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# COHERENT LATTICES

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**Abstract**

The notion of coherent lattices is introduced and established relations between a coherent lattice and that of a generalized Stone lattice, Boolean algebra, quasi-complemented lattice, and normal lattice. A set of equivalent conditions is given for every sublattice of a lattice to become a coherent lattice. Some equivalent conditions are given for every interval of a lattice to become a coherent sublattice. Coherent lattices are characterized with the help of certain properties of filters and dense elements.

**Keywords:** Coherent lattice, generalized Stone lattice, Boolean algebra, quasi-complemented lattice, normal lattice.

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## 1. Introduction

In 1968, the theory of relative annihilators was introduced in lattices by Mark Mandelker [11] and he characterized distributive lattices in terms of their relative annihilators. Later many authors introduced the concept of annihilators in

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the structures of rings as well as lattices and characterized many algebraic structures in terms of annihilators. Speed [15] and Cornish [4, 6] made an extensive study of annihilators in distributive lattices. Cornish introduced the notion of normal lattices [4] and characterized the normal lattices using minimal prime ideals and congruences. In [4], he introduced the notion of quasi-complemented lattices and characterized the class of quasi-complemented lattices using the annulets and congruences. In [4], he introduced the notion of generalized Stone lattices and studied the interconnections among generalized Stone lattices, normal lattices and quasi-complemented lattices. The theory of pseudo-complements in lattices, and particularly in distributive lattices was developed by Stone [16], Frink [8], and George Gratzer [9]. Later many authors like Speed [15], and Frink [8] etc., extended the study of pseudo-complements to characterize Stone lattices. In [3], Chajda, Halaš and Kühr extensively studied the structure of pseudo-complemented semilattices. In [12], the authors investigated extensively certain properties of $D$-filters of distributive lattices. In this paper, the authors given a set of equivalent conditions for a quasi-complemented lattice to become a Boolean algebra by using the $D$-filters. In [13], the authors investigated the properties of prime $D$-filters and then characterized the minimal prime $D$-filters of distributive lattices using certain congruences.

In this note, the concept of coherent lattices is introduced and proved that every generalized Stone lattice is a coherent lattice. Some equivalent conditions are given for every coherent lattice to become a generalized Stone lattice. Boolean algebras are characterized in terms of annulets and principal ideals of distributive lattices and then a set of equivalent conditions is given for every coherent lattice to become a Boolean algebra. A sufficient condition is given for every quasi-complemented lattice to become a coherent lattice. A sufficient condition is given for every coherent lattice to become a normal lattice. Properties of coherent lattices are generalized to the case of direct product of coherent lattices.

A set of equivalent conditions is given for every sublattice of a lattice to become a coherent sublattice. Some equivalent conditions are derived for every interval of a lattice to become a coherent lattice. Coherent lattices are characterized with the help of the properties of filters and $D$-filters of distributive lattices.

## 2. Preliminaries

The reader is referred to [1, 4, 6, 7, 10, 12, 13, 14] and [15] for the elementary notions and notations of distributive lattices. Some of the preliminary definitions and results are presented for the ready reference of the reader.

**Definition** [1]**.** An algebra $(L, \wedge, \vee)$ of type $(2, 2)$ is called a distributive lattice if for all $x, y, z \in L$, it satisfies the following properties $(1), (2), (3)$ and $(4)$ along with $(5)$ or $(5')$

(1) $x \wedge x = x$, $x \vee x = x$,

(2) $x \wedge y = y \wedge x$, $x \vee y = y \vee x$,

(3) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$, $(x \vee y) \vee z = x \vee (y \vee z)$,

(4) $(x \wedge y) \vee x = x$, $(x \vee y) \wedge x = x$,

(5) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,

(5′) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

A non-empty subset $A$ of a lattice $L$ is called an ideal (filter) of $L$ if $a \vee b \in A$ $(a \wedge b \in A)$ and $a \wedge x \in A$ $(a \vee x \in A)$ whenever $a, b \in A$ and $x \in L$. Define a relation $\leq$ on a lattice $L$ by $x \leq y$ if and only if $x \vee y = y$ or equivalently $x \wedge y = x$. Then $(L, \leq)$ is a partially ordered set. The set $(a] = \{x \in L \mid x \leq a\}$ (resp. $[a) = \{x \in L \mid a \leq x\}$) is called a principal ideal (resp. principal filter) generated by $a$. The set $\mathcal{I}(L)$ of all ideals of a distributive lattice $L$ with 0 forms a complete distributive lattice. The set $\mathcal{F}(L)$ of all filters of a distributive lattice $L$ with 1 forms a complete distributive lattice. A proper ideal (resp. filter) $P$ of a distributive lattice $L$ is said to be *prime* if for any $x, y \in L$, $x \wedge y \in P$ (resp. $x \vee y \in P$) implies $x \in P$ or $y \in P$. A proper ideal (resp. filter) $P$ of a lattice $L$ is called *maximal* if there exists no proper ideal (resp. filter) $Q$ such that $P \subset Q$. A proper ideal (resp. filter) $P$ of a distributive lattice is *minimal* [4] if there exists no prime ideal (resp. filter) $Q$ such that $Q \subset P$.

