x_2)$. We get $a_1 \sqcap a_2 = h(x_1) \sqcap h(x_2) = h(x_1 \sqcap x_2) \in h(F)$. If $b \in h(D)$ and $a_1 \in h(F)$ such that $a_1 \sqsubseteq b$, then there are $x \in D$ and $x_1 \in F$ such that $a_1 = h(x_1) \sqsubseteq b = h(x)$. Since h is quasi-injective, we get $x_1 \sqsubseteq x$ and $x \in F$. Thus $b = h(x) \in h(F)$, and $h(F)$ is a filter of $h(\underline{D})$. Similarly, $h(I)$ is an ideal of $h(\underline{D})$. $F_1, F_2 \subseteq F_1 \vee F_2$ implies $h(F_1) \vee^h h(F_2) \subseteq h(F_1 \vee F_2)$. If $b \in h(F_1 \vee F_2)$, then $b = h(x)$ for some $x \in F_1 \vee F_2$. Thus, there are $x_1 \in F_1$ and $x_2 \in F_2$ such that $x_1 \sqcap x_2 \sqsubseteq x$. It follows that $h(x_1) \sqcap h(x_2) = h(x_1 \sqcap x_2) \sqsubseteq h(x) = b$, and $b \in h(F_1) \vee^h h(F_2)$.

$h(F_1 \cap F_2) \subseteq h(F_1) \cap h(F_2)$. If $b \in h(F_1) \cap h(F_2)$, then there are $x_1 \in F_1$ and $x_2 \in F_2$ such that $h(x_1) = b = h(x_2)$. We have $b \sqcap b = h(x_1) \vee h(x_2) = h(x_1 \vee x_2) \in h(F_1 \cap F_2)$, and $b \sqcap b \in h(F_1 \cap F_2)$, so $b \in h(F_1 \cap F_2)$; since $h(F_1 \cap F_2)$ is a filter. The proof for ideals is obtained similarly. ■

Proposition 4.6. *Let $h : D \to M$ be a dBa homomorphism. The following hold.*

(1) *τ and ν are homomorphisms of bounded lattices.*

(2) *If h is quasi-injective, then τ and ν are injective. If h is onto, then τ and ν are onto. τ (resp. ν) is an isomorphism iff h is a quasi-isomorphism.*

(3) *If $\underline{D}$ and $\underline{M}$ are complete dBas and h a homomorphism of complete dBas, then τ and ν are complete lattices homomorphisms.*

Proof. Let $h : D \to M$ be a dBa homomorphism. Let $F_1, F_2 \in F(\underline{D})$.

(1) By (3) of Lemma 4.1 τ and ν are lattices homomorphisms.

(2) If h is quasi-injective and $\tau(F_1) = \tau(F_2)$, then by (2) of Lemma 4.1 $F_1 = F_2$.

If h is onto and $G \in F(\underline{M})$, then $h^{-1}(G)$ is in $F(\underline{D})$ and $\tau(h^{-1}(G)) = G$, since $G = h(h^{-1}(G))$.

If h is a quasi-isomorphism, then h is quasi-injective and onto. Thus τ and ν are injective and onto, and therefore isomorphisms.

If τ is an isomorphism, then for $a_1, a_2 \in D$,

$$a_1 \sqsubseteq a_2 \Leftrightarrow F(a_1) \sqsubseteq F(a_2) \Leftrightarrow \tau(F(a_1) \subseteq \tau(F(a_2))$$

$$\Leftrightarrow F(h(a_1)) \subseteq F(h(a_2)) \Leftrightarrow h(a_1) \sqsubseteq h(a_2).$$

Thus h is a quasi-isomorphism.

(3) Assume that $\underline{D}$ and $\underline{M}$ are complete dBas.

Let $\{F_j : j \in J\} \subseteq F(\underline{D})$. Since τ is isotone, we get $\bigvee_{i\in J} \tau(F_j) \subseteq \tau(\bigvee_{j\in J} F_j)$ and $\tau(\bigcap_{j\in J} F_j) \subseteq \bigcap_{j\in J} \tau(F_j)$. Let $y \in M$. If $y \in \tau(\bigvee_{j\in J} F_j)$, then there is $x \in \bigvee_{j\in J} F_j$ such that $h(x) \sqsubseteq y$. Thus there are $i_1, \ldots, i_n \in J, n \geq 1$ and $x_{i_l} \in F_{i_l}, l = 1, \ldots, n$ such that $x_{i_1} \sqcap \cdots \sqcap x_{i_n} \sqsubseteq x$. Furthermore, h is compatible with $\sqcap$ and $\sqsubseteq$ is transitive, so $h(x_{i_1}) \sqcap \cdots \sqcap h(x_{i_n}) \sqsubseteq h(x) \sqsubseteq y$. Therefore $y \in \tau(F_{i_1}) \vee \cdots \vee \tau(F_{i_n}) \subseteq \bigvee_{j\in J} \tau(F_j)$.

