Discussiones Mathematicae
General Algebra and Applications xx (xxxx) 1–25

SOME ALGEBRAIC CHARACTERISATIONS OF GENERALISED MIDDLE BOL LOOPS

Benard Osoba[1]

Bells University of Technology
Ota, Ogun State, Nigeria

e-mail: benardomth@gmail.com, b_osoba@bellsuniversity.edu.ng

Abdulkareem Abdulafeez Olalekan

Department of Mathematics
Lagos State University Ojo
Lagos State, 102101 Nigeria

e-mail: abdulafeez.abdulkareem@lasu.edu.ng

Yakub Tunde Oyebo

Department of Mathematics
Lagos State University Ojo
Lagos State, 102101 Nigeria

e-mail: oyeboyt@yahoo.com, yakub.oyebo@lasu.edu.ng

and

Anthony Oyem

Department of Mathematics
Universityof Lagos, Akoka, Lagos

e-mail: tonyoyem@yahoo.com

Abstract

In this article, some algebraic characterisations of generalised middle Bol loop (GMBL) using its parastrophes and holomorph were studied. In particular, it was shown that if the generalised map α is bijective such $\alpha : e \to e$, then the $(12)-$ parastrophe of GMBL is a GMBL. The conditions for $(13)-$ and $(123)-$parastrophes of a GMBL to be GMBL of exponent two were unveiled. We further established that a commutative $(13)-$ and

[1] All correspondence to be addressed to this author.

(123)−parastrophes of GMBL has an inverse properties. (23)− parastrophe of Q was shown to be super $\alpha-$elastic property if it has a middle symmetric while (132)−parastrophe of Q satisfies left $\alpha-$symmetric. It is further shown that a commutative (13)− and (123)− parastrophes of Q are generalised Moufang loops of exponent two. Also, commutative (132)− and (23)− parastophes of Q are shown to be Steiner loops. A necessary and sufficient condition for holomorph of generalised middle Bol loop to be GMBL was presented. The holomorph of a commutative loop was shown to be a commutative generalised middle Bol loop if and only if the loop is a GMBL.

Keywords: loop, parastrophe, Holomorph, Generalised middle Bol loop.

2020 Mathematics Subject Classification: Primary 20N05; Secondary 08A05.

1 Introduction

1.1 Quasigroups and Loops

Let Q be a non -empty set. Define a binary operation ” ·” on Q. If $x \cdot y \in Q$ for all $x, y \in Q$, then the pair $(Q, \cdot)$ is called a groupoid or magma. If the equations $a \cdot x = b$ and $y \cdot a = b$ have unique solutions $x, y \in Q$ for all $a, b \in Q$ then $(Q, \cdot)$ is called a quasigroup. Let $(Q, \cdot)$ be a quasigroup and let there exist a unique element $e \in Q$ called the identity element such that for all $x \in Q, x \cdot e = e \cdot x = x$, then $(Q, \cdot)$ is called a loop. At times, we shall write xy instead of $x \cdot y$ and stipulate that ” · ” has lower priority than juxtaposition among factors to be multiplied. Let $(Q, \cdot)$ be a groupoid and a be a fixed element in Q, then the left and right translations L_a and R_a of a are respectively defined by $xL_a = a \cdot x$ and $xR_a = x \cdot a$ for all $x \in Q$. It can now be seen that a groupoid

$(Q, \cdot)$ is a quasigroup if its left and right translation mappings are permutations. Since the left and right translation mappings of a quasigroup are bijective, then the inverse mappings L_a^{-1} and R_a^{-1} exist.

Let

$$a \backslash b = bL_a^{-1} = aM_b \qquad \text{and} \qquad a/b = aR_b^{-1} = bM_a^{-1}$$

and note that

$$a \backslash b = c \Longleftrightarrow a \cdot c = b \qquad \text{and} \qquad a/b = c \Longleftrightarrow c \cdot b = a.$$

Thus, for any quasigroup $(Q, \cdot)$, we have two new binary operations; right division $(/)$ and left division $(\backslash)$. M_a is the middle translation for any fixed $a \in Q$. Consequently, $(Q, \backslash)$ and $(Q, /)$ are also quasigroups. Using the operations $(\backslash)$ and $(/)$, the definition of a loop can be restated as follows.

Definition 1.1. A *loop* $(Q, \cdot, /, \backslash, e)$ is a set Q together with three binary operations $(\cdot)$, $(/)$, $(\backslash)$ and one nullary operation e such that

(i) $a \cdot (a\backslash b) = b$, $(b/a) \cdot a = b$ for all $a, b \in Q$,

(ii) $a\backslash a = b/b$ or $e \cdot a = a \cdot e = a$ for all $a, b \in Q$.

We also stipulate that $(/)$ and $(\backslash)$ have higher priority than $(\cdot)$ among factors to be multiplied. For instance, $a \cdot b/c$ and $a \cdot b\backslash c$ stand for $a(b/c)$ and $a(b\backslash c)$ respectively.

In a loop $(Q, \cdot)$ with identity element e, the *left inverse element* of $x \in Q$ is the element $xJ_\lambda = x^\lambda \in Q$ such that

$$x^\lambda \cdot x = e$$

while the *right inverse element* of $x \in G$ is the element $xJ_\rho = x^\rho \in G$ such that

$$x \cdot x^\rho = e.$$

It is well known that every quasigroup $(Q\cdot)$ belongs to a set of six quasigroups, called adjugates by (Fisher, Yates [5] 1934), conjugates by (Stein , 1957) and parastrophes by (Belousov [4], 1967)

A binary groupoid (Q, A) with a binary operation "A" such that in the equality $A(x_1, x_2) = x_3$ knowledge of any 2 elements of x_1, x_2, x_3 uniquely specifies remaining one is called a binary quasigroup. It follows that any quasigroup (Q, A), associate $(3! - 1)$ quasigroups called parastrophes of quasigroup $(Q, A); A(x_1, x_2) = x_3 \iff A^{(12)}(x_2, x_1) = x_3 \iff A^{(13)}(x_3, x_2) = x_1 \iff A^{(23)}(x_1, x_2) = x_2 \iff A^{(123)}(x_2, x_3) = x_1 \iff A^{(132)}(x_3, x_1) = x_2$.[see (Shcherbacov [32], 2008)]. For more on quasigroups and loops, check [31, 33].

1.2 Middle Bol Loop and its Generalisation

Definition 1.2. A loop $(Q, \cdot)$ is called a middle Bol loop if

$$(x/y)(z\backslash x) = (x/(zy))x \text{ or } (x/y)(z\backslash x) = x((zy)\backslash x) \tag{1}$$

for all $x, y \in Q$.

Middle Bol loop were first studied in the work of V. D. Belousov [4], where he gave identity (1) characterizing loops that satisfy the universal anti-automorphic inverse property. After this beautiful characterisation by Belousov and the laying of foundations for a classical study of this structure, Gwaramija in [6] gave isostrophic connection between right(left) with middle Bol loop.

Grecu [16] showed that the right multiplication group of a middle Bol loop coincides with the left multiplication group of the corresponding right Bol loop.

After that, middle Bol loops resurfaced in literature in 1994 and 1996 when Syrbu [20, 21] considered them in relation to the universality of the elasticity law. In 2003, Kuznetsov [15], while studying gyrogroups (a special class of Bol loops) established some algebraic properties of middle Bol loop and designed a method of constructing a middle Bol loop from a gyrogroup.

In 2010, Syrbu [22] studied the connections between structure and properties of middle Bol loops and of the corresponding left Bol loops. It was noted that two middle Bol loops are isomorphic if and only if the corresponding left (right) Bol loops are isomorphic, and a general form of the autotopisms of middle Bol loops was deduced. Relations between different sets of elements, such as nucleus, left (right, middle) nuclei, the set of Moufang elements, the center of a middle Bol loop and left Bol loop were established. In 2012, Grecu and Syrbu [17] proved that two middle Bol loops are isotopic if and only if the corresponding right (left) Bol loops are isotopic.

In 2012, Drapal and Shcherbacov [18] rediscovered the middle Bol identities in a new way. In 2013, Syrbu and Grecu [19] established a necessary and sufficient condition for the quotient loop of a middle Bol loop and of its corresponding right Bol loop to be isomorphic. In 2014, Grecu and Syrbu [24] established that the commutant (centrum) of a middle Bol loop is an AIP-subloop and gave a necessary and sufficient condition when the commutant is an invariant under the existing isostrophy between middle Bol loop and the corresponding right Bol loop and the same authors presented a study of loops with invariant flexibility law under the isostrophy of loop [23].

In 2017, Jaiyéọlá et al. [8] presented the holomorphic structure of middle Bol loop and showed that the holomorph of a commutative loop is a commutative middle Bol loop if and only if the loop is a middle Bol loop and its automorphism group is abelian. Adeniran et al. [1], Jaiyéọlá and Popoola [13] studied generalised Bol loops. It was revealed in [10] that isotopy-isomorphy is a necessary and sufficient condition for any distinct quasigroups to be parastophic invariance relative to the associative law.

(Osoba et al. [25] and [26]) investigate further the multiplication group of middle Bol loop in relation to left Bol loop and the relationship of multiplication groups and isostrophic quasigroups respectively while Jaiyéọlá [11, 12] studied second Smarandache Bol loops. The Smarandache nuclei of second Smarandache Bol loops was further studied by Osoba [27].