For any non-empty subset $A$ of a distributive lattice $L$ with 0, the annulet [6] of $a$ is define as the set $(a)^* = \{x \in L \mid x \wedge a = 0\}$. For any $a \in L$, $(a)^*$ is an ideal of the lattice $L$. An element $x \in L$ is called *dense* if $(x)^* = \{0\}$ and the set of all dense elements of a lattice is denoted by $D$.

**Proposition 1** [15]**.** *Let $L$ be a distributive lattice with 0. For any $a, b, c \in L$,*

(1) $a \leq b$ *implies* $(b)^* \subseteq (a)^*$,

(2) $(a \vee b)^* = (a)^* \cap (b)^*$,

(3) $(a \wedge b)^{**} = (a)^{**} \cap (b)^{**}$,

(4) $(a)^{***} = (a)^*$,

(5) $(a)^* = L$ *if and only if* $a = 0$.

**Theorem 2** [10]**.** *A prime ideal $P$ of a distributive lattice is a minimal prime ideal if and only if to each $x \in P$ there exists $y \notin P$ such that $x \wedge y = 0$* (or equivalently, for any $x \in L$, $x \notin P$ if and only if $(x)^* \subseteq P$).

A distributive lattice $L$ with 0 is called a *normal lattice* [4] if every prime ideal contains a unique minimal prime ideal. A distributive lattice $L$ with 0 is called a *quasi-complemented lattice* [7] if to each $x \in L$, there exists $x' \in L$ such that $x \wedge x' = 0$ and $x \vee x'$ is dense. A distributive lattice with 0 and dense elements is quasi-complemented if and only if to each $x \in L$, there exists $x' \in L$ such that $(x)^{**} = (x')^{*}$. A distributive lattice $L$ with 0 is called a *generalized Stone lattice* [7] if it satisfies the property: $(x)^{*} \vee (x)^{**} = L$ for all $x \in L$. The pseudo-complement $b^*$ of an element $b$ is the element satisfying

$$a \wedge b = 0 \;\Leftrightarrow\; a \wedge b^* = a \;\Leftrightarrow\; a \leq b^*$$

where $\leq$ is the induced order of $L$. Every pseudo-complemented distributive lattice is a quasi-complemented lattice. In a pseudo-complemented distributive lattice, we have $(x^*] = (x)^*$ for any $x \in L$.

**Theorem 3** [4]**.** *Following are equivalent in a distributive lattice $L$ with 0:*

(1) *$L$ is normal;*

(2) *for any $x, y \in L$, $x \wedge y = 0$ implies $(x)^* \vee (y)^* = L$;*

(3) *for any $x, y \in L$, $(x)^* \vee (y)^* = (x \wedge y)^*$.*

**Theorem 4** [4]**.** *A distributive lattice $L$ with 0 is a generalized Stone lattice if and only if it satisfies the following conditions:*

(1) *$L$ is quasi-complemented,*

(2) *$L$ is normal.*

A lattice $L$ is called relatively complemented if for any $a, b \in L$, the interval $[a, b]$ is a complemented lattice. A lattice $L$ is relatively complemented if $[0, a]$ is complemented for any $a \in L$.

**Theorem 5** [14]**.** *A distributive lattice $L$ with 0 is relatively complemented if and only if every prime ideal of $L$ is a minimal prime ideal.*

A filter $F$ of a distributive lattice $L$ is called a *$D$-filter* [12] if $D \subseteq F$. A prime $D$-filter of a distributive lattice is *minimal* if it is the minimal element in the poset of all prime $D$-filters. A prime $D$-filter of a distributive lattice is minimal [13] if and only if to each $x \in P$, there exists $y \notin P$ such that $x \vee y \in D$. For any non-empty subset $A$ of a lattice, we define $A^{\circ} = \{x \in L \mid x \vee y \in D \text{ for all } y \in A\}$. Clearly $A^{\circ}$ is a $D$-filter of $L$. For $A = \{a\}$, we consider $\{a\}^{\circ}$ by $(a)^{\circ}$.