If $y \in \bigcap_{j\in J} \tau(F_j)$, then for all $j \in J$, there is $x_j \in F_j$ such that $h(x_j) \sqsubseteq y$. Since h is a homomorphism of complete dBas we get $\bigvee_{j\in J} h(x_j) = h(\bigvee_{j\in J} x_j) \sqsubseteq y$ and $\bigvee_{j\in J} x_j \in \bigcap_{j\in J} F_j$, so $y \in \tau(\bigcap_{j\in J} F_j)$. ■

5. POWER OF DOUBLE BOOLEAN ALGEBRAS

It is known that if $\underline{L} := (L, \leq)$ is a (complete) lattice, and X a non-empty set, then the set L^X of all maps from X to L forms a (complete) lattice when ordered pointwise, i.e., for all $f, g \in L^X$,

$$f \leq g :\Leftrightarrow f(x) \leq g(x) \text{ for all } x \in X.$$

For $U \subseteq L$ we set $L(x, U) = \{f \in L^X \mid f(x) \in U\}$. If $U$is a filter (resp. an ideal) then $L(x, U)$ is a filter (resp. an ideal). We will now consider the set D^X of all maps from X to D, where $\underline{D}$ is a double Boolean algebra and X a non-empty set. The dBa operations are extended to D^X, pointwise. For $f, g \in D^X$ and $x \in X$.

- $(f \sqcap g)(x) = f(x) \sqcap g(x)$,
- $(f \sqcup g)(x) = f(x) \sqcup g(x)$,
- $(\neg f)(x) = \neg f(x)$,
- $(\lrcorner f)(x) = \lrcorner f(x)$,
- $(f \vee g)(x) = f(x) \vee g(x)$,
- $(f \wedge g)(x) = f(x) \wedge g(x)$.

We denote by ϕ_a the constant map $X \to D, x \mapsto a$. For any $a, b \in D$, we have $\phi_{a\sqcap b} = \phi_a \sqcap \phi_b, \phi_{a\sqcup b} = \phi_a \sqcup \phi_b$, $\neg(\phi_a) = \phi_{\neg a}$ and $\lrcorner(\phi_a) = \phi_{\lrcorner a}$. The algebra $\underline{D}^X := (D^X, \sqcap, \sqcup, \neg, \lrcorner, \phi_\bot, \phi_\top)$ is a dBa called a **power** of the dBa $\underline{D}$. The corresponding quasi-order is given by

$$f \sqsubseteq g :\Leftrightarrow f(x) \sqsubseteq g(x) \text{for all } x \in X.$$

For any $a \in D$, it holds $a \sqsubseteq b \Leftrightarrow \phi_a \sqsubseteq \phi_b$.

To keep the notation simple, we set $L := D^X$. Our aims is to investigate the relationship between $\underline{D}$ and $\underline{L}$. We show that if $\underline{D}$ is complete, then $\underline{L}$ is complete and $\underline{D}$ is contextual, fully contextual (resp. trivial) if and only if $\underline{L}$ is contextual, fully contextual (resp. trivial). We will also show that the lattice $\mathcal{F}(\underline{D})$ (resp. $\mathcal{I}(\underline{D})$) embeds in $\mathcal{F}(\underline{L})$ (resp. $\mathcal{I}(\underline{L})$).

Proposition 5.1. *Let $U \subseteq D$ and $x \in X$. The following hold.*

(1) (a) $L_\sqcap = \{f \in L \mid f(X) \subseteq D_\sqcap\}$, (b) $L_\sqcup = \{f \in L \mid f(X) \subseteq D_\sqcup\}$.

(2) *$\underline{D}$ is contextual if and only if $\underline{L}$ is contextual.*

(3) *$\underline{D}$ is fully contextual if and only if $\underline{L}$ is fully contextual.*

(4) *If $\underline{D}$ is complete, then $\underline{L}$ is also complete.*

(5) *U is a (proper) filter (resp. an ideal) of $\underline{D}$ if and only if $L(x, U)$ is a (proper) filter (resp. an ideal) ideal of $\underline{L}$.*

(6) *U is a prime/maximal filter (resp. ideal) if and only if $L(x, U)$ is a prime/ maximal filter (resp. ideal).*

(7) *$\underline{D}$ is trivial if and only if $\underline{L}$ is trivial.*

Proof. (1) (a) Let $Y = \{f \in L \mid \forall x \in X, f(x) \in D_\sqcap\}$, by definition we have $Y \subseteq L_\sqcap$. Let $f \in L_\sqcap$, then $f \sqcap f = f$ and $(f \sqcap f)(x) = f(x)\ \forall x \in X$, that is $f(x) \sqcap f(x) = f(x)\ \forall x \in X$ and $f \in Y$, therefore $L_\sqcap \subseteq Y$. Thus $Y = L_\sqcap$. Dually we can show that (b) holds.

(2) Assume that $\underline{D}$ is contextual. Let's show that $\underline{L}$ is contextual. Let $f, g \in L$ such that $f \sqsubseteq g$ and $g \sqsubseteq f$, we show that $f = g$. Let $x \in X$, then $f(x) \sqsubseteq g(x)$ and $g(x) \sqsubseteq f(x)$. Since $\underline{D}$ is contextual, we get $f(x) = g(x)$, so $f(x) = g(x)$ for all $x \in X$ and therefore $f = g$. Thus $\underline{L}$ is contextual. Conversely, assume that $\underline{L}$ is contextual. We show that $\underline{D}$ is contextual. If $a, b \in D$ such that $a \sqsubseteq b$ and $b \sqsubseteq a$, then $\phi_a, \phi_b \in L$ (1) and $\phi_a \sqsubseteq \phi_b$ and $\phi_b \sqsubseteq \phi_a$ (2). Since $\underline{L}$ is contextual, (1) and (2) yield $\phi_a = \phi_b$, that is $a = b$ (due to ϕ_a and ϕ_b are constant maps), therefore $\underline{D}$ is contextual. Thus (2) holds.