(Jaiyéọlá et al. [7], 2015) in furtherance to their exploit obtained new algebraic identities of middle Bol loop, where necessary and sufficient conditions for a bi-variate mapping of a middle Bol loop to have RIP, LIP, RAIP, LAIP and flexible property were presented. Additional algebraic properties of middle Bol loop were announced in (Jaiyéọlá et al. [9], 2021).

The new algebraic connections between right and middle Bol loops and their cores were unveiled by (Osoba and Jaiyéọlá (2022),[28]). More results on the algebraic properties of middle Bol loops using its parastrophes was presented by (Oyebo and Osoba, [30]). The paper revealed some of the algebraic properties the parastrophic structures of middle Bol loop shared with its underline structure. The connections between middle Bol loop and right Bol loop with their crypto-automorphism features were unveiled in [29] by Oyebo et.al. In [14], Bryant-Schneider group of middle Bol loop with some of the isostrophy-group invariance results was linked. It was further shown that some subgroups of the Bryant-Schneider group of a middle Bol loop were isomorphic to the automorphism and pseudo-aumorphism groups of its corresponding right (left) Bol loop.

A generalised middle Bol loop characterised by

$$(x/y)(z^{\alpha}\backslash x^{\alpha}) = x(z^{\alpha}y\backslash x^{\alpha}) \tag{2}$$

was first introduced in [2], as a consequence of a generalised Moufang loop with universal $\alpha-$elastic property where the map $\alpha : Q \mapsto Q$ is a homomorphism. Thus, if $\alpha : x \mapsto x$, then identity of generalised middle Bol loop reduces to the identity of middle Bol loop. The authors in [3], presented the basic algebraic properties of generalised middle Bol loop, where it revealed the necessary and sufficient conditions for the identity to satisfies left(right) inverse and $\alpha-$alternative property was also presented.

Furtherance to earlier studies, this paper investigates some structural characterisation of generalised middle Bol loop using its parastrophes and holomorph. The second section provides preliminaries for necessary background of the study. Section 3 contains the main results where the parastrophic characterisation of generalised middle Bol loop is presented. It is shown that a $(12)-$parastrophe of a generalised middle Bol is also a generalised middle Bol loop and further established the conditions for $(13)-$ and $(123)-$parastrophes of Q to be GMBL. We further investigate the algebraic properties of the parastrophes to obtain some of the related properties and identities they share with the underline structure. Interestingly, some new identities are found. In the fourth section, the holomorphic characterisations of generalised middle Bol loop is studied and the necessary and sufficient condition is found.

2 Preliminaries

Definition 2.1. A loop $(Q, \cdot, /, \backslash)$ is called a generalised middle Bol loop if is satisfies the identity

$$(x/y)(z^{\alpha}\backslash x^{\alpha}) = (x/(z^{\alpha}y))x^{\alpha} \tag{3}$$

Definition 2.2. For any non-empty set Q, the set of all permutations on Q forms a group $SYM(Q)$ called the symmetric group of Q. Let $(Q, \cdot)$ be a loop and let

$A, B, C \in SYM(Q)$. If

$$xA \cdot yB = (x \cdot y)C \ \forall \ x, y \in Q$$

then the triple (A, B, C) is called an autotopism (ATP) and such triples form a group $AUT(Q, \cdot)$ called the autotopism group of $(Q, \cdot)$. Also, suppose that

$$xA \cdot yB = (y \cdot x)C \ \forall \ x, y \in Q$$

then the triple (A, B, C) is called anti-autotopism (AATP). If $A = B = C$, then A is called an automorphism of $(Q, \cdot)$ which form a group $AUM(Q, \cdot)$ called the automorphism group of $(Q, \cdot)$.

Definition 2.3. A groupoid (quasigroup) $(Q, \cdot)$ is said to have the

1. left inverse property (LIP) if there exists a mapping $J_\lambda : x \mapsto x^\lambda$ such that $x^\lambda \cdot xy = y$ for all $x, y \in Q$.
2. right inverse property (RIP) if there exists a mapping $J_\rho : x \mapsto x^\rho$ such that $yx \cdot x^\rho = y$ for all $x, y \in Q$.
3. inverse property (IP) if it has both the LIP and RIP. for all $x, y \in Q$.
4. flexibility or elasticity if $xy \cdot x = x \cdot yx$ holds for all $x, y \in Q$.
5. $\alpha-$elastic if $xy \cdot x^\alpha = x \cdot yx^\alpha$ holds for all $x, y \in Q$.
6. super $\alpha-$elastic if $(x \cdot y^\alpha) \cdot x^\alpha = x \cdot (y^\alpha \cdot x^\alpha)$ holds for all $x, y \in Q$.
7. cross inverse property (CIP) if there exist mapping $J_\lambda : x \mapsto x^\lambda$ or $J_\rho : x \mapsto x^\rho$ such that $xy \cdot x^\rho = y$ or $x \cdot yx^\rho = y$ or $x^\lambda \cdot yx = y$ or $x^\lambda y \cdot x = y$ for all $x, y \in Q$.

Definition 2.4. A loop $(Q, \cdot)$ is said to be

1. commutative loop if $R_x = L_x$ and a commutative square loop if $R_x^2 = L_x^2$ for all $x, y \in Q$
2. an automorphic inverse property loop (AIPL) if $(xy)^{-1} = x^{-1}y^{-1}$ for all $x, y \in Q$
3. an anti- automorphic inverse property loop (AAIPL) if $(xy)^{-1} = y^{-1}x^{-1}$ for all $x, y \in Q$.

Definition 2.5. [31] Moufang loops are loops satisfying the identities $(xy \cdot z)y = x(y \cdot zy), yz \cdot xy = y(zx \cdot y)$ and $(yz \cdot y)x = y(z \cdot yx)$

Definition 2.6. A groupoid (quasigroup) $(Q,\cdot)$ is

1. right symmetric if $yx\cdot x=y$ for all $x,y\in Q$
2. left symmetric if $x\cdot xy=y$ for all $x,y\in Q$
3. middle symmetric if $x\cdot yx=y$ or $xy\cdot x=y$ for all $x,y\in Q$
4. idempotent if $x\cdot x=x$ for all $x\in Q$
5. right $\alpha-$symmetric if $y^\alpha x\cdot x=y^\alpha$ for all $x,y\in Q$
6. left $\alpha-$symmetric if $x\cdot xy^\alpha=y^\alpha$ for all $x,y\in Q$
7. middle $\alpha-$symmetric if $x\cdot y^\alpha x=y^\alpha$ or $xy^\alpha\cdot x=y$ for all $x,y\in Q$
8. super middle $\alpha-$symmetric if $x\cdot(y^\alpha\cdot x^\alpha)=y^\alpha$ or $(x\cdot y^\alpha)\cdot x^\alpha=y^\alpha$ for all $x,y\in Q$

Definition 2.7. A quasigroup $(Q,\cdot)$ is totally symmetric if any relation $xy=z$ implies any other such relation can be obtained by permuting x, y and z.

Definition 2.8. [31] If a totally symmetric quasigroup $(Q,\cdot)$ is a loop, then it is called Steiner loop.

Theorem 2.1. [31] A quasigroup $(Q,\cdot)$ is totally symmetric if and only if it is commutative ($xy=yx$) for all $x,y\in Q$) and is right or left symmetric

Theorem 2.2. [31] A loop $(Q,\cdot)$ is totally symmetric if and only if $(Q,\cdot)$ is an IP loop of exponent 2.

Corollary 2.1. [31] Every T.S. quasigroup is a commutative I.M. quasigroup.

Definition 2.9. Let $(Q,\cdot)$ be a loop. The pair $(H,\circ)=H(Q,\cdot)$ given by

$$H=A(Q)\times Q,\ \text{where}\ A(Q)\le AUT(Q,\cdot)\ \text{such that}\ (\phi,x)\circ(\psi,y)=(\phi\psi,x\psi\cdot y)$$

for all $(\phi,x),(\psi,y)\in H$ is called the $A(H)-$ Holomorph of $(Q,\cdot)$

Lemma 2.1. [8] Let $(L,\cdot,/,\backslash)$ be a loop with holomorph $G(L,\cdot)$. Then, $G(L,\cdot)$ is a commutative if and only if $A(L,\cdot)$ is an abelian group and $(\psi,\phi^{-1},I_e)\in AATP(L,\cdot)$ for all $\phi,\psi\in A(L)$

Definition 2.10. [32] Let $(Q,\cdot)$ be quasigroup with e_l and e_r identity elemente. $(Q,\cdot)$ is called:

1. a left loop if $e_l\cdot x=x\ \forall x\in Q$
2. a right loop if $x\cdot e_r=x\ \forall x\in Q$
3. a loop if $e_l\cdot x=x\cdot e_r=x\ \forall x\in Q$

A quasigroup $(Q,\cdot)$, for which $e_l=e_r$ is called a loop. In more general note $e_l=e_r=e$

3 Main Results

3.1 Some algebraic connections between identities (2) and (3)

Here, we uncovered some characterisations of the two identities of GBML: (2) and (3), and further established that they are equivalent.

Lemma 3.1. Let $(Q,\cdot)$ be a loop. Let x, y, z be arbitrary elements in Q.