**Proposition 6.** [13] *Let $A, B$ be two subsets of a distributive lattice $L$. Then*

(1) *$A \subseteq B$ implies $B^{\circ} \subseteq A^{\circ}$,*

(2) *$A \subseteq A^{\circ\circ}$,*

(3) *$A^{\circ\circ\circ} = A^{\circ}$,*

(4) *$A^{\circ} = L$ if and only if $A \subseteq D$.*

Throughout this note, all lattices are bounded and distributive unless otherwise mentioned.

## 3. Coherent lattices

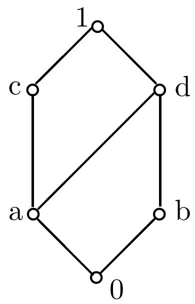
In this section, the notion of coherent lattices is introduced. Relations of the coherent lattices with the classes of generalized Stone lattices, Boolean algebras, quasi-complemented lattices, normal lattices are investigated. Coherent lattices are characterized with the help of filters and $D$-filters.

**Definition.** A lattice $L$ is called a *coherent lattice* if, for all $x, y \in L$,

$$x \vee y \in D \text{ implies } (x)^{**} \vee (y)^{**} = L.$$

Since every non-zero element of a chain (totally ordered set) is a dense element, every chain is a coherent lattice. Obviously, every dense lattice (i.e., $(x)^* = \{0\}$ for all $0 \neq x \in L$) is coherent. In the following example, we observe a non-trivial example of a coherent lattice.

**Example 7.** Consider the following bounded and finite distributive lattice $L = \{0, a, b, c, d, 1\}$ whose Hasse diagram is given by:




Observed that $(a)^* = \{0, b\}$, $(b)^* = \{0, a, c\}$, $(c)^* = \{0, b\}$ and $(d)^* = \{0\}$. Clearly $d, 1$ are the only dense elements in $L$. Also $(a)^{**} = \{0, a, c\}$, $(b)^{**} = \{0, b\}$, $(c)^{**} = \{0, a, c\}$ and $(d)^{**} = L$. It can be routinely verified that $L$ is coherent.

**Proposition 8.** *Every generalized Stone lattice is a coherent lattice.*

***Proof.*** Assume that $L$ is a generalized Stone lattice. Let $x, y \in L$ be such that $x \vee y \in D$. By Theorem 4, $L$ is quasi-complemented and normal. Then there exist $x', y' \in L$ such that $(x)^{**} = (x')^*$ and $(y)^{**} = (y')^*$. Hence, we get

$$\begin{aligned}
(x)^{**} \vee (y)^{**} &= (x')^* \vee (y')^* \\
&= (x' \wedge y')^* && \text{since } L \text{ is normal} \\
&= (x' \wedge y')^{***} && \text{by Proposition 1(4)} \\
&= \{(x')^{**} \cap (y')^{**}\}^* && \text{by Proposition 1(3)} \\
&= \{(x)^* \cap (y)^*\}^* \\
&= \{(x \vee y)^*\}^* \\
&= L && \text{since } x \vee y \in D
\end{aligned}$$

Hence $(x)^{**} \vee (y)^{**} = L$ for all $x, y \in L$ with $x \vee y \in D$. Thus $L$ is coherent. ■

In the following theorem, a set of equivalent conditions is given for a coherent lattice to become a generalized Stone lattice.

**Theorem 9.** *Let $L$ be a coherent lattice. Then the following are equivalent:*

(1) *$L$ is a generalized Stone lattice;*

(2) *every prime $D$-filter is a minimal prime $D$-filter;*

(3) *every maximal filter is a minimal prime $D$-filter;*

(4) *$L$ is quasi-complemented.*

***Proof.*** (1)⇒(2): Assume that $L$ is a generalized Stone lattice. Let $P$ be a prime filter of $L$. Let $d \in D$ and $x \in P$. Since $L$ is a generalized Stone lattice, we get $(x)^* \vee (x)^{**} = L$. Hence $d \in (x)^* \vee (x)^{**}$. Thus $a \vee b = d \in D$ for some $a \in (x)^*$ and $b \in (x)^{**}$. Hence $a \wedge x = 0$ and $(x)^* \subseteq (b)^*$. Suppose $a \in P$. Then $0 = a \wedge x \in P$, which is a contradiction. Thus, we must have $a \notin P$. Now,

$$\begin{aligned} a \vee b \in D \quad &\Rightarrow \quad (a)^* \cap (b)^* = \{0\} \\ &\Rightarrow \quad (a)^* \cap (x)^* = \{0\} \qquad \text{since } (x)^* \subseteq (b)^* \\ &\Rightarrow \quad (a \vee x)^* = \{0\} \end{aligned}$$

which means that $a \vee x \in D$. Therefore $P$ is a minimal prime $D$-filter of $L$.