(3) Assume that $\underline{D}$ is fully contextual. Then $\underline{D}$ is contextual and by (2) $\underline{L}$ is also contextual. Let $f \in L_\sqcap, g \in L_\sqcup$ such that $f \sqcup f = g \sqcap g$ $(*)$. We search a unique $h \in L$ such that $h \sqcap h = f$ and $h \sqcup h = g$. Since $(*)$ holds, for all $x \in X, f(x) \sqcup f(x) = g(x) \sqcap g(x)$, furthermore $f(x) \in D_\sqcap$ and $g(x) \in D_\sqcup$ (due to

$f \in L_{\sqcap}$ and $g \in L_{\sqcup}$). By the fact that $\underline{D}$ is fully contextual, it follows that for all $y \in X$, there is a unique $z_y \in D$ such that $z_y \sqcap z_y = f(y)$ and $z_y \sqcup z_y = g(y)$. Setting $h : X \to D, y \mapsto h(y) = z_y$, h is well defined (due to z_y unique), and for all $y \in X, h(y) \sqcap h(y) = f(y)$ and $h(y) \sqcup h(y) = g(y)$. It follows that $h \sqcap h = f$ and $h \sqcup h = g$. Thus $\underline{L}$ is fully contextual.

Conversely, we suppose that $\underline{L}$ is fully contextual. Let's show that $\underline{D}$ is fully contextual. Since $\underline{L}$ is fully contextual, it is contextual and by (2) $\underline{L}$ is contextual. It remains to show that for any $a \in D_{\sqcap}, b \in D_{\sqcup}$ such that $a \sqcup a = b \sqcap b$, there is a unique $c \in D$ such that $c \sqcap c = a$ and $c \sqcup c = b$. If $a \in D_{\sqcap}, b \in D_{\sqcup}$ such that $a \sqcup a = b \sqcap b$, then $\phi_a \in L_{\sqcap}, \phi_b \in L_{\sqcup}$ with $\phi_{b \sqcup b} = \phi_b \sqcap \phi_b$, since $\phi_a \sqcup \phi_b = \phi_{a \sqcup b}$ and $\phi_a \sqcap \phi_b = \phi_{a \sqcap b}$. Thefrefore $\phi_a \sqcup \phi_a = \phi_b \sqcap \phi_b$ and $\underline{L}$ fully contextual, we deduce that there is a unique $f \in L$ such that $f \sqcup f = \phi_b$ $(*_1)$ and $f \sqcap f = \phi_a$ $(*_2)$. Let's show that f is a constant map. Let $x_1, x_2 \in X$ such that $x_1 \neq x_2$, we get $f(x_1) \sqcup f(x_1) = b = f(x_2) \sqcup f(x_2)$ $(*_3)$ (using $(*_1)$ and $f(x_1) \sqcap f(x_1) = a = f(x_2) \sqcap f(x_2)$ $(*_4)$ (using $(*_2)$). $(*_3)$ and $(*_4)$ together with (14) of Proposition 2.2 yield $f(x_1) \sqsubseteq f(x_2)$ and $f(x_2) \sqsubseteq f(x_1)$, therefore $f(x_1) = f(x_2)$ (due to $\underline{D}$ contextual); therefore f is a constant map and there is aunique $c \in D$ such that for any $x \in X$, $f(x) = c$, so $f = \phi_c$, and from $(*_1)$ and $(*_2)$ we get $c \sqcup c = b$ and $c \sqcap c = a$. It follows that $\underline{D}$ is fully contextual.

(4) Assume that $\underline{D}$ is complete. Let $Y \subseteq L$. If $Y \subseteq L_{\sqcap}$, set $\sqcap Y = \bigsqcap_{f \in Y} f$ and $(\bigsqcap_{f \in Y} f)(x) = \bigsqcap_{f \in Y} f(x)$; if $Y \subseteq L_{\sqcup}$, set $\sqcup Y = \bigsqcup_{f \in Y} f$ and $(\bigsqcup_{f \in Y} f)(x) = \bigsqcup_{f \in Y} f(x)$. Since $\underline{D}$ is complete, $\bigsqcap_{f \in Y} f \in L_{\sqcap}$ for any $Y \subseteq L_{\sqcap}$, and $\bigsqcup_{f \in Y} f \in L_{\sqcup}$ for any $Y \subseteq L_{\sqcup}$. Thus $\underline{L}$ is complete.

(5) We show the case of ideals and that of filters is obtained similarly. Assume that U is an ideal of $\underline{D}$, then $\phi_{\perp} \in L(x, U)$ and $L(x, U) \neq \emptyset$. If $f, g \in L(x, U)$, then $(f \sqcup g)(x) = f(x) \sqcup g(x) \in U$; so $f \sqcup g \in L(x, U)$. If $f \sqsubseteq g$ and $g \in L(x, U)$ then $f(x) \sqsubseteq g(x)$ and $g(x) \in U$; so $f \in L(x, U)$. Thus $L(x, U)$ is an ideal. Conversely, assume that $L(x, U)$ is an ideal of $\underline{L}$, then $\phi_{\perp} \in L(x, U)$ and $\perp \in U$; so U is a non-emptyset. Let $a, b \in U$, then $\phi_a, \phi_b \in L(x, U)$ and $\phi_a \sqcup \phi_b = \phi_{a \sqcup b} \in L(x, U)$; therefore $a \sqcup b \in U$. Let $b \in U, a \in D$ such that $a \sqsubseteq b$, then $\phi_a \sqsubseteq \phi_b$ and $\phi_a \in L(x, U)$ (due to $L(x, U)$ is an ideal); so $a \in U$. Therefore U is an ideal of $\underline{D}$.