1. If $(Q,\cdot)$ obeys identity (2) such that $\alpha : e \mapsto e$, then
   (a) $(x/y)\cdot x^\alpha = x\cdot(y\backslash x^\alpha)$.
   (b) $y^\lambda\cdot(z^\alpha)^\rho = (z^\alpha\cdot y)^\rho$.
   (c) $y^\lambda = y^\rho$.
2. If $(Q,\cdot)$ obeys identity (3) such that $\alpha : e \mapsto e$, then
   (a) $x\cdot(z^\alpha\backslash x^\alpha) = (x/z^\alpha)\cdot x^\alpha$.
   (b) $y^\lambda\cdot(z^\alpha)^\rho = (z^\alpha\cdot y)^\lambda$.
   (c) $(z^\alpha)^\lambda = (z^\alpha)^\rho$.
3. If $(Q,\cdot)$ obeys identity (2) such that α is bijective and $\alpha : e \mapsto e$, then
   (a) $(x/y)\cdot x^\alpha = x\cdot(y\backslash x^\alpha)$.
   (b) $y^\lambda\cdot z^\rho = (z\cdot y)^\rho$.
   (c) $y^\lambda = y^\rho$.
4. If $(Q,\cdot)$ obeys identity (3) such that α is bijective and $\alpha : e \mapsto e$, then
   (a) $x\cdot(z\backslash x^\alpha) = (x/z)\cdot x^\alpha$.
   (b) $y^\lambda\cdot z^\rho = (z\cdot y)^\lambda$.
   (c) $z^\lambda = z^\rho$.
5. Let $\alpha : e \mapsto e$. Then, $(Q,\cdot)$ obeys identity (2) if and only if $(Q,\cdot)$ obeys identity (3) and $(x/y)\cdot x^\alpha = x\cdot(y\backslash x^\alpha)$.
6. Let α be bijective such that $\alpha : e \mapsto e$. Then, $(Q,\cdot)$ obeys identity (2) if and only if $(Q,\cdot)$ obeys identity (3).

Proof. 1. Assume that $(Q,\cdot)$ obeys the identity (2) such that $\alpha : e \mapsto e$.

(a) Put $z = e$ in (2) to get $(x/y)\cdot(e^\alpha\backslash x^\alpha) = x\cdot((e^\alpha\cdot y)\backslash x^\alpha)$ which gives $(x/y)\cdot x^\alpha = x\cdot(y\backslash x^\alpha)$.

(b) In (2), put $x = e$ to get $(e/y)\cdot(z^\alpha\backslash e^\alpha) = e\cdot((z^\alpha\cdot y)\backslash e^\alpha)$ to get $y^\lambda\cdot(z^\alpha)^\rho = (z^\alpha\cdot y)^\rho$.

(c) In (b), put $z = e$ to get $y^\lambda = y^\rho$.

2. Assume that $(Q,\cdot)$ obeys the identity (3) such that $\alpha : e \mapsto e$. Do similarly step as 1 to prove (a), (b) and (c).

3. Assume that $(Q,\cdot)$ obeys identity (2) such that α is bijective and $\alpha : e \mapsto e$. Then the proofs of (a), (b) and (c) follow up from 1.

4. Assume that $(Q,\cdot)$ obeys identity (3) such that α is bijective and $\alpha : e \mapsto e$. Then the proofs of (a), (b) and (c) follow up from 2.

5. Let $\alpha : e \mapsto e$. If $(Q,\cdot)$ obeys identity (2), then it obeys identity (3) because it satisfies $(x/y)\cdot x^{\alpha} = x \cdot (y\backslash x^{\alpha})$ by 1. The converse follows by reversing the process.

6. This follows from 5.

■

Henceforth, we shall assume that in a generalised middle Bol loop identity (2) or (3), the map $\alpha : Q \to Q^{i}$, where $i = (12), (13), (23), (123),\ (132)$, is a bijective map such that $\alpha : e \mapsto e$. Note that $J : x \mapsto x^{-1}$.

3.2 Parastrophes of Generalised Middle Bol Loop

We now look at characterisation of the parastrophe of identity 2

Lemma 3.2. Let $(Q,\cdot)$ be a quasigroup with e_l and e_r be the identity elements:

(a)
1. (12)-parastrophe of a left loop is right loop
2. (12)-parastrophe of a right loop is a left loop
3. (12)-parastrophe of a loop is also loop

(b)
1. (13)-parastrophe of a left loop is a not loop
2. (13)-parastrophe of right loop is a right loop
3. (13)-parastrophe of loop is a loop if and only if $|x| = 2$ for all $x \in Q$.

(c)
1. (23)-parastrophe of a left loop is a left loop
2. (23)-parastrophe of right loop is not a loop
3. (23)-parastrophe of loop is a loop if and only if $|x| = 2$ for all $x \in Q$

(d)
1. (123)-parastrophe of a left loop is a not loop
2. (123)-parastrophe of right loop is a left loop
3. (123)-parastrophe of loop is a loop if and only if $|x| = 2$ for all $x \in Q$

(e)
1. (132)-parastrophe of a left loop is a right loop

2. (132)-parastrophe of right loop is not a loop
3. (132)-parastrophe of loop is a loop if and only if $|x| = 2$ for all $x \in Q$

Proof. (a) " $\circ_{(12)}$ " denotes the operation of (12)-parastrophe of Q. If $(Q, \cdot)$ is a left loop, then $e_l \cdot x = x$ this implies that (12)$-$parastrophe of Q is $x \circ_{(12)} e_r = x$ for all $x \in Q$. $(Q, \cdot)$ is right loop if $x \circ_{(12)} e_r = x \Rightarrow$ (12)$-$parastrophe of Q is $e_l \circ_{(12)} x = x$ for all $x \in Q$. Therefore, (12)-parastrophe of Q is a loop.

(b) (13)$-$parastrophe of a left loop is given as $x \circ_{(13)} x = e_l$. This is only possible iff $|x| = 2$ for all $x \in Q$. Conversely, suppose that (13)$-$parastrophe of a left loop is of exponent 2, this implies that $x^\lambda = x$, then $x^\lambda \cdot x = e_l$ Also, if $(Q, \cdot)$ is right loop, then (13)$-$parastrophe of Q is also loop, that is $x \circ_{(13)} e_r = x$. Thus, $x^\lambda = x^\rho = x$ Therefore, (13)$-$parastrophe of Q is a loop if and only if $|x| = 2$. Similar results are obtained for (c), (d) and (e).

■

Theorem 3.1. Let $(Q, \cdot, /, \backslash)$ be a generalised middle Bol loop. Then, (12)$-$parastrophe of Q is also a generalised middle Bol loop

Proof. Let

$$a \cdot b = x(z^\alpha y \backslash x^\alpha) \tag{4}$$

in equation (2) where $a = x/y \Rightarrow x = ay \underbrace{\Rightarrow}_{\text{by (12)-permuation}} y \circ_{(12)} a = x \Rightarrow a = y\backslash^{(12)} x$. And $b = z^\alpha \backslash x^\alpha \Rightarrow z^\alpha b = x^\alpha \underbrace{\Rightarrow}_{\text{take (12)-permuation}} b \circ_{(12)} z^\alpha = x^\alpha \Rightarrow b = x^\alpha /^{(12)} z^\alpha$.

Substitute for a and b into equation (4), give

$$(y\backslash^{(12)} x) \cdot (x^\alpha /^{(12)} z^\alpha) = x(z^\alpha y \backslash x^\alpha) \tag{5}$$

Applying (12)$-$permutation on equation (5), to get

$$(x^\alpha /^{(12)} z^\alpha) \circ_{(12)} (y\backslash^{(12)} x) = \big((y \circ_{(12)} z^\alpha)\backslash x^\alpha\big) \circ_{(12)} x \tag{6}$$

Let

$$(y \circ_{(12)} z^\alpha)\backslash x^\alpha = c \Rightarrow (y \circ_{(12)} z^\alpha) \cdot c = x^\alpha \underbrace{\Rightarrow}_{\text{take (12)-permuation}} c \circ_{(12)} (y \circ_{(12)} z^\alpha) = x^\alpha \Rightarrow$$

$$c = x^\alpha /^{(12)} (y \circ_{(12)} z^\alpha)$$

Put c into equation (6) and make the substitution $x \leftrightarrow x^\alpha$, $z^\alpha \leftrightarrow y$, one obtains

$$(x/^{(12)} y) \circ_{(12)} (z^\alpha \backslash^{(12)} x^\alpha) = \big(x/^{(12)}(z^\alpha \circ_{(12)} y)\big) \circ_{(12)} x^\alpha$$

■

Lemma 3.3. Let $(Q, \cdot, /, \backslash)$ be a generalised middle Bol loop. Then, the $(13)-$parastrophe of Q is given by

$$(x \circ_{(13)} y)/^{(13)}(x^\alpha \backslash^{(13)} z^\alpha) = x/^{(13)}\big[x^\alpha \backslash^{(13)}(z^\alpha /^{(13)} y)\big] \qquad (7)$$

Proof. Let

$$a \cdot b = x(z^\alpha y \backslash x^\alpha) \qquad (8)$$

in equation (2), where

$$a = x/y \Rightarrow x = ay \underbrace{\Rightarrow}_{\text{taking (13)-permuation}} a = x \circ_{(13)} y \qquad (9)$$

and

$$b = z^\alpha \backslash x^\alpha \Rightarrow z^\alpha b = x^\alpha \underbrace{\Rightarrow}_{\text{take (13)-permuation}} z^\alpha = x^\alpha \circ_{(13)} b \Rightarrow x^\alpha \backslash^{(13)} z^\alpha = b \qquad (10)$$

Let $c = z^\alpha y$ in identity (2), this implies that $z^\alpha = \underbrace{c \circ_{(13)} y}_{\text{(13)-permutation}} \Rightarrow c = z^\alpha /^{(13)} y$.

