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# A NOTE ON INTRA REGULARITY ON SEMIGROUPS OF PARTIAL TRANSFORMATIONS WITH INVARIANT SET

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**Abstract**

Let $X$ be any non-empty set and $P(X)$ denote the semigroup (under the composition of functions) of partial transformations on a set $X$. Let $Y$ be a fixed non-empty subset of $X$ and

$$\overline{PT}(X,Y) = \{\alpha \in P(X) : (\operatorname{dom}\alpha \cap Y)\alpha \subseteq Y\}.$$

Then $\overline{PT}(X,Y)$ is a semigroup consisting of all mappings in $P(X)$ that leave $Y \subseteq X$ invariant. In this paper, we present criteria for checking the intra-regularity of elements in $\overline{PT}(X,Y)$ and apply these results to quantify intra-regular elements in $\overline{PT}(X,Y)$, when $X$ is finite.

**Keywords:** partial transformation semigroup, intra regularity, invariant set.

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## 1. Introduction and preliminaries

Let $S$ be a semigroup. An element $a$ of $S$ is said to be *intra-regular* if there exist $x, y \in S$, such that $a = xa^2y$. The notion of intra regularity was introduced in Croisot's theory of decompositions [2, Section 4.1].

Let $X$ be a non-empty set and $T(X)$ denote the semigroup (under the composition of mappings) of all transformations from $X$ into itself. It is known as *full transformation semigroup*. The study of algebraic properties on semigroups in such types was started by Doss [4] in 1955. The author completely described its Green's relations. Particularly, a characterization of $\mathcal{J}$-relation can be directly used to identify intra regularity, because $\alpha \in T(X)$ is intra-regular if and only if $\alpha \ \mathcal{J} \ \alpha^2$. Several other properties of $T(X)$ have been researched extensively by the fact that any semigroup can be embedded in $T(X)$ for some an appropriate set $X$.

Let $Y$ be a non-empty subset of $X$. In 1966, Magill [7] introduced a subsemigroup of $T(X)$, defined by

$$\overline{T}(X,Y) = \{\alpha \in T(X) \mid Y\alpha \subseteq Y\}.$$

In addition, $\overline{T}(X,Y)$ is a generalization of $T(X)$, since $\overline{T}(X,X) = T(X)$. This fact inspired Homyam and Sanwong [5] to give a complete description of Green's relations on $\overline{T}(X,Y)$. Later in 2013, Choomanee *et al.* [1] used these results to provide characterization and number of intra-regular elements on $\overline{T}(X,Y)$.

For any non-empty set $X$, the super semigroup of all transformation semigroups on $X$ is a *partial transformation semigroup*, which is defined by

$$P(X) = \{\alpha : A \to X \mid A \subseteq X\}.$$

Its Green's relations was shown in [6]. Similarly, the characterization of intra-regular was explored immediately.

For a fixed non-empty subset $Y$ of $X$, in analogy with $\overline{T}(X,Y)$, consider

$$\overline{PT}(X,Y) = \{\alpha \in P(X) \mid (\operatorname{dom}\alpha \cap Y)\alpha \subseteq Y\},$$

where $\operatorname{dom}\alpha$ and $Y\alpha$ denote the domain of $\alpha$ and $(\operatorname{dom}\alpha \cap Y)\alpha$, respectively. Since $\overline{PT}(X,X) = P(X)$, we may regard $\overline{PT}(X,Y)$ as a generalization of $P(X)$. Note that $id_X$, the identity map on $X$, belongs to $\overline{PT}(X,Y)$.

We now provide important preliminaries for this paper. Some basic mathematical terminologies and relevant notations used in what follows on semigroups are prescribed. Further, we refer to [2, 3, 6] for more information. Indeed, throughout this paper, the functions are written on the right, i.e., in the composition $\alpha\beta$, $\alpha$ is applied first. For any $\alpha \in P(X)$, the notations $\operatorname{dom}\alpha$ and $\operatorname{im}\alpha$ denote the *domain* of $\alpha$ and the *range* of $\alpha$, respectively. Additionally, for any $x \in \operatorname{im}\alpha$, $x\alpha^{-1}$ denotes the set of inverse images of $x$ under $\alpha$, i.e., $x\alpha^{-1} = \{z \in \operatorname{dom}\alpha : z\alpha = x\}$. In addition, the following notation is applied

$$\alpha = \begin{pmatrix} X_i \\ a_i \end{pmatrix}.$$

Here, the script $i$ belongs to some (unmentioned) index set $I$, the abbreviation $\{a_i\}$ denotes $\{a_i : i \in I\}$ and that $\operatorname{im}\alpha = \{a_i\}$ and $a_i\alpha^{-1} = X_i \subseteq \operatorname{dom}\alpha$ where $\bigcup_{i\in I} X_i = \operatorname{dom}\alpha$. More specifically, when $\alpha \in \overline{PT}(X,Y)$, we have $Y\alpha \subseteq Y$. Thus, the domain of $\alpha$ can be divided into three parts as follows

$$\alpha = \begin{pmatrix} A_i & B_j & C_k \\ a_i & b_j & c_k \end{pmatrix},$$

where $A_i \cap Y \neq \emptyset$, $B_j, C_k \subseteq X \setminus Y$; $a_i, b_j \in Y$ and $c_k \in X \setminus Y$. Here, $I, J$ and $K$ can be empty.

In this paper, we describe the necessary and sufficient conditions for elements of $\overline{PT}(X,Y)$ to be intra-regular. The results recapture the known results on $P(X)$ when we focus on $Y = X$. Moreover, they are used to deduce the results for $T(X,Y)$, when elements with $X$ as their domain are considered. We also apply the results to quantify the intra-regular elements in the $\overline{PT}(X,Y)$ when $X$ is a finite set.

## 2. MAIN RESULTS

This section provides criteria for checking intra regularity of elements in $\overline{PT}(X,Y)$. By somewhat abusing the notation, we use $A\alpha$ to denote $(\operatorname{dom}\alpha \cap A)\alpha$ for any $A \subseteq X$. Note for $\alpha, \beta \in \overline{PT}(X,Y)$, we have $\operatorname{dom}(\alpha\beta) \subseteq \operatorname{dom}\alpha$, $\operatorname{im}(\alpha\beta) \subseteq \operatorname{im}\beta$, and $|\operatorname{im}\alpha| \leq |\operatorname{dom}\alpha|$.

**Lemma 1.** *Let $\alpha, \beta \in \overline{PT}(X,Y)$. Then, $\alpha = \gamma\beta\mu$ for some $\gamma, \mu \in \overline{PT}(X,Y)$ if and