Now assume that U is a proper ideal of $\underline{D}$, then $\top \notin U$ and $L(x, U)$ is an ideal. Since $\top \notin U$, we get $\phi_{\top} \notin L(x, U)$. Thus $L(x, U)$ is a proper ideal of $\underline{L}$. Conversely, assume that $L(x, U)$ is a proper ideal of $\underline{L}$, then $\phi_{\top} \notin L(x, U)$, that is $\top \notin U$; therefore U is a proper ideal of $\underline{D}$.

(6) (a) Assume that U is a prime ideal of $\underline{D}$. $L(x, U)$ is an ideal of $\underline{L}$ by (5). Let $f, g \in L$ such that $f \sqcap g \in L(x, U)$, then $(f \sqcap g)(x) = f(x) \sqcap g(x) \in U$; since U is prime we get $f(x) \in U$ or $g(x) \in U$; it follows that $f \in L(x, U)$ or $g \in L(x, U)$. Therefore $L(x, U)$ is a prime ideal of $\underline{L}$. Conversely, assume that $L(x, U)$ is a

prime ideal of $\underline{L}$. Let $a, b \in D$ such that $a \sqcap b \in I$, then $\phi_{a\sqcap b} = \phi_a \sqcap \phi_b \in L(x, U)$; since $L(x, U)$ is a prime ideal, we get $\phi_a \in L(x, U)$ or $\phi_b \in L(x, U)$ and $a \in U$ or $b \in U$; therefore U is a prime ideal.

(b) Assume that U is a maximal ideal of $\underline{D}$. We will show that $L(x, U)$ is also a maximal ideal of $\underline{L}$. Since U is a proper ideal, $L(x, U)$ is also a proper ideal. Let $f \in L \setminus L(x, U)$, then $f(x) \notin U$. Let's show that $\langle L(x, U) \cup \{f\}\rangle = L$, that is for $g \in L$ there exists $h \in L(x, U)$ such that $g \sqsubseteq h \sqcup f$.

Let $g \in L$. Since U is maximal, $\langle I \cup \{f(x)\}\rangle = D$, and there exists $i_x \in U$ such that $g(x) \sqsubseteq i_x \sqcup f(x)$. Let $h : X \to D$, defined by $h(x) = i_x$ and $h(y) = g(y)$ if $y \neq x$. Then $h \in L$ and $g \sqsubseteq h \sqcup f$. Thus $L = \langle L(x, U) \cup \{f\}\rangle$ and $L(x, U)$ is a maximal ideal.

Conversely, assume that $L(x, U)$ is a maximal ideal of $\underline{L}$. Then U is a proper ideal of $\underline{D}$. Let $a \in D \setminus U$, then $\phi_a \notin L(x, U)$ and $\langle L(x, U) \cup \{\phi_a\}\rangle = L$ $(*)$ (because $L(x, U)$ is a maximal ideal). Let $b \in D$, from $(*)$ there is $f \in L(x, U)$ such that $\phi_b \sqsubseteq f \sqcup \phi_a$; so $f(x) \in U$ and $b \sqsubseteq f(x) \sqcup a$; therefore $D \subseteq \langle I \cup \{a\}\rangle$. Thus U is a maximal ideal of $\underline{D}$.

(7) Assume that $\underline{D}$ is trivial, then $\bot \sqcup \bot = \top \sqcap \top$. Thus for all $x \in X$,

$$(\phi_\bot \sqcup \phi_\bot)(x) = \bot \sqcup \bot = \top \sqcap \top = (\phi_\top \sqcap \phi_\top)(x).$$

Therefore $\phi_\bot \sqcup \phi_\bot = \phi_\top \sqcap \phi_\top$ and $\underline{L}$ is trivial. Conversely, if $\underline{L}$ is trivial, then $\phi_\top \sqcap \phi_\top = \phi_\bot \sqcup \phi_\bot$ and $\top \sqcap \top = \bot \sqcup \bot$. Therefore $\underline{D}$ is trivial. Thus (7) holds. ■

Recall that for any $a \in D$, ϕ_a is the constant function of value a. For any $F \in F(\underline{D})$, $I \in I(\underline{D})$ we set $I^* = \{\phi_y \mid y \in I\}$, $F^* = \{\phi_y \mid y \in F\}$ and $I(\underline{D})^* = \{I^* \mid I \in I(\underline{D})\}$, $F(\underline{D})^* = \{F^* \mid F \in F(\underline{D})\}$.

We define also $\Phi : D \to L$ by $\Phi(a) = \phi_a$. It is easy to see that his function is a quasi-embedding. The following lemma gives some properties of the operator *.

Lemma 5.1. *Let $F, F_i, i = 1, 2$ be filters and I, I_i, $i = 1, 2$ ideals on $\underline{D}$. The following hold.*

(1) (i) *F^* is a filter of $\underline{\phi(D)}$ and $F^* \subseteq L(x, F)$ for every $x \in X$.* (ii) *I^* is an ideal of $\underline{\phi(D)}$ and $I^* \subseteq L(x, I)$.*

(2) ** is an increasing map on the lattice of filters (resp. ideals).*

(3) (i) $(\bigcap_{j\in J} F_j)^* = \bigcap_{j\in J} F_j^*$, (ii) $\bigvee_{j\in J} F_j^* = (\bigvee_{j\in J} F_j)^*$.