Also, let $d = c \backslash x^\alpha \Rightarrow c \cdot d = x^\alpha \underbrace{\Rightarrow}_{\text{by taking (13)-permuation}} x^\alpha \circ_{(13)} d = c \Rightarrow d = x^\alpha \backslash^{(13)} c$. Then, substituting c into d, we have

$$d = x^\alpha \backslash^{(13)}(z^\alpha /^{(13)} y) \qquad (11)$$

Let $s = x \cdot d \Rightarrow x = s \circ_{(13)} d \Rightarrow s = x/^{(13)} d \underbrace{\Rightarrow}_{\text{substitute } d \text{ into } s}$

$$s = x/^{(13)}\big[x^\alpha \backslash^{(13)}(z^\alpha /^{(13)} y)\big] \qquad (12)$$

Now, according to identity (2), we have $a \cdot b = s \Rightarrow \underbrace{s \circ_{(13)} b}_{\text{(13)-permutaion}} = a \Rightarrow a/^{(13)} b = s$. Substituting (9), (10) and (12) into the last equality, we have

$$(x \circ_{(13)} y)/^{(13)}(x^\alpha \backslash^{(13)} z^\alpha) = x/^{(13)}\big[x^\alpha \backslash^{(13)}(z^\alpha /^{(13)} y)\big]$$

which is the $(13)-$parastrophe of Q as required. ■

Theorem 3.2. Let $(Q, \cdot, /, \backslash)$ be a generalised middle Bol loop. Then, the following hold in $(13)-$parastrophe of Q

1. $(L_x, L_{x^\alpha}^{-1}, L_{x^\alpha}^{-1} M_x^{-1}) \in AATP(Q, /^{(13)})$
2. $t^\lambda \circ_{(13)} (t \circ_{(13)} y) = y$ that is left inverse property for all $t \in Q$

3. $L_xR^{-1}_{(x^\alpha)^\rho} = \lambda JL^{-1}_{x^\alpha}M^{-1}_x$

4. $L_xM_x = L^{-1}_{x^\alpha}M^{-1}_{x^\alpha}$

5. $x/^{(13)}(x^\alpha)^\rho = (x \cdot_{(13)} y)/^{(13)}(x\backslash^{(13)}y)$ for all $x, y \in Q$

6. $y = (y^\lambda)^\lambda$ for all $y \in Q$

7. $L_xR^{-1}_{(x^\alpha)^\lambda} = \lambda JL_{(x^\alpha)^\lambda}M^{-1}_x$

Proof. 1. From equation (7) of Lemma 3.3, we have $yL_x/^{(13)}z^\alpha L^{-1}_{x^\alpha} = (z^\alpha/^{(13)}y)L^{-1}_{x^\alpha}M^{-1}_x \Rightarrow$

$(L_x, L^{-1}_{x^\alpha}, L^{-1}_{x^\alpha}M^{-1}_x) \in AATP(Q, /^{(13)})$

2. Let $x = e \Rightarrow e^\alpha \to e$ the identity element in Q, in equation (7), we have

$$ey/^{(13)}z^\alpha = e/^{(13)}(z^\alpha/^{(13)}y) \Rightarrow y/^{(13)}z^\alpha = (z^\alpha/^{(13)}y)^\lambda \Rightarrow y = (z^\alpha/^{(13)}y)^\lambda \circ_{(13)} z^\alpha \tag{13}$$

Let $t = z^\alpha/^{(13)}y \Rightarrow z^\alpha = t \circ_{(13)} y$, put z^α and t in (13), give $y = t^\lambda \circ_{(13)} (t \circ_{(13)} y)$ for all $t \in Q$.

3. Let $z = e$ and $e^\alpha \mapsto e$ in equation (7), we have $(x\circ_{(13)}y)/(x^\alpha)^\rho = x/^{(13)}(x^\alpha\backslash^{(13)}y^\lambda) \Rightarrow yL_xR^{-1}_{(x^\alpha)^\rho} = y\lambda L^{-1}_{x^\alpha}M^{-1}_x \Rightarrow L_xR^{-1}_{(x^\alpha)^\rho} = \lambda JL^{-1}_{x^\alpha}M^{-1}_x$

4. Let $z = x$ in equation (7), we have $x \circ_{(13)} y = x/^{(13)}\big(x^\alpha\backslash^{(13)}(x^\alpha/^{(13)}y) \Rightarrow yL_x = yM^{-1}_{x^\alpha}L^{-1}_{x^\alpha}M^{-1}_x \Rightarrow L_xM_x = L^{-1}_{x^\alpha}M^{-1}_{x^\alpha}$

5. Let $z = y$ and $y^\alpha \mapsto y$ in equation (7), give $x/^{(13)}(x^\alpha)^\rho = (x\circ_{(13)}y)/^{(13)}(x\backslash^{(13)}y)$.

6. Let $z = x = e$ in equation (7), we obtain $(y^\lambda)^\lambda = y$.

7. Apply 2 to equation (7), to get $(x\circ_{(13)}y)/^{(13)}((x^\alpha)^\lambda\circ_{(13)}z^\alpha) = x/^{(13)}((x^\alpha)^\lambda\circ_{(13)} (z^\alpha/^{(13)}y))$. Let $z^\alpha \mapsto e$ to get $x \circ_{(13)} y/^{(13)}(x^\alpha)^\lambda = x/^{(13)}((x^\alpha)^\lambda \circ_{(13)} y^\lambda) \Rightarrow yL_xR^{-1}_{(x^\alpha)^\lambda} = y\lambda JL_{(x^\alpha)^\lambda}M^{-1}_x \Rightarrow L_xR^{-1}_{(x^\alpha)^\lambda} = \lambda JL_{(x^\alpha)^\lambda}M^{-1}_x$

■

Corollary 3.1. In (13)−parastrophe of a generalised middle Bol loop $(Q, \cdot, /, \backslash)$, the following hold:

1. $(x^\alpha)^\rho = (x^\alpha)^\lambda\ \forall x \in Q$

2. $x^\rho = x^\lambda\ \forall x \in Q$

Proof. From 7 of Theorem 3.2, we have $L_xR^{-1}_{(x^\alpha)^\lambda} = \lambda J L^{-1}_{x^\alpha} M^{-1}_x$. Recall from 3 of Theorem 3.2, $L_xR^{-1}_{(x^\alpha)^\rho} = \lambda J L^{-1}_{x^\alpha} M^{-1}_x$. This implies that $L_xR^{-1}_{(x^\alpha)^\rho} = L_xR^{-1}_{(x^\alpha)^\lambda} \Rightarrow R^{-1}_{(x^\alpha)^\rho} = R^{-1}_{(x^\alpha)^\lambda} \Rightarrow (x^\alpha)^\rho = (x^\alpha)^\lambda$. Since α is bijective, we have $x^\rho = x^\lambda \ \forall x \in Q$ ■

Remark 3.1. The above Corollary shows that in $(13)-$parastrophe of a generalised middle Bol loop $(Q, \cdot, /, \backslash)$, the right and the left inverse properties coincide. So, the $(13)-$parastrophe satisfies IP if it is commutative. Also, if $|Q^{(13)}| = 2$, then $x^\rho = x^\lambda = x \ \forall x \in Q$. Thus, (13)-parastrophe of Q is a loop.

Corollary 3.2. A commutative $(13)-$parastrophe of a generalised middle Bol loop $(Q, \cdot, /, \backslash)$, satisfies AAIPL if $|x| = 2 \ \forall\ x \in Q$

Proof. Based on the Remark (3.1), the identity (7) become $(x \circ_{(13)} y) \circ_{(13)} (x^\alpha)^{-1} \circ_{(13)} (z^\alpha)^{-1} = x \circ_{(13)} \big[(x^\alpha)^{-1} \circ_{(13)} (z^\alpha \circ_{(13)} y^{-1})\big]^{-1}$. Let $x = e$ to get $(e \circ_{(13)} y) \circ_{(13)} (e^\alpha)^{-1} \circ_{(13)} (z^\alpha)^{-1} = e \circ_{(13)} \big[(e^\alpha)^{-1} \circ_{(13)} (z^\alpha \circ_{(13)} y^{-1})\big]^{-1}$, then $y \circ_{(13)} (z^\alpha)^{-1} = (z^\alpha \circ_{(13)} y^{-1})^{-1} \Rightarrow y^{-1} \circ_{(13)} (z^\alpha)^{-1} = (z^\alpha \circ_{(13)} y)^{-1}$ ■

Corollary 3.3. A commutative $(13)-$parastrophe of a generalised middle Bol loop $(Q, \cdot, /, \backslash)$ is a Steiner loop if it is a loop of exponent two.

Proof. This is a consequence of 2 of theorem 3.2 and the Corollary 3.1. ■

Theorem 3.3. A commutative $(13)-$parastrophe, of exponent two, of a generalised middle Bol loop $(Q, \cdot, /, \backslash)$ is a Moufang loop.