(2)⇒(3): Since every maximal filter is a prime $D$-filter, it is clear.

(3)⇒(4): Assume condition (3). Let $x \in L$. Suppose $0 \notin [x) \vee (x)^\circ$. Then there exists a maximal filter $M$ such that $[x) \vee (x)^\circ \subseteq M$. Hence $x \in M$ and $(x)^\circ \subseteq M$. By condition (3), $M$ will be a minimal prime $D$-filter. Since $(x)^\circ \subseteq M$, we get $x \notin M$, which is a contradiction. Hence $0 \in [x) \vee (x)^\circ$. Thus $x \wedge a = 0$ for some $a \in (x)^\circ$. Hence $x \vee a \in D$. Therefore $L$ is quasi-complemented.

(4)⇒(1): Assume that $L$ is quasi-complemented. Let $x \in L$. Since $L$ is quasi-complemented, there exists $x' \in L$ such that $x \wedge x' = 0$ and $x \vee x' \in D$. Since $L$ is coherent, we get $(x)^{**} \vee (x')^{**} = L$. Since $x \wedge x' = 0$, we get $(x')^{**} \subseteq (x)^*$. Hence $L = (x)^{**} \vee (x')^{**} \subseteq (x)^{**} \vee (x)^*$. Therefore $L$ is a generalized Stone lattice. ■

Since every pseudo-complemented lattice is quasi-complemented, the following corollary is a direct consequence of the above theorem:

**Corollary 10.** *Let $L$ be a pseudo-complemented lattice. Then $L$ is a coherent lattice if and only if it is a generalized Stone lattice.*

**Corollary 11.** *A quasi-complemented and coherent lattice is normal.*

***Proof.*** Follows from Theorem 9 and Theorem 4. ■

**Corollary 12.** *Any quasi-complemented and normal lattice is coherent.*

***Proof.*** Follows from Theorem 9 and Theorem 4. ∎

**Theorem 13.** *A lattice $L$ is Boolean if and only if $(x] \vee (x)^* = L$ for all $x \in L$.*

***Proof.*** Assume that $L$ is a Boolean algebra. Let $x \in L$. Since $L$ is Boolean, there exists $x' \in L$ such that $x \wedge x' = 0$ and $x \vee x' = 1$. Hence $x' \in (x)^*$ and $(x] \vee (x'] = (1] = L$. Thus $(x'] \subseteq (x)^*$. Therefore $(x] \vee (x)^* = L$.

Conversely, assume the condition. Let $x \in L$. Hence $(x)^* \vee (x] = L$. Thus $1 \in (x)^* \vee (x]$. Hence $a \vee x = 1$ for some $a \in (x)^*$. Since $a \in (x)^*$, we get $a \wedge x = 0$. Thus $a$ is the complement of $x$ in $L$. Therefore $L$ is a Boolean algebra. ∎

**Proposition 14.** *Every Boolean algebra is a generalized Stone lattice.*

***Proof.*** Assume that $L$ is a Boolean algebra. Let $x \in L$. Suppose $(x)^* \vee (x)^{**} \neq L$. Then there exists prime ideal $P$ such that $(x)^* \vee (x)^{**} \subseteq P$. Hence $(x)^* \subseteq P$ and $x \in (x)^{**} \subseteq P$. Since $L$ is Boolean, there exists $x' \in L$ such that $x \wedge x' = 0$ and $x \vee x' = 1$. Suppose $x' \in P$. Then $1 = x \vee x' \in P$, which is a contradiction. Hence $x' \notin P$. Therefore $P$ is minimal. Since $(x)^* \subseteq P$, we get $x \notin P$ which is a contradiction. Thus $(x)^* \vee (x)^{**} = L$. Hence $L$ is a generalized Stone lattice. ∎

**Corollary 15.** *Every Boolean algebra is a coherent lattice.*

***Proof.*** It follows from Proposition 8 and Proposition 14. ∎

In general, the converse of Proposition 14 is not true. For, consider the coherent lattice given in Example 7. Observe that $L$ is not a Boolean algebra because the element $a$ has no complement in $L$. However, a set of equivalent conditions is given for a coherent lattice to become a Boolean algebra.

**Theorem 16.** *A coherent lattice $L$ is a Boolean algebra if and only if it satisfies the following conditions;*

(1) *$L$ is quasi-complemented;*

(2) *every principal ideal is an annihilator ideal.*

***Proof.*** Given that $L$ is a coherent lattice. Assume that $L$ satisfies conditions (1) and (2). Since $L$ is quasi-complemented and coherent, by Theorem 9, we get that $L$ is a generalized Stone lattice. Let $x \in L$. Then $(x)^* \vee (x)^{**} = L$. By (2), we get $(x] = (x)^{**}$. Hence $(x] \vee (x)^* = L$. By Theorem 13, $L$ is a Boolean algebra.