(4) (i) $(\bigcap_{j\in J} I_j)^* = \bigcap_{j\in J} I_j^*$, (ii) $\bigvee_{j\in J} I_j^* = (\bigvee_{j\in J} I_j)^*$.

Proof. It is an easy check. ■

The following proposition is useful for the proof of Theorem 5.2.

Proposition 5.2. Let $\{F_j, j \in J\} \subseteq F(\underline{D})$ and $\{I_j, j \in J\} \subseteq I(\underline{D})$. The following hold.

(1) $I_p \subseteq I_q$ if and only if $L(x, I_p) \subseteq L(x, I_q)$, for any $p, q \in J$.

(2) $F_p \subseteq F_q$ if and only if $L(x, F_p) \subseteq L(x, F_q)$, for any $p, q \in J$.

(3) (a) $L(x, \bigvee_{i \in J} F_j) = \bigvee_{j \in J} L(x, F_j)$, (b) $L(x, \bigcap_{i \in J} F_j) = \bigcap_{j \in J} L(x, F_j)$,

(4) (a) $L(x, \bigvee_{j \in J} I_j) = \bigvee_{j \in J} L(x, I_j)$, (b) $L(x, \bigcap_{j \in J} I_j) = \bigcap_{j \in J} L(x, I_j)$.

Proof. An easy check using the constant functions give the proof of (1), (2), (3) (b) and (4) (b).

Claim. For every fiters F_1 and F_2, we have $L(x, F_1 \vee F_2) = L(x, F_1) \vee L(x, F_2)$. Since $F_1, F_2 \subseteq F_1 \vee F_2$, using (2) we get $L(x, F_1) \vee L(x, F_2) \subseteq L(x, F_1 \vee F_2)$.

It remains to show that $L(x, F_1 \vee F_2) \subseteq L(x, F_1) \vee L(x, F_2)$. Let $f \in L(x, F_1 \vee F_2)$, then $f(x) \in F_1 \vee F_2$; so there are $y_i \in F_i$, $i = 1, 2$ such that $y_1 \sqcap y_2 \sqsubseteq f(x)$. Let $f_i : X \to D, t \mapsto f_i(t)$, $i = 1, 2$ be defined by

$$f_i(t) = \begin{cases} y_i & \text{if } t = x, \\ f(t) & \text{if } t \neq x. \end{cases}$$

Therefore $f_i \in L(x, F_i)$, $i = 1, 2$ and it is easy to see that $f_1 \sqcap f_2 \sqsubseteq f$. Thus $L(x, F_1 \vee F_2) \subseteq L(x, F_1) \vee L(x, F_2)$.

Second, the above claim can be generalize to a finite number of filters by induction $(**)$.

Thirth, an easy check give a generalization of (3) (a) as follow. If J is a non-empty set and $\{A_j \subseteq D : j \in J\}$ a family of non-empty subsets of D, then

$$L\Big(x, \bigcup_{j \in J} A_j\Big) = \bigcup_{j \in J} L(x, A_j) \ (*_1).$$

Fourth, we finalize our proof. From $(*)$ of Remark 3.5 we get

$$\begin{aligned} L\Big(x, \bigvee_{j \in J} F_j\Big) &= L\Big(x, \bigcup_{(i_1, \ldots, i_n) \in \tilde{J}} (F_{i_1} \vee \cdots \vee F_{i_n})\Big) \\ &= \bigcup_{(i_1, \ldots, i_n) \in J^*} L(x, F_{i_1} \vee \cdots \vee F_{i_n}) \ \ (\text{by}(*_1)) \\ &= \bigcup_{(i_1, \ldots, i_n) \in \tilde{J}} [L(x, F_{i_1}) \vee \cdots \vee L(x, F_{i_n})] \ (\text{by } (**)) \\ &= \bigvee_{j \in J} L(x, F_j) \ (\text{aplying}(*) \text{ of Remark 3.5}). \end{aligned}$$

Thus (3) (a) holds. ■

We consider the following two algebras:

$$\underline{\mathcal{F}(\underline{D})^*} = (F(\underline{D})^*; \wedge, \vee, F(\perp)^*, F(\top)^*\}), \underline{\mathcal{I}(\underline{D})^*} = (I(\underline{D})^*; \wedge, \vee, I(\perp)^*, I(\top)^*\}).$$

From Lemma 5.1, we can see that they are lattices of filters (resp. ideals) of $\underline{\phi(D)}$.

Theorem 5.2. *Let $x \in X$.*

(1) $\psi_x : F(\underline{D}) \to F(\underline{L}), F \mapsto L(x, F)$ (*resp.* $\eta_x : I(\underline{D}) \to I(\underline{L}), I \mapsto L(x, I)$) *is an embedding of complete lattices.*

(2) $^* : F(\underline{D}) \to F(\underline{D})^*, F \mapsto F^*$ (*resp.* $^* : I(\underline{D}) \to I(\underline{D})^*, I \mapsto I^*$) *is an isomorphism of complete lattices.*

Proof. It is a direct consequence of Proposition 5.2 and Lemma 5.1 ■

Recall that for any $F \in F(\underline{D})$, $F^\phi = \{f \in L \mid \exists y \in F, \phi_y \sqsubseteq f\} \in F(\underline{L})$ and for any $I \in I(\underline{D})$, $I^\phi = \{f \in L : \exists y \in I, f \sqsubseteq \phi_y\} \in I(\underline{L})$.

Remark 5.3. If X is finite, then for any $F \in F(\underline{D})$, $I \in I(\underline{D})$, the following hold.