Proof. From Remark (3.1), we have the identity (7) to be $(x \circ_{(13)} y) \circ_{(13)} (x^\alpha)^{-1} \circ_{(13)} (z^\alpha)^{-1} = x \circ_{(13)} \big[(x^\alpha)^{-1} \circ_{(13)} (z^\alpha \circ_{(13)} y^{-1})\big]^{-1}$. Since $|Q^{(13)}| = 2$, we have $(x \circ_{(13)} y) \circ_{(13)} (x^\alpha \circ_{(13)} z^\alpha) = x \circ_{(13)} \big[x^\alpha \circ_{(13)} (z^\alpha \circ_{(13)} y)\big] \Rightarrow z^\alpha L_{x^\alpha} L_{xy} = z^\alpha R_y L_{x^\alpha} L_x \Rightarrow z^\alpha R_{x^\alpha} L_{xy} = z^\alpha L_y L_{x^\alpha} L_x \Rightarrow (x \circ_{(13)} y) \circ_{(13)} (z^\alpha \circ_{(13)} x^\alpha) = x \circ_{(13)} ((y \circ_{(13)} z^\alpha) \circ_{(13)} x^\alpha)$ ■

Corollary 3.4. In $(13)-$parastrophe, of exponent two, of a generalised middle Bol loop $(Q, \cdot, /, \backslash)$ is a GMBL

Proof. Follow from Theorem 3.3, we have $(x \circ_{(13)} y) \circ_{(13)} [(x^\alpha)^{-1} \circ_{(13)} (z^\alpha)]^{-1} = x \circ_{(13)} \big[(x^\alpha)^{-1} \circ_{(13)} (z^\alpha \circ_{(13)} y^{-1})\big]^{-1}$. Use $y^{-1} = y$ and Corollary 3.2 to get $(x \circ_{(13)} y^{-1}) \circ_{(13)} ((z^\alpha)^{-1} \circ_{(13)} x^\alpha) = x \circ_{(13)} \big[(z^\alpha \circ_{(13)} y)\big]^{-1} \circ_{(13)} x^\alpha \Rightarrow (x /^{(13)} y) \circ_{(13)} (z^\alpha \backslash^{(13)} x^\alpha) = x \circ_{(13)} [(z^\alpha \circ_{(13)} y) \backslash^{(13)} x^\alpha]$

■

Lemma 3.4. Let $(Q, \cdot, /, \backslash)$ be a generalised middle Bol loop. Then, the $(23)-$parastrophe of Q is given by

$$(y /^{(23)} x) \backslash^{(23)} (z^\alpha \circ_{(23)} x^\alpha) = x \backslash^{(23)} \big[(z^\alpha \backslash^{(23)} y) \circ_{(23)} x^\alpha\big] \tag{14}$$

Proof. Let

$$a \cdot b = x(z^{\alpha} y \backslash x^{\alpha}) \tag{15}$$

in an identity (2), where

$$a = x/y \Rightarrow x = a \cdot y \underbrace{\Rightarrow}_{(23)\text{-permutation}} y = a \circ_{(23)} x \Rightarrow a = y/^{(23)} x \tag{16}$$

and

$$b = z^{\alpha} \backslash x^{\alpha} \underbrace{\Rightarrow}_{(23)\text{-permutation}} z^{\alpha} \circ_{(23)} b = x^{\alpha} \Rightarrow z^{\alpha} \circ_{(23)} x^{\alpha} = b \tag{17}$$

Let $c = z^{\alpha} y$ in identity (2), then $\underbrace{z^{\alpha} \circ_{(23)} c}_{(23)\text{-permutation}} = y \Rightarrow c = z^{\alpha} \backslash^{(23)} y$. Let $d = c \backslash x^{\alpha} \Rightarrow c \circ_{(23)} d = x^{\alpha} \Rightarrow c \circ_{(23)} x^{\alpha} = d$, put c into d to get

$$d = (z^{\alpha} \backslash^{(23)} y) \circ_{(23)} x^{\alpha}. \tag{18}$$

Also, let $t = x \cdot d \underbrace{\Rightarrow}_{(23)\text{-permutation}} x \circ_{(23)} t = d \Rightarrow t = x \backslash^{(23)} d$. Substitute d into t

$$t = x \backslash^{(23)} \left[(z^{\alpha} \backslash^{(23)} y) \circ_{(23)} x^{\alpha} \right] \tag{19}$$

Now, going by the identity (2), we have $a \cdot b = t \underbrace{\Rightarrow}_{(23)\text{-permutation}} a \circ_{(23)} t = b \Rightarrow a \backslash^{(23)} b = t$. Then, substituting equations (16), (17) and (19) in the equality $a \backslash^{(23)} b = t$, gives

$$(y/^{(23)} x) \backslash^{(23)} (z^{\alpha} \circ_{(23)} x^{\alpha}) = x \backslash^{(23)} \left[(z^{\alpha} \backslash^{(23)} y) \circ_{(23)} x^{\alpha} \right] \tag{20}$$

which is the $(23)-$parastrophe of Q. ∎

Theorem 3.4. Let $(Q, \cdot, /, \backslash)$ be a generalised middle Bol loop. Then, the following holds in $(23)-$parastrophe of Q

1. $(L_x^{-1}, R_{x^{\alpha}}, R_{x^{\alpha}} L_x^{-1}) \in AATP(Q, \backslash^{(23)})$ for all $x \in Q$
2. $(z \circ_{(23)} t) \circ_{(23)} t = z$ for all $z, t \in Q$
3. if $Q^{(23)}$ is middle symmetric then, $x \circ_{(23)} (z^{\alpha} \circ_{(23)} x^{\alpha}) = (x \circ_{(23)} z^{\alpha}) \circ_{(23)} x^{\alpha}$ that is, super $\alpha-$elastic
4. $R_x^{-1} M_{x^{\alpha}} = R_{x^{\alpha}} L_x^{-1}$
5. $\rho J R_{x^{\alpha}} L_x^{-1} = R_{x^{\alpha}} L_{x^{\lambda}}^{-1}$

6. $\rho J R_{x^\alpha}^{-1} M_{x^\alpha} = R_{x^\alpha} L_{x^\lambda}^{-1}$

Proof. 1. this follows from equation (14),

$$yR_x^{-1}\backslash^{(23)} z^\alpha R_{x^\alpha} = (z^\alpha\backslash^{(23)} y)R_{x^\alpha}L_x^{-1} \Rightarrow (R_x^{-1}, R_{x^\alpha}, R_{x^\alpha}L_x^{-1}) \in AATP(Q,\backslash)$$

2. Put $x = e$ such that $e^\alpha \mapsto e$ is the identity map in (14), give $y\backslash^{(23)} z^\alpha = z^\alpha\backslash^{(23)} y \Rightarrow y \circ_{(23)} (z^\alpha\backslash^{(23)} y) = z^\alpha$. Let $t = z^\alpha\backslash^{(23)} y \Rightarrow z^\alpha \circ_{(23)} t = y$. Put y into the last equality to get $(z^\alpha \circ_{(23)} t) \circ_{(23)} t = z^\alpha$ for any $t \in Q$.

3. Put $y = x$ in (14), we have

$$z^\alpha \circ_{(23)} x^\alpha = x\backslash^{(23)}[(z^\alpha\backslash^{(23)} x) \circ_{(23)} x^\alpha] \Rightarrow$$
$$x^\alpha \circ_{(23)} (z^\alpha \circ_{(23)} x) = (z^\alpha\backslash^{(23)} x) \circ_{(23)} x^\alpha \Rightarrow$$
$$z^\alpha R_{x^\alpha} L_x = z^\alpha M_x R_{x^\alpha} \underbrace{\Rightarrow}_{\text{Use middle symmetric as } L_x = M_x \text{ to get}} z^\alpha R_{x^\alpha} L_x = z^\alpha L_x R_{x^\alpha}$$

or $x \circ_{(23)} (z^\alpha \circ_{(23)} x^\alpha) = (x \circ_{(23)} z^\alpha) \circ_{(23)} x^\alpha$

4. Put $z = e$ and $e^\alpha \mapsto e$, the identity element in (14), we have $(y/^{(23)} x)\backslash^{(23)} x^\alpha = x\backslash^{(23)}(y \circ_{(23)} x^\alpha) \Rightarrow yR_x^{-1}M_{x^\alpha} = yR_{x^\alpha}L_x^{-1} \Rightarrow R_x^{-1}M_{x^\alpha} = R_{x^\alpha}L_x^{-1}$

5. $y = e$ in (14), we have

$$x^\lambda\backslash^{(23)}(z^\alpha \circ_{(23)} x^\alpha) = x\backslash^{(23)}((z^\alpha)^\rho \circ_{(23)} x^\alpha) \Rightarrow$$
$$z^\alpha \rho J R_{x^\alpha} L_x^{-1} = z^\alpha R_{x^\alpha} L_{x^\lambda}^{-1} \Rightarrow \rho J R_{x^\alpha} L_x^{-1} = R_{x^\alpha} L_{x^\lambda}^{-1}$$

6. Use 4 and 5.

■

Corollary 3.5. A commutative $(23)-$parastrophe of a generalised middle Bol loop $(Q, \cdot, /, \backslash)$ is totally symmetric.

Proof. This is a consequence of the right symmetric property 2 of Theorem 3.4. ■

Theorem 3.5. Let the $(23)-$parastrophe of a generalised middle Bol loop $(Q, \cdot, /, \backslash)$ be commutative and of exponent two, then $L_{x^\alpha} L_x = R_x R_{x^\alpha}$ for all $x \in Q$.