Conversely, assume that $L$ is Boolean. Clearly $L$ is quasi-complemented. Let $x \in L$. By Proposition 14, $L$ is a generalized Stone lattice. Hence $(x)^{**} \vee (x)^* = L$. By Theorem 13, we get $(x] \vee (x)^* = L$. It is clear that $(x] \cap (x)^* = \{0\}$ and $(x)^{**} \cap (x)^* = \{0\}$. Since the lattice $\mathcal{I}(L)$ of all ideals of the lattice $L$ is distributive, by the cancellation property, we get $(x] = (x)^{**}$. Thus every principal ideal is an annihilator ideal. ∎

**Corollary 17.** *A generalized Stone lattice is a Boolean algebra if and only if every principal ideal of the lattice is an annihilator ideal.*

***Proof.*** It follows from Theorem 9 and Theorem 16. ∎

**Proposition 18.** *A lattice in which every maximal ideal is non-dense is a coherent lattice.*

***Proof.*** Let $L$ be a lattice in which every maximal ideal is non-dense. Let $x, y \in L$ be such that $x \vee y \in D$. Suppose $(x)^{**} \vee (y)^{**} \neq L$. Then there exists a maximal ideal $M$ such that $(x)^{**} \vee (y)^{**} \subseteq M$. Hence $(x)^{**} \subseteq M$ and $(y)^{**} \subseteq M$. Thus $M^* \subseteq (x)^*$ and $M^* \subseteq (y)^*$, which gives $M^* \subseteq (x)^* \cap (y)^* = (x \vee y)^* = \{0\}$ because of $x \vee y \in D$. Thus $M$ is dense, which is a contradiction. Hence $(x)^{**} \vee (y)^{**} = L$. Therefore $L$ is a coherent lattice. ∎

The converse of Proposition 18 is not true. For, consider the coherent lattice given in Example 7. Clearly $L$ is possessing two maximal ideals, namely $M_1 = \{0, a, c\}$ and $M_2 = \{0, a, b, d\}$. Observe that $M_2^* = \{0\}$ but $M_1^* = \{0, b\} \neq \{0\}$. The following result about the direct products of coherent lattices is of intrinsic interest. First we need the following lemma whose proof is routine.

**Lemma 19.** *For any positive integer $n$, let $L_1, L_2, \ldots, L_n$ be $n$ lattices. For any $a_1 \in L_1, a_2 \in L_2, \ldots, a_n \in L_n$, the following properties hold:*

(1) $(a_1, a_2, \ldots, a_n)^* = (a_1)^* \times (a_2)^* \times \cdots \times (a_n)^*$,

(2) $(a_1, a_2, \ldots, a_n)^* \vee (b_1, b_2, \ldots, b_n)^* = (a_1 \vee b_1, a_2 \vee b_2, \ldots, a_n \vee b_n)^*$,

(3) $(a_1, a_2, \ldots, a_n)^{**} = (a_1)^{**} \times (a_2)^{**} \times \cdots \times (a_n)^{**}$.

***Proof.*** The proof of (1) and (2) is routine.

(3) Let us denote $(x)_i = (x_1, x_2, \ldots, x_n)$ where $x_1 \in L_1, x_2 \in L_2, \ldots, x_n \in L_n$. For any $(a)_i \in L_1 \times L_2 \times \cdots \times L_n$, we have

$$\begin{aligned}
(x)_i \in (a)_i^{**} &\Leftrightarrow (a)_i^* \subseteq (x)_i^* \\
&\Leftrightarrow (a_1)^* \times (a_2)^* \times \cdots \times (a_n)^* \subseteq (x_1)^* \times (x_2)^* \times \cdots \times (x_n)^* \\
&\Leftrightarrow (a_i)^* \subseteq (x_i)^* \qquad \text{for } i = 1, 2, \ldots, n \\
&\Leftrightarrow (x_i)^{**} \subseteq (a_i)^{**} \qquad \text{for } i = 1, 2, \ldots, n \\
&\Leftrightarrow (x)_i \in (x)_i^{**} \subseteq (a)_i^{**}
\end{aligned}$$

Therefore $(a_1, a_2, \ldots, a_n)^{**} = (a_1)^{**} \times (a_2)^{**} \times \cdots \times (a_n)^{**}$. ∎

**Theorem 20.** *Let $L_1, L_2, \ldots, L_n$ (where $n$ is a positive integer) be a finite family of lattices. Then the product lattice $L_1 \times L_2 \times \cdots \times L_n$ (with point-wise operations) is coherent if and only if $L_1, L_2, \ldots, L_n$ are coherent.*