(1) F^ϕ is the least filter of $\underline{L}$ containing F^* and $F^\phi = \bigcap_{x \in X} L(x, F)$.

(2) I^ϕ is the least ideal of $\underline{L}$ containing I^* and $I^\phi = \bigcap_{x \in X} L(x, I)$.

(3) $\phi(\underline{D})$ is a subalgebra of $\underline{L}$ isomorphic to $\underline{D}$.

(4) Each filter F^* of $\phi(\underline{D})$ is a base of $\tau(F)$ in $\underline{L}$.

(5) Each ideal I^* of $\phi(\underline{D})$ is a base of $\nu(I)$ in $\underline{L}$.

Theorem 5.3. *Let $\underline{L}$ and ϕ as above, the following hold.*

(1) $\tau : F(\underline{D}) \to F(\underline{L}), F \mapsto \tau(F) = F^\phi$ *is an embedding of lattices. Furthermore, if $\underline{D}$ is a complete dBa, then τ is an embedding of complete lattices.*

(2) $\nu : I(\underline{D}) \to I(\underline{L}), I \mapsto \nu(I) = I^\phi$ *is an embedding of lattices. Furthermore, if $\underline{D}$ is complete dBa, then ν is an embedding of complete lattices.*

Proof. It is an easy consequence of Proposition 4.6. ■

In the sequel, $X = D$ and $L = D^D$. For any $a \in D$, we denote by Φ_a, Ψ_a, η_a, h_a the elements of L defined by: $\Phi_a(x) = a \sqcap x$, $\Psi_a(x) = a \sqcup x, \eta_a(x) = a \wedge x$, $h_a(x) = a \vee x$ and P, Q, M and N the sets:

$$P = \{\eta_a \mid a \in D_\sqcup\}, Q = \{h_a \mid a \in D_\sqcap\},$$
$$M = \{\Phi_a \mid a \in D_\sqcap\} \text{ and } N = \{\Psi_a \mid a \in D_\sqcup\}.$$

The following two lemmas give some properties of the above maps.

Lemma 5.4. *Let $\underline{D}$ be a dBa, $a \in D$ and Φ_a, Ψ_a, h_a and η_a as above. The following hold.*

(1) η_a, h_a, Φ_a *and* Ψ_a *are isotone maps.*

(2) (i) η_a *(resp.* Ψ_a*) is compatible with* $\sqcup$ *and* $\wedge$, (ii) h_a *(resp.* Φ_a*) is compatible with* $\sqcap$ *and* $\vee$.

(3) (i) $\Phi_a \sqsubseteq \eta_a \sqsubseteq \Psi_a$, (ii) $\Phi_a \sqsubseteq h_a \sqsubseteq \Psi_a$.

(4) *If* $a \sqsubseteq b$, *then* (i) $\Phi_a \sqsubseteq \Phi_b$, (ii) $\Psi_a \sqsubseteq \Psi_b$, (iii) $h_a \sqsubseteq h_b$, (iv) $\eta_a \sqsubseteq \eta_b$.

Proof. (1) It is an easy check using the compatibility of $\sqsubseteq$ and the operations of $\underline{D}$.

(2) (i) If $x, y \in D$, then $\eta_a(x) \sqcup \eta_a(y) = (a \wedge x) \sqcup (a \wedge y) = a \wedge (x \sqcup y)$ (by (i) of Proposition 2.3 $= \eta_a(x \sqcup y)$; so η_a is compatible with $\sqcup$. For $\wedge$ we have

$\eta_a(x) \wedge \eta_a(y)$

$= a \wedge x \wedge a \wedge y = (a \wedge a) \wedge (x \wedge y)$ (by commutativity and associativity of$\wedge$)

$= (a \sqcup a) \wedge (x \wedge y) = [(x \wedge y) \wedge a] \sqcup [((x \wedge y) \wedge a]$ (by (i) of Proposition 2.3)

$= a \wedge (x \wedge y)$ (due to $a \wedge x \wedge y \in D_\sqcup$) $= \eta_a(x \wedge y)$.

Hence η_a is conmpatible with $\wedge$. From the properties of a dba Ψ is compatible with $\sqcup$ and $\wedge$.

(3) follows from (14) of Proposition 2.2.

(4) follows also from the compatibility of the relation $\sqsubseteq$ with operations. ■

Lemma 5.5. *Let $\underline{D}$ be a dBa, let $a, b \in D$, and $\underline{L} := (D^D; \sqcap, \sqcup, \neg, \lrcorner, \phi_\bot, \phi_\top)$. The following hold.*

(1) (i) $\eta_a \sqcup \eta_b = \eta_{a\sqcup b}$, (ii) $\eta_a \wedge \eta_b = \eta_{a\wedge b} = \eta_a \circ \eta_b$, (iii) $\eta_a \wedge \eta_a = \eta_a \sqcup \eta_a = \eta_a$.

(2) (i) $h_a \sqcap h_b = h_{a\sqcap b}$, (ii) $h_a \vee h_b = h_{a\vee b} = h_a \circ h_b$, (iii) $h_a \vee h_a = h_a = h_a \sqcap h_a$.

(3) (i) $\Phi_a \sqcap \Phi_b = \Phi_{a\sqcap b} = \Phi_a \circ \Phi_b$, (ii) $\Phi_a \vee \Phi_b = \Phi_{a\vee b}$, (iii) $\Phi_a \sqcap \Phi_a = \Phi_a = \Phi_a \vee \Phi_a$.