Proof. Recall (6) in Theorem 3.4, we have $\rho J R_x^{-1} M_{x^\alpha} = R_{x^\alpha} L_{x^\lambda}^{-1}$. Since $Q^{(23)}$ is commutative, then it implies that it has a middle symmetric property as $L_x =$

M_x. Applying the middle symmetric identity gives $\rho J R_x^{-1} L_{x^\alpha} = R_{x^\alpha} L_{x^\lambda}^{-1}$. Then, for all $t \in Q$, we have

$$t^\rho R_x^{-1} L_{x^\alpha} = t R_{x^\alpha} L_{x^\lambda}^{-1} \Rightarrow x^\alpha \circ_{(23)} (t^\rho / x) = x^\lambda \backslash^{(23)} (t \circ_{(23)} x^\alpha) \Rightarrow$$
$$x^\lambda \circ_{(23)} [x^\alpha \circ_{(23)} (t^\rho / x)] = t \circ_{(23)} x^\alpha$$

Let $t^\rho /^{(23)} x = s \Rightarrow t^\rho = s \circ_{(23)} x$. Then, $x^\lambda \circ_{(23)} (x^\alpha \circ_{(23)} s) = (s \circ_{(23)} x) \circ_{(23)} x^\alpha$. Using the fact that $|Q^{(23)}| = 2$ for all $x \in Q$, one obtains $s L_{x^\alpha} L_x = s R_x R_{x^\alpha} \Rightarrow L_{x^\alpha} L_x = R_x R_{x^\alpha}$ for all $x \in Q$ ■

Corollary 3.6. If $(23)-$parastrophe of a generalised middle Bol loop $(Q, \cdot, /, \backslash)$ is commutative and $x^\alpha \mapsto x$, then $L_x^2 = R_x^2$ for all $x \in Q$.

Proof. Consequence of Theorem 3.5. ■

Lemma 3.5. Let $(Q, \cdot, /, \backslash)$ be a generalised middle Bol. Then, the $(123)-$parastrophe of Q is given by

$$(z^\alpha /^{(123)} x^\alpha) \backslash^{(123)} (y \circ_{(123)} x) = \left[(y \backslash^{(123)} z^\alpha) /^{(123)} x^\alpha\right] \backslash^{(123)} x \tag{21}$$

Proof. Let $a \cdot b = x \cdot (z^\alpha y \backslash x^\alpha)$ in equation (2) where

$$a = x/y \Rightarrow a \cdot y = x \underbrace{\Rightarrow}_{\text{(123)-permutation}} y \circ_{(123)} x = a \tag{22}$$

$$b = z^\alpha \backslash x^\alpha \Rightarrow z^\alpha \circ b = x^\alpha \underbrace{\Rightarrow}_{\text{(123)-permutation}} b \circ_{(123)} x^\alpha = z^\alpha \Rightarrow b = z^\alpha /^{(123)} x^\alpha \tag{23}$$

Let $c = z^\alpha \cdot y$ in equation (2), then, we have $y \circ_{(123)} c = z^\alpha \underbrace{\Rightarrow}_{\text{(123)-permutation}} c = y \backslash^{(123)} z^\alpha$. Also, let $d = c \backslash x^\alpha \Rightarrow c \cdot d = x^\alpha \Rightarrow d \circ_{(123)} x^\alpha = c \Rightarrow d = c /^{(123)} x^\alpha$. Substitute c into d, give

$$d = (y \backslash^{(123)} z^\alpha) /^{(123)} x^\alpha \tag{24}$$

Next, let $t = x \cdot d \underbrace{\Rightarrow}_{\text{(123)-permutation}} d \circ_{(123)} t = x \Rightarrow t = d \backslash^{(123)} x$. Substitute (24) into t give

$$t = \left[(y \backslash^{(123)} z^\alpha) /^{(123)} x^\alpha\right] \backslash^{(123)} x \tag{25}$$

Going by the identity (2), we have $a \cdot b = t \underbrace{\Rightarrow}_{\text{(123)-permutation}} b \circ_{(123)} t = a \Rightarrow b \backslash^{(123)} a = t$. Substitute (22), (23) and (25) into the equality $b \backslash^{(123)} a = t$, gives the $(123)-$parastrophe as

$$(z^\alpha /^{(123)} x^\alpha) \backslash^{(123)} (y \circ_{(123)} x) = \left[(y \backslash^{(123)} z^\alpha) /^{(123)} x^\alpha\right] \backslash^{(123)} x$$

■

Theorem 3.6. Let $(Q,\cdot,/,\backslash)$ be a generalised middle Bol loop. Then, the following hold in $(123)-$parastrophe of Q

1. $(L_x^{-1}, R_x, R_x^{-1}M_x) \in AATP(Q, \backslash^{(123)})$
2. $(y \circ_{(123)} t) \circ_{(123)} t^{\rho} = y$, i.e right inverse property
3. $(z^{\alpha}/^{(123)}x^{\alpha})[(x^{\alpha})^{\lambda}\backslash^{(123)}x] = z^{\alpha} \circ_{(123)} x$
4. $R_x L^{-1}_{(x^{\alpha})^{\lambda}} = \rho J R^{-1}_{x^{\alpha}} M_x$
5. $R_x M_x^{-1} = M_{x^{\alpha}} R^{-1}_{x^{\alpha}}$
6. $(x \circ_{(123)} t) \circ_{(123)} x = (x\backslash^{(123)}t)\backslash^{(123)}x$ for all $x, t \in Q$

Proof. 1. From equation (21) , we have

$$z^{\alpha}R^{-1}_{x^{\alpha}}\backslash^{(123)}yR_x = (y\backslash^{(123)}z^{\alpha})R^{-1}_{x^{\alpha}}M_x \Rightarrow$$
$$(R^{-1}_{x^{\alpha}}, R_x, R^{-1}_{x^{\alpha}}M_x) \in AATP(Q, \backslash^{(123)})$$

2. Let $x^{\alpha} \mapsto x$ and put $x = e$, the identity element in equation (21), we have

$$((z^{\alpha}/^{(123)}e^{\alpha})\backslash^{(123)}(y \circ_{(123)} e) = \big((y\backslash^{(123)}z^{\alpha})/^{(123)}e\big)\backslash^{(123)}e^{\alpha} \Rightarrow z^{\alpha}\backslash^{(123)}y =$$
$$(y\backslash^{(123)}z^{\alpha})^{\rho} \Rightarrow z^{\alpha} \circ_{(123)} (y\backslash^{(123)}z^{\alpha})^{\rho} = y$$

Let $t = y\backslash^{(123)}z^{\alpha} \Rightarrow y \circ_{(123)} t = z^{\alpha}$ for any $t \in Q$, this implies that $(y \circ_{(123)} t) \circ_{(123)} t^{\rho} = y$.

3. Set $y = z^{\alpha}$ in equation (21), we have $(z^{\alpha}/^{(123)}x^{\alpha})\backslash^{(123)}(z^{\alpha} \circ_{(123)} x) = (x^{\alpha})^{\lambda}\backslash^{(123)}x \Rightarrow (z^{\alpha}/^{(123)}x^{\alpha})[(x^{\alpha})^{\lambda}\backslash^{(123)}x] = z^{\alpha} \circ_{(123)} x$

4. Put $z \to e$ in equation (21), to get $(x^{\alpha})^{\lambda}\backslash^{(123)}y\circ_{(123)}x) = (y^{\rho}/^{(123)}x^{\alpha})\backslash^{(123)}x \Rightarrow yR_x L^{-1}_{(x^{\alpha})^{\lambda}} = y\rho J R^{-1}_{x^{\alpha}} M_x \Rightarrow R_x L^{-1}_{(x^{\alpha})^{\lambda}} = \rho J R^{-1}_{x^{\alpha}} M_x$

5. Set $z = x$ in equation (21), give $y \circ_{(123)} x = \big((y\backslash^{(123)}x^{\alpha})/^{(123)}x^{\alpha}\big)\backslash^{(123)}x \Rightarrow yR_x = yM_{x^{\alpha}}R^{-1}_{x^{\alpha}}M_x \Rightarrow R_x M_x^{-1} = M_{x^{\alpha}}R^{-1}_{x^{\alpha}}$

■

Corollary 3.7. A commutative $(123)-$parastrophe of a generalised middle Bol loop $(Q,\cdot,/,\backslash)$ has an inverse property.

Proof. This is a consequence of 2 of Theorem 3.6. ■

Corollary 3.8. A commutative $(123)-$parastrophe of a generalised middle Bol loop $(Q,\cdot,/,\backslash)$ has AAIP if $|Q^{(123)}| = 2$

Proof. Applying Corollary 3.7 to (21) $(z^\alpha \circ_{(123)} (x^\alpha)^{-1})^{-1} \circ_{(123)} (y \circ_{(123)} x) = \left[(y^{-1} \circ_{(123)} z^\alpha) \circ_{(123)} (x^\alpha)^{-1}\right]^{-1} \circ_{(123)} x$. Put $x = e$ and $y = y^{-1}$ to get $(z^\alpha)^{-1} \circ_{(123)} y^{-1} = (y^{-1} \circ_{(123)} z^\alpha)^{-1}$ ∎

Corollary 3.9. A commutative $(123)-$parastrophe, of exponent 2, of a generalised middle Bol loop $(Q, \cdot, /, \backslash)$ is Steiner loop.

Proof. Follows from Corollary 3.7. ∎

Theorem 3.7. Let $Q^{(123)}$ be a commutative $(123)-$parastrophe of a generalised middle Bol loop $(Q, \cdot, /, \backslash)$ of exponent two, then $Q^{(123)}$ is a Moufang loop.