***Proof.*** Assume that $L_1 \times L_2 \times \cdots \times L_n$ is a coherent lattice. Let $D_1, D_2, \ldots, D_n$ be the sets containing dense elements of $L_1, L_2, \ldots, L_n$ respectively. Let $a, b \in L_1$ be such that $a \vee b \in D_1$. Choose $d_2 \in D_2, d_3 \in D_3, \ldots, d_n \in D_n$. Then

$$(a, d_2, d_3, \ldots, d_n) \vee (b, d_2, d_3, \ldots, d_n) = (a \vee b, d_2, d_3, \ldots, d_n) \in D_1 \times D_2 \times \cdots \times D_n.$$

Since $L_1 \times L_2 \times \cdots \times L_n$ is coherent, we get

$$(a, d_2, d_3, \ldots, d_n)^{**} \vee (b, d_2, d_3, \ldots, d_n)^{**} = L_1 \times L_2 \times \cdots \times L_n.$$

Let $z \in L_1$. Then $(z, d_2, d_3, \ldots, d_n) \in L_1 \times L_2 \times \cdots \times L_n$. Hence, there exists $(s_1, s_2, \ldots, s_n) \in (a, d_2, d_3, \ldots, d_n)^{**}$ and $(t_1, t_2, \ldots, t_n) \in (b, d_2, d_3, \ldots, d_n)^{**}$ such that

$$(z, d_2, d_3, \ldots, d_n) = (s_1, s_2, \ldots, s_n) \vee (t_1, t_2, \ldots, t_n).$$

Therefore $z = s_1 \vee t_1$ where $s_1 \in (a)^{**}$ and $t_1 \in (b)^{**}$. Hence $(a)^{**} \vee (b)^{**} = L_1$. Therefore $L_1$ is coherent. Similarly, it can be proved that $L_2, L_3, \ldots, L_n$ are coherent lattices. Converse follows from the fact that $(a_1, a_2, \ldots, a_n)^{**} = (a_1)^{**} \times (a_2)^{**} \times \cdots \times (a_n)^{**}$ for any $(a_1, a_2, \ldots, a_n) \in L_1 \times L_2 \times \cdots \times L_n$. ■

**Lemma 21.** *If a lattice $L$ is relatively complemented, then every chain has at most three elements.*

***Proof.*** Assume that $L$ is relatively complemented. Suppose there exist three elements $x, y, z \in L - \{0\}$ such that $0 < x < y < z$. Clearly $x \in [0, x \vee y]$. Since $L$ is relatively complemented, there exists $t \in L$ such that $x \wedge t = 0$ and $x \vee t = x \vee y = y$. Since $x \wedge t = 0$, by the assumption, we get $y = x \vee t = 1$. This is absurd. Therefore every chain of $L$ has at most three elements. ■

A sublattice $S$ of a lattice $L$ is called a *$D$-sublattice* if $0 \in S$ and $S \cap D \neq \emptyset$. An ideal $J$ of a lattice is called $D$-ideal if $J \cap D \neq \emptyset$. Clearly every $D$-ideal is a $D$-sublattice.

**Theorem 22.** *The following assertions are equivalent in a lattice $L$:*

(1) *every $D$-sublattice is coherent;*

(2) *for any $x, y \in L - \{0\}$, $x \wedge y = 0$ implies $x \vee y = 1$;*

(3) *$L$ is a dense lattice or $L$ is relatively complemented.*

***Proof.*** (1)⇒(2): Assume that every $D$-sublattice of $L$ is coherent. Let $x, y \in L - \{0\}$ be such that $x \wedge y = 0$. Suppose that $x \vee y \neq 1$. Choose $1 \neq z \in L$ such that $x \vee y < z$. Now, consider the sublattice $L_1 = \{0, x, y, x \vee y, z\}$. Clearly $x \vee y \in D_1$ and so $L_1$ is a $D$-sublattice. Now, $(x)^{**} \vee (y)^{**} = \{0, x\} \vee \{0, y\} = L_1 - \{z\} \neq L_1$. Hence $L_1$ is not coherent which contradicts the assumption. Therefore $x \vee y = 1$.

(2)⇒(3): Assume condition (2). Suppose $L$ is non-dense. Then $\{0\}$ is not a prime ideal of $L$. Let $P$ be a prime ideal of $L$. Suppose $P$ is not minimal. Then there exists minimal prime ideal $M$ such that $M \subset P$. Choose $0 \neq x \in M$. Then $(x)^* \cap P = \{0\}$, otherwise $y \in (x)^* \cap P$. Then $x \wedge y = 0$ and $y \in P$. By the hypothesis, we get $x \vee y = 1$. Since $x \in P$, we get $1 = x \vee y \in P$ which is a contradiction. Hence $(x)^* \cap P = \{0\} \subseteq M$. Since $M$ is prime and $M \subset P$, we must have $(x)^* \subseteq M$. This contradicts the fact that $M$ is minimal. Therefore $P$ is minimal. By Theorem 5, $L$ is relatively complemented.

(3)⇒(1): Assume condition (3). Let $L_1$ be a $D$-sublattice of $L$ and $D_1$ is the set of all dense elements of $L_1$. If $L$ is dense, then we are through. Suppose $L$ is relatively complemented. By Lemma 21, every chain in $L$ has at most three elements. Let $x, y \in L_1$ be such that $x \vee y \in D_1$. Suppose $x \in D_1$ or $y \in D_1$. Then clearly $(x)^{**}_{L_1} \vee (y)^{**}_{L_1} = L_1$. Suppose $x \notin D_1$ and $y \notin D_1$. Suppose $0 < x \leq x \vee y$. If $x = x \vee y$, then $x \in D_1$ which is a contradiction. Hence $0 < x < x \vee y$. Since every chain has at most three elements, $x \vee y$ will be the greatest element of $L_1$. Hence $x \vee y \in (x)^{**}_{L_1} \vee (y)^{**}_{L_1}$. Therefore $(x)^{**}_{L_1} \vee (y)^{**}_{L_1} = L_1$. ∎

**Theorem 23.** *The following assertions are equivalent in a lattice $L$:*

(1) *$L$ is coherent;*

(2) *each proper $D$-ideal is a coherent sublattice;*

(3) *for each $d \in D$, $[0, d]$ is a coherent sublattice.*

***Proof.*** (1)⇒(2): Assume that $L$ is a coherent lattice. Let $J$ be a $D$-ideal of $L$ with $J \neq L$. Suppose $x, y \in J$ be such that $x \vee y \in D_J \subseteq D$. Since $L$ is coherent, we get $(x)^{**} \vee (y)^{**} = L$. Write $(a)^{**}_J = J \cap (a)^{**}$ for any $a \in J$. Clearly $(a)^{**}_J$ is an ideal in $J$ with $(a)^*_J$ is an annulet of $a$ in $J$. Now, we get

$$J = J \cap L = J \cap \{(x)^{**} \vee (y)^{**}\} = \{J \cap (x)^{**}\} \vee \{J \cap (y)^{**}\} = (x)^{**}_J \vee (y)^{**}_J$$

which yields that $J$ is a coherent sublattice of $L$.

(2)⇒(3): It is obvious because of $[0, d]$ is a proper $D$-ideal for any $d \in D$.

(3)⇒(1): By taking $d = 1$, the proof follows. ∎

**Definition.** For any non-empty subset $A$ of a lattice $L$, define

$$A^\tau = \{x \in L \mid (a)^{**} \vee (x)^{**} = L \text{ for all } a \in A\}$$

Clearly $\{0\}^\tau = D$ and $L^\tau = D$. For any $a \in L$, we denote $(\{a\})^\tau$ by $(a)^\tau$. Then it is obvious that $(0)^\tau = D$ and $(1)^\tau = L$.

**Proposition 24.** *For any non-empty subset $A$ of $L$, $A^\tau$ is a $D$-filter of $L$.*

***Proof.*** Clearly $D \subseteq A^\tau$. Let $x, y \in A^\tau$. For any $a \in A$, we get $(x \wedge y)^{**} \vee (a)^{**} = \{(x)^{**} \cap (y)^{**}\} \vee (a)^{**} = \{(x)^{**} \vee (a)^{**}\} \cap \{(y)^{**} \vee (a)^{**}\} = L \cap L = L$. Hence $x \wedge y \in A^\tau$. Again, let $x \in A^\tau$ and $x \leq y$. Then $(x)^{**} \vee (a)^{**} = L$ for any $a \in A$ and $(x)^{**} \subseteq (y)^{**}$. For any $c \in A$, we get $L = (x)^{**} \vee (c)^{**} \subseteq (y)^{**} \vee (c)^{**}$. Hence $y \in A^\tau$. Therefore $A^\tau$ is a $D$-filter of $L$. ∎

The following lemma is a direct consequence of the above definition.

**Lemma 25.** *For any two non-empty subsets $A$ and $B$ of a lattice $L$, the following properties hold:*

(1) $A^\tau = \bigcap_{a\in A} (a)^\tau$,

(2) $A \cap A^\tau \subseteq D$,

(3) $A \subseteq B$ *implies* $B^\tau \subseteq A^\tau$,

(4) $A \subseteq A^{\tau\tau}$,

(5) $A^{\tau\tau\tau} = A^\tau$,

(6) $A^\tau = L$ *if and only if* $A \subseteq D$.

In case of filters, we have the following result.

**Proposition 26.** *For any two filters $F, G$ of a lattice $L$, $(F \vee G)^\tau = F^\tau \cap G^\tau$.*

***Proof.*** Clearly $(F \vee G)^\tau \subseteq F^\tau \cap G^\tau$. Conversely, let $x \in F^\tau \cap G^\tau$. Let $c \in F \vee G$ be an arbitrary element. Then $c = i \wedge j$ for some $i \in F$ and $j \in G$. Now $(x)^{**} \vee (c)^{**} = (x)^{**} \vee (i \wedge j)^{**} = (x)^{**} \vee \{(i)^{**} \cap (j)^{**}\} = \{(x)^{**} \vee (i)^{**}\} \cap \{(x)^{**} \vee (j)^{**}\} = L \cap L = L$. Thus $x \in (F \vee G)^\tau$ and therefore $(F \vee G)^\tau = F^\tau \cap G^\tau$. ■

The following corollary is a direct consequence of the above results.

**Corollary 27.** *Let $L$ be a lattice and $a, b \in L$. Then the following hold:*

(1) $a \leq b$ *implies* $(a)^\tau \subseteq (b)^\tau$,

(2) $(a \wedge b)^\tau = (a)^\tau \cap (b)^\tau$,

(3) $(a)^\tau = L$ *if and only if $a$ is dense,*

(4) $a \in (b)^\tau$ *implies* $a \vee b \in D$,

(5) $(a)^* = (b)^*$ *implies* $(a)^\tau = (b)^\tau$.

For any filter $F$ of a lattice $L$, it can be easily observed that $F^\tau \subseteq F^\circ$. However, we derive a set of equivalent conditions for every filter to satisfy the reverse inclusion which leads to a characterization of coherent lattices.

**Theorem 28.** *The following assertions are equivalent in a lattice $L$:*

(1) *$L$ is a coherent lattice;*

(2) *for any two filters $F, G$ of $L$, $F \cap G \subseteq D$ if and only if $F \subseteq G^\tau$;*

(3) *for any filter $F$ of $L$, $F^\tau = F^\circ$;*

(4) *for any $a \in L$, $(a)^\tau = (a)^\circ$.*

***Proof.*** (1)⇒(2): Assume that $L$ is coherent. Let $F$ and $G$ be two filters of $L$. Suppose $F \cap G \subseteq D$. Let $x \in F$. For any $a \in G$, we get $x \vee a \in F \cap G \subseteq D$. Hence $x \vee a \in D$. Since $L$ is coherent, we get $(x)^{**} \vee (a)^{**} = L$ for all $a \in G$. Thus $x \in G^{\tau}$. Therefore $F \subseteq G^{\tau}$. Conversely, suppose that $F \subseteq G^{\tau}$. Let $x \in F \cap G$. Then $x \in F \subseteq G^{\tau}$. Hence $x \in G \cap G^{\tau} \subseteq D$. Therefore $F \cap G \subseteq D$.

(2)⇒(3): Assume condition (2). Let $F$ be a filter of $L$. Clearly $F^{\tau} \subseteq F^{\circ}$. Conversely, let $x \in F^{\circ}$. Hence, for any $a \in F$, we have

$$\begin{aligned} x \vee a \in D &\Rightarrow [x) \cap [a) \subseteq D \\ &\Rightarrow [x) \subseteq [a)^{\tau} \subseteq (a)^{\tau} \qquad\qquad \text{by (2)} \\ &\Rightarrow (x] \subseteq \bigcap_{a \in F} (a)^{\tau} = F^{\tau} \\ &\Rightarrow x \in F^{\tau} \end{aligned}$$

which concludes that $F^{\circ} \subseteq F^{\tau}$. Therefore $F^{\circ} = F^{\tau}$.

(3)⇒(4): Assume condition (3). Let $a \in L$. Clearly $(a)^{\tau} \subseteq (a)^{\circ}$. Conversely, let $x \in (a)^{\circ}$. Since $([a))^{\circ} = (a)^{\circ}$, by (3), we get $x \in ([a))^{\circ} = ([a))^{\tau}$. Since $\{a\} \subseteq [a)$, we get $x \in ([a))^{\tau} \subseteq (\{a\})^{\tau} = (a)^{\tau}$.

(4)⇒(1): Assume condition (4). Let $a, b \in L$ and suppose $a \vee b \in D$. Then $a \in (b)^{\circ} = (b)^{\tau}$. Hence $(a)^{**} \vee (b)^{**} = L$. Therefore $L$ is a coherent lattice. ■

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