(4) (i) $\Psi_a \sqcup \Psi_b = \Psi_{a\sqcup b} = \Psi_a \circ \Psi_b$, (ii) $\Psi_a \wedge \Psi_b = \Psi_{a\wedge b}$, (iii) $\Psi_a \sqcup \Psi_a = \Psi_a = \psi_a \wedge \Psi_a$.

(5) (i) $h_a \sqcap (h_b \vee h_c) = (h_a \sqcap h_b) \vee (h_a \sqcap h_c)$ *and* $h_a \vee (h_a \sqcap h_a) = (h_a \vee h_b) \sqcap (h_a \vee h_c)$.

(ii) $\eta_a \sqcup (\eta_b \wedge \eta_c) = (\eta_a \sqcup \eta_b) \wedge (\eta_a \sqcup \eta_c)$ *and* $\eta_a \wedge (\eta_a \sqcup \eta_a) = (\eta_a \wedge \eta_b) \sqcup (\eta_a \wedge \eta_c)$.

(iii) $\Psi_a \wedge (\Psi_b \sqcup \Psi_c) = (\Psi_a \wedge \Psi_b) \sqcup (\Psi_a \wedge \Psi_c)$ *and* $\Psi_a \sqcup (\Psi_b \wedge \Psi_c) = (\Psi_a \sqcup \Psi_b) \wedge (\Psi_a \sqcup \Psi_c)$.

(iv) $\Phi_a \vee (\Phi_b \sqcap \Phi_c) = (\Phi_a \wedge \Phi_b) \sqcap (\Psi_a \vee \Phi_c)$ *and* $\Phi_a \sqcap (\Phi_b \vee \Phi_c) = (\Phi_a \sqcap \Phi_b) \vee (\Phi_a \sqcup \Phi_c)$.

(6) (i) $\eta_a \circ \Psi_a = \phi_{a\sqcup a}$, (ii) $\Psi_a \circ \Phi_a = \phi_{a\sqcup a}$, (iii) $\Phi_a \circ \Psi_a = \phi_{a\sqcap a}$, (iv) $h_a \circ \Phi_a = \phi_{a\sqcap a}$.

Proof. Recall that for every $f, g \in L = D^D$ and $a, x \in D$, $(f \sqcap g)(x) = f(x) \sqcap g(x), (f \sqcup g)(x) = f(x) \sqcup g(x), (\neg f)(x) = \neg(f(x)), (\lrcorner f)(x) = \lrcorner f(x)$, $\phi_a(x) = a$ and $\phi_\perp$ (resp. $\phi_\top$) is the least (resp. greatest) element of $\underline{L}$. Let $a, b, x \in D$.

(1) For (1) (i) we have

$$(\eta_a \sqcup \eta_b)(x) = \eta_a(x) \sqcup \eta_b(x) = \eta_x(a) \sqcup \eta_x(b) = \eta_x(a \sqcup b) \text{ by (2) of Lemma 5.4} \\ = \eta_{a \sqcup b}(x).$$

So (1) (i) holds. For (1) (ii), we have

$$\begin{aligned}(\eta_a \wedge \eta_b)(x) &= \eta_a(x) \wedge \eta_b(x) = \eta_x(a) \wedge \eta_x(b) \\ &= \eta_x(a \wedge b) \text{ (by (2) of Lemma 5.4 and } \eta_a \text{ compatible with } \wedge) \\ &= \eta_{a \wedge b}(x) = \eta_a \circ \eta_b(x) \text{(using associativity of } \wedge);\end{aligned}$$

therefore (1) (ii) holds.

For (1) (iii) we have $\eta_{a \sqcup a}(x) = x \wedge (a \sqcup a) = (x \wedge a) \sqcup (x \wedge a)$ (by (ii) of Proposition 2.3) $= (a \wedge x)$ (due to $a \wedge x \in D_\sqcup$) $= \eta_a(x)$.

(2), (3) and (4) are obtained similarly to (1).

(5) is obtained from (1), (2) and (3).

(6) Let $x \in D$. (i) we have $\eta_a \circ \Psi_a(x) = a \wedge (a \sqcup x) = a \sqcup a$ (using (ii) of Proposition 2.3) $= \phi_{a \sqcup a}(x)$ and $\Psi_a \circ \eta_a(x) = a \sqcup (a \wedge x) = a \sqcup a = \phi_{a \sqcup a}(x)$ (using axiom (7b). Hence $\eta_a \circ \Psi_a = \Psi_a \circ \eta_a = \phi_{a \sqcup a}$.

(ii) $\Psi_a \circ \Phi_a(x) = a \sqcup (a \sqcap x) = a \sqcup a$ (by axiom (5b)) $= \phi_{a \sqcup a}(x)$ and $\Phi_a \circ \Psi_a(x) = a \sqcap (a \sqcup x) = a \sqcap a$ (by axiom (5a)) $= \phi_{a \sqcap a}(x)$. ■

Furthermore, for any $a \in D_\sqcap$ (resp. $D_\sqcup$) we set $\Phi_a^c := \Phi_{\neg a}, h_a^c := h_{\neg a}$ (resp. $\Psi_a^c := \Psi_{\lrcorner a}$, $\eta_a^c = \eta_{\lrcorner a}$). Lemma 5.5 allows us to consider the folowing algebras:

$$\underline{P} := (P; \wedge, \sqcup, ^c, \eta_{\perp \sqcup \perp}, \eta_\top),\ \underline{Q} := (Q; \sqcap, \vee, ^c, h_\perp, h_{\top \sqcap \top}),$$

$$\underline{M} := (M; \wedge, \sqcup, ^c, \Phi_\perp, \Phi_{\neg \perp}) \text{ and } \underline{N} := (N; \wedge, \sqcup, ^c, \Psi_{\lrcorner \top}, \Psi_\top).$$