Proof. Using the Corollary 3.7 on identity (21), we have $(z^\alpha \circ_{(123)} (x^\alpha)^{-1})^{-1} \circ_{(123)} (y \circ_{(123)} x) = \left[(y^{-1} \circ_{(123)} z^\alpha) \circ_{(123)} (x^\alpha)^{-1}\right]^{-1} \circ_{(123)} x$. Since $|Q^{(123)}| = 2$, we have $(z^\alpha \circ_{(123)} x^\alpha) \circ_{(123)} (y \circ_{(123)} x) = \left[(y \circ_{(123)} z^\alpha) \circ_{(123)} (x^\alpha)\right] \circ_{(123)} x \Rightarrow z^\alpha L_{x^\alpha} R_{yx} = z^\alpha L_y R_{x^\alpha} R_x \Rightarrow z^\alpha L_{x^\alpha} R_{yx} = z^\alpha L_y L_{x^\alpha} R_x \Rightarrow (x^\alpha \circ_{(123)} z^\alpha) \circ (y \circ_{(123)} x) = (x^\alpha \circ_{(123)} (z^\alpha \circ_{(123)} y)) \circ_{(123)} x \Rightarrow (x^\alpha \circ_{(123)} z^\alpha) \circ (y \circ_{(123)} x) = x^\alpha \circ_{(123)} ((z^\alpha \circ_{(123)} y) \circ_{(123)} x)$
∎

Corollary 3.10. A commutative $(123)-$parastrophe of a generalised middle Bol loop $(Q, \cdot, /, \backslash)$ is a GMBL of exponent two.

Proof. Follow from Corollaries 3.7 and 3.8 and (21), we get $(z^\alpha \circ_{(123)} (x^\alpha)^{-1})^{-1} \circ_{(123)} (y \circ_{(123)} x) = \left[(y^{-1} \circ_{(123)} z^\alpha) \circ_{(123)} (x^\alpha)^{-1}\right]^{-1} \circ_{(123)} x$. So, use $y^{-1} = y$ and take the following steps: $x \leftrightarrow x^\alpha$, $z^\alpha \leftrightarrow y$, one obtains

$$(x/^{(12)}y) \circ_{(12)} (z^\alpha \backslash^{(12)} x^\alpha) = \left(x/^{(12)}(z^\alpha \circ_{(12)} y)\right) \circ_{(12)} x^\alpha$$

which is the same as (3) ∎

Lemma 3.6. Let $(Q, \cdot, /, \backslash)$ be a generalised middle Bol loop. Then, the $(132)-$parastrophe of Q is given by

$$(x^\alpha \circ_{(132)} z^\alpha)/^{(132)}(x\backslash^{(132)}y) = \left[x^\alpha \circ_{(132)} (y/^{(132)}z^\alpha)\right]/^{(132)}x \tag{26}$$

Proof. Let $a \cdot b = x \cdot (z^\alpha y \backslash x^\alpha)$ in equation (2) where

$$x/y = a \Rightarrow x = a \cdot y \quad \underbrace{\Rightarrow}_{(132)\text{-permutation}} \quad x \circ_{132} a = y \Rightarrow a = x\backslash^{(132)}y \tag{27}$$

and

$$z^\alpha \backslash x^\alpha = b \Rightarrow z^\alpha \cdot b = x^\alpha \quad \underbrace{\Rightarrow}_{(132)\text{-permutation}} \quad x^\alpha \circ_{(132)} z^\alpha = b \tag{28}$$

Let $c = z^\alpha \cdot y \underbrace{\Rightarrow}_{\text{(132)-permutation}} c \circ_{(132)} z^\alpha = y \Rightarrow c = y/^{(132)}z^\alpha$. Also, let $d = c\backslash x^\alpha \Rightarrow c \cdot d = x^\alpha \underbrace{\Rightarrow}_{\text{(132)-permutation}} x^\alpha \circ_{(132)} c = d$. Substitute c into to d to get $d = x^\alpha \circ_{(132)} (y/^{(132)}z^\alpha)$. Let $t = x \cdot d \underbrace{\Rightarrow}_{\text{taking (132)-permutation}} t \circ_{(132)} x = d \Rightarrow t = d/^{(132)}x$. Hence, putting d into t, we have

$$t = \left[x^\alpha \circ_{(132)} (y/^{(132)}z^\alpha)\right]/^{(132)}x \tag{29}$$

Now, going by the identity (2), we have $a \cdot b = t \underbrace{\Rightarrow}_{\text{taking (132)-permutation}} t \circ_{(132)} a = b \Rightarrow b/^{(132)}a = t$. Substitute equations (27), (28) and (29) into the equality $b/^{(132)}a = t$, we have

$$(x^\alpha \circ_{(132)} z^\alpha)/^{(132)}(x\backslash^{(132)}y) = \left[x^\alpha \circ_{(132)} (y/^{(132)}z^\alpha)\right]/^{(132)}x$$

which is the $(132)-$ parastrophe of Q. ■

Theorem 3.8. Let $(Q, \cdot, /, \backslash)$ be a generalised middle Bol loop. Then, the following holds in $(132)-$parastrophe of Q

1. $(L_{x^\alpha}, L_x^{-1}, L_{x^\alpha}R_x^{-1}) \in AATP(Q, /^{(132)})$ for all $x \in Q$
2. $z^\alpha = t \circ_{132} (t \circ_{132} z^\alpha)$ i.e $\alpha-$left symmetric property
3. $(x^\alpha \circ_{(132)} z^\alpha) \circ_{(132)} x = x^\alpha \circ_{(132)} (x/^{(132)}z^\alpha)$ or $M_x^{-1}L_{x^\alpha} = L_{x^\alpha}R_x$
4. $L_{x^\alpha}R_{x^\rho}^{-1} = \lambda J L_{x^\alpha}R_x^{-1}$
5. $L_x^{-1}M_{x^\alpha}^{-1} = L_{x^\alpha}R_x^{-1}$

Proof. 1. From equation (26), we have $z^\alpha L_{x^\alpha}/^{(132)}yL_x^{-1} = (y/^{(132)}z^\alpha)L_{x^\alpha}R_x^{-1} \Rightarrow (L_{x^\alpha}, L_x^{-1}, L_{x^\alpha}R_x^{-1}) \in AATP(Q, /^{(132)})$ for all $x \in Q$

2. Let $x^\alpha \mapsto e$ in (26), give $z^\alpha/^{(132)}y = y/^{(132)}z^\alpha$, by setting $t = y/^{(132)}z^\alpha \Rightarrow y = (z^\alpha \circ_{(132)} t) \Rightarrow z^\alpha = t \circ_{(132)} (t \circ_{(132)} z^\alpha)$

3. Put $y = x$ in (26), to get $(x^\alpha \circ_{(132)} z^\alpha) \circ_{(132)} x = x^\alpha \circ_{(132)} (x/^{(132)}z^\alpha) \Rightarrow z^\alpha M_x^{-1}L_{x^\alpha} = z^\alpha L_{x^\alpha}R_x \Rightarrow M_x^{-1}L_{x^\alpha} = L_{x^\alpha}R_x$ for all $x \in Q$

4. Put $y = e$ in (26), we have $(x^\alpha \circ_{(132)} z^\alpha)/^{(132)}x^\rho = (x^\alpha \circ_{(132)} (z^\alpha)^\lambda)/^{(132)}x \Rightarrow z^\alpha L_{x^\alpha}R_{x^\rho}^{-1} = (z^\alpha)\lambda J L_{x^\alpha}R_x^{-1} \Rightarrow L_{x^\alpha}R_{x^\rho}^{-1} = \lambda J L_{x^\alpha}R_x^{-1}$.

5. Put $z = e$, we have $x/^{(132)}(x\backslash^{(132)}y) = (x \circ_{(132)} y)/^{(132)}x \Rightarrow yL_xM_x^{-1} = yL_xR_x^{-1} \Rightarrow L_x^{-1}M_{x^\alpha}^{-1} = L_{x^\alpha}R_x^{-1}$ for all $x \in Q$.

■

Corollary 3.11. Let $(Q,\cdot,/,\backslash)$ be a generalised middle Bol loop. Then, a commutative $(132)-$parastrophe of Q is totally symmetric.

Proof. This is a consequence, of 2, of Theorem 3.8. ■

3.3 Holomorphic Structure of Generalised Middle Bol Loop

Theorem 3.9. $(Q,\cdot,/,\backslash)$ is a generalised middle Bol loop if and only if $(JM_x^{-1}, JM_{x^\alpha}, JM_{x^\alpha}L_x)$ is an autotopism.

Proof. Suppose $(Q,\cdot)$ is a generalised middle Bol loop, then

$$\begin{aligned} x(y^\alpha z\backslash x^\alpha) = (x/z)(y^\alpha\backslash x^\alpha) &\Leftrightarrow zM_x^{-1}\cdot y^\alpha M_{x^\alpha} = (y^\alpha\cdot z)M_{x^\alpha}L_x \\ &\Leftrightarrow zM_x^{-1}\cdot y^\alpha M_{x^\alpha} = (zJ\cdot y^\alpha J)JM_{x^\alpha}L_x \\ &\Leftrightarrow zJM_x^{-1}\cdot y^\alpha JM_{x^\alpha} = (z\cdot y^\alpha)JM_{x^\alpha}L_x \end{aligned}$$

Thus, $(JM_x^{-1}, JM_{x^\alpha}, JM_{x^\alpha}L_x)\in ATP(Q,\cdot)$ ■

Theorem 3.10. Let $(Q,\cdot,/,\backslash)$ be a loop with holomorph $(H(Q),*)$. Then, $(H(Q),*)$ is a generalised middle Bol loop if and only if $(x\tau)\cdot(y\cdot z^\alpha\tau)\backslash x^\alpha = (x^\alpha\tau/z^\alpha\tau)\cdot(y\backslash x)$ for all $x,y,z\in Q,\tau\in A(Q)$.

Proof. We need to show the necessary and sufficient condition for the holomorph of a generalised middle Bol loop to ba a generalised middle Bol loop.

$$(x^\alpha/z^\alpha)(y\backslash x) = x((y\cdot z^\alpha)\backslash x^\alpha) \tag{30}$$

$$\text{Let } (\phi,x)*(\psi,y) = (\theta,z) \Rightarrow (\phi,x) = (\theta,z)/(\psi,y), \text{ so }, \tag{31}$$
$$(\phi\psi, x\psi\cdot y) = (\theta,z)$$
$$\Rightarrow \phi = \theta\psi^{-1}, x = (z/y)\psi^{-1}.$$
$$\Rightarrow (\theta,z)/(\psi,y) = \big(\theta\psi^{-1},(z/y)\phi^{-1}\big) = (\phi,x). \tag{32}$$
$$\text{Also}, (\phi,x)*(\psi,y) = (\theta,z) \Rightarrow (\psi,y) = (\phi,x)\backslash(\theta,z).$$
$$\text{Thus, } (\phi\psi, x\psi\cdot y) = (\theta,z) \Rightarrow \psi = \phi^{-1}\theta, y = (x\phi^{-1}\theta)\backslash z$$
$$\Rightarrow (\psi,y) = (\phi^{-1}\theta,(x\phi^{-1}\theta)\backslash z) = (\phi,x)\backslash(\theta,z) \tag{33}$$

$$
\begin{aligned}
((\phi,x)/(\psi,y))*((\theta,z^\alpha)\backslash(\phi,x^\alpha)) &= (\phi,x)*[((\psi,y)*(\theta,z^\alpha))\backslash(\phi,x^\alpha)]\\
\text{RHS} &= (\phi,x)*[((\psi,y)*(\theta,z^\alpha))\backslash(\phi,x^\alpha)]\\
&= (\phi,x)*\left((\psi\theta)^{-1}\phi,(y\theta\cdot z^\alpha)\theta^{-1}\psi^{-1}\phi\backslash x^\alpha\right)\\
&= (\phi,x)*\left((\psi\theta)^{-1}\phi, y\psi^{-1}\phi\cdot z^\alpha\theta^{-1}\psi^{-1}\phi\backslash x^\alpha\right)\\
&= \big(\phi\theta^{-1}\psi^{-1}\phi,(x\theta^{-1}\psi^{-1}\phi)\cdot(y\psi^{-1}\phi\cdot z^\alpha\theta^{-1}\psi^{-1}\phi\backslash x^\alpha)\big)\\
\text{LHS } &= ((\phi,x)/(\psi,y))*((\theta,z^\alpha)\backslash(\phi,x^\alpha))\\
&= \big(\phi\theta^{-1},(x^\alpha/z^\alpha)\theta^{-1}\big)*\big(\psi^{-1}\phi,(y\psi^{-1}\phi)\backslash x\big)\\
&\big(\phi\theta^{-1}\psi^{-1}\psi,(x^\alpha/z^\alpha)\theta^{-1}\psi^{-1}\phi\cdot(y\psi^{-1}\phi)\backslash x\\
&RHS = LHS\\
\Leftrightarrow \big((x\theta^{-1}\psi^{-1}\phi)\cdot(y\psi^{-1}\phi\cdot z^\alpha\theta^{-1}\psi^{-1}\phi\backslash x^\alpha)\big) &= \big((x^\alpha/z^\alpha)\theta^{-1}\psi^{-1}\phi\cdot(y\psi^{-1}\phi)\backslash x\big)
\end{aligned}
$$

Let $\tau=\theta^{-1}\psi^{-1}\phi$, then $(x\tau)\cdot(y\theta\tau\cdot z^\alpha\tau)\backslash x^\alpha)=(x^\alpha/z^\alpha)\tau\cdot(y\theta\tau)\backslash x$.
Replacing y by $y(\theta\tau)^{-1}$, we have

$$
\begin{aligned}
(x\tau)\cdot(y(\theta\tau)^{-1}\theta\tau\cdot z^\alpha\tau)\backslash x^\alpha &= (x^\alpha/z^\alpha)\tau\cdot(y(\theta\tau)^{-1}\theta\tau)\backslash x\\
\Leftrightarrow (x\tau)\cdot(y\cdot z^\alpha\tau)\backslash x^\alpha &= (x^\alpha/z^\alpha)\tau\cdot(y\backslash x)\\
\Leftrightarrow (x\tau)\cdot(y\cdot z^\alpha\tau)\backslash x^\alpha &= (x^\alpha\tau/z^\alpha\tau)\cdot(y\backslash x)
\end{aligned}
$$

■

Corollary 3.12. Let $(Q,\cdot,/,\backslash)$ be a loop with holomorph $H(Q,\cdot)$. Then, $H(Q,\cdot)$ is a commutative generalised middle Bol loop if and only if $(\tau^{-1}M_{x^\alpha}^{-1}\tau, M_x, M_x^\alpha L_{x\tau})\in ATP(Q,\cdot)$

Proof. From the consequence of Theorem 3.10, we have

$$z^\alpha\tau^{-1}M_{x^\alpha}^{-1}\tau\cdot yM_x=(y\cdot z^\alpha)M_{x^\alpha}L_{x\tau} \tag{34}$$

$$\Leftrightarrow (\tau^{-1}M_{x^\alpha}^{-1}\tau, M_x, M_{x^\alpha}L_{x\tau})\in ATP(Q,\cdot) \tag{35}$$

■

Theorem 3.11. Let $(Q,\cdot,/,\backslash)$ be a commutative generalised middle Bol loop with a holomorph $(H,*)=H(Q,\cdot)$. If :

1. $\tau=\tau(a,b)=R_{(b\backslash a)}R_b^{-1}$ for each $\tau\in A(Q)$ and for any $a,b\in Q$
2. $M_x^{-1}R_{s\tau}=R_sR_x^{-1}R_{x\tau}$ for all $s,x\in Q$ and $\tau\in A(Q)$, then $H(Q,\cdot)$ is a GMBL.

Proof. From Corollary 3.12, observe that $(\tau^{-1}M_{x^\alpha}^{-1}\tau, M_x, M_x^\alpha L_{x\tau}) = (\tau^{-1}, M_x, I_e)\circ (M_{x^\alpha}^{-1}, M_x, M_{x^\alpha}L_x)\circ(\tau, M_x^{-1}, L_x^{-1}L_{x\tau})$. Where I_e is an identity map.

$$\text{Consider one hand },(\tau^{-1}, M_x, I_e) \in ATP(Q,\cdot) \Leftrightarrow a\tau^{-1}\cdot bM_x = ab$$
$$\Leftrightarrow a\tau^{-1}\cdot b\backslash x = ab$$
$$\Leftrightarrow a\tau^{-1}R_{b\backslash x} = aR_b$$
$$\Leftrightarrow \tau^{-1}R_{b\backslash x} = R_b \Leftrightarrow \tau = \tau(a,b) = R_{b\backslash a}R_b^{-1}$$

Also,

$$(\tau, M_x^{-1}, L_x^{-1}L_{x\tau}) \in ATP(Q,\cdot)$$
$$\Leftrightarrow s\tau\cdot yM_x^{-1} = (sy)L_x^{-1}L_{x\tau}$$
$$\Leftrightarrow yM_x^{-1}L_{s\tau} = yL_sL_x^{-1}L_{x\tau} \Leftrightarrow M_x^{-1}R_{s\tau} = R_sR_x^{-1}R_{x\tau}$$

■

Corollary 3.13. Let $(Q,\cdot,/,\backslash)$ be a commutative loop such that $M_x^{-1}R_{s\tau} = R_sR_x^{-1}R_{x\tau}$ for all $x, s\in Q$ and $\tau\in A(Q)$. $(H,*) = H(Q,\cdot)$ is commutative GMBL if and only if

1. $(Q,\cdot)$ is a generalised middle Bol loop
2. $\tau = \tau(a,b) = R_{b\backslash a}R_b^{-1}$ for arbitrarily fixed $a,b\in Q$ and for each $\tau\in A(Q)$

Proof. It is straightforward. ■

Conclusion

In this research, we have been able to shown that the two identities of GMBL are equivalent if the generalising map α is bijective such that it fixes the identity element. Also, among all the five parastrophes of GMBL, $(12)-$parastrophe of GMBL is a GMBL and $(13)-$ and $(123)-$ parastrophes of Q are GMBL of exponent two. In line with Lemma 3.2, it can be seen that $(13)-$ and $(123)-$parastrophes of GMBL of exponent two are loop. It is noted that $(23)-$ and $(132)-$parastrophes of GMBL with commutative property are totally symmetric. The work further reveals that, in $(13)-$parastrophe of Q, the right inverse element coincides with left inverse element if α is bijective such that $\alpha : e\to e$ which is one of the general property of middle Bol loop revealed by Kuznetsov in [15].

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Received
Revised
Accepted