We consider also the following maps:

- $\Phi^* : D_\sqcap \to M, a \mapsto \Phi_a$; $\quad \Psi^* : D_\sqcup \to N, a \mapsto \Psi_a$; $\quad \eta^* : D_\sqcup \to P, a \mapsto \eta_a$,
- $h^* : D_\sqcap \to Q, a \mapsto h_a$ and $\Psi_a^* : D_\sqcup \to D_\sqcup, x \mapsto \Psi_a(x)$,
- $\eta_a^* : D_\sqcup \to D_\sqcup, x \mapsto a \wedge x$, $h_a^* : D_\sqcap \to D_\sqcap, x \mapsto x \wedge a$ and $\Phi_a^* : D_\sqcap \to D_\sqcap, x \mapsto a \sqcap x$.

The following Theorem shows that the algebras $\underline{P}$ and $\underline{M}$ (resp. $\underline{Q}$ and $\underline{N}$) are Boolean algebras isomorphic to $\underline{D_\sqcup}$ (resp. $\underline{D_\sqcap}$).

Theorem 5.6. *Let $\underline{D}$ be a dBa, $a \in D$, $\underline{L} = \underline{D}^D$, $\underline{P}, \underline{Q}, \underline{M}$ and $\underline{N}$ as above. Then the following hold.*

(1) *$\underline{P}$ and $\underline{N}$ are Boolean algebras isomorphic to $D_{\sqcup}$.*

(2) *$\underline{M}$ and $\underline{P}$ are Boolean algebras isomorphic to $\underline{D}_{\sqcap}$.*

(3) *The maps Φ_a^*, h_a^*, Ψ_a^*, η_a^* are homomorphisms of distributive lattices.*

Proof. (1) From Lemma 5.5, P, Q, M and N are distributive and complemented lattices. So they are Boolean algebras. Hence (1) and (2) hold.

Now we show that $\underline{P}$ is isomorphic to $\underline{D}_{\sqcap}$. Le $\eta^* : D_{\sqcup} \to P, x \mapsto \eta_a$. η^* is well defined. For any $a \in D_{\sqcup}$, $\eta^*(\lrcorner a) = \eta_{\lrcorner a} = (\eta_a)^c = (\eta^*(a))^c$; further, by (1) (i) and (1) (ii) of Lemma 5.5 we deduce that η^* is a homomorphism of Boolean algebras. By definition, η^* is onto. Let $a, b \in D_{\sqcup}$ such that $\eta^*(a) = \eta^*(b)$, then $\eta_a(a) = \eta_b(a)$, and $\eta_a(b) = \eta_b(b)$; therefore $a \wedge a = b \wedge b$; so $a \wedge a = a \sqcup a, b \wedge b = b \sqcup b = b$. It follows that $a = b$ and η^* is one-to-one. Thus η^* is an isomorphism. A similar argument shows that the rest are also isomorphisms.

(3) follows from (1) and (2) of Lemma 5.4. ■

Corollary 5.4. *Let $\underline{D}$ be a dBa, $\underline{M}, \underline{P}, \underline{Q}, \underline{N}$ be the Boolean algebras of Theorem* 5.6. *The following hold.*

(1) *The map $\Phi : D \to L, a \mapsto \Phi_a$ and $h : D \mapsto L, a \mapsto h_a$ are compatible with $\sqsubseteq, \vee$ and $\sqcap$.*

(2) *The maps $\Psi : D \to L, a \mapsto \Psi_a$ and $\eta : D \to L, a \mapsto \eta_a$ are compatible with $\sqcup, \wedge$ and $\sqsubseteq$.*

Proof. It follows from Lemma 5.5. ■

6. CONCLUSION

In this work, we have described filter (ideal) generated by an arbitrary subset of a double Boolean algebra and we have shown that its principal filters (resp. ideals) form a bounded sublattice of the lattice of filters (resp. ideals) and are (non necessary isomorphic) Boolean algebras. We have also shown that $\mathcal{F}(\underline{D})$ (resp. $\mathcal{I}(\underline{D})$) and $\mathcal{F}(\underline{D_{\sqcap}})$ (resp. $\mathcal{I}(\underline{D_{\sqcup}})$) are isomorphic algebraic lattices. We have also shown for dBas that are not Boolean algebras, primary filters (resp. ideals) are exactly ultrafilters (resp. maximal ideals) and primary filters (resp. ideals) need not be a prime filter (resp. ideal). We have investigated some properties of homomorphisms of dBas. For a double Boolean algebra $\underline{D}$ and an arbitrary non-empty set X, we have that $\underline{D}$ embeds in the power $\underline{L} := \underline{D}^X$ of $\underline{D}$, and $\underline{D}$ is contextual, fully contextual (resp. trivial) if and only if $\underline{L}$ is contextual, fully contextual (resp. trivial); and if $\underline{D}$ is complete, then $\underline{L}$ is complete. We have also seen that the lattice $\mathcal{F}(\underline{D})$ (resp. $\mathcal{I}(\underline{D})$ embeds in $\mathcal{F}(\underline{L})$ (resp. $\mathcal{I}(\underline{L})$). The purpose of our future work is to investigate the variety of dBas and some useful properties of protoconcepts algebras with possible applications in formal concepts analysis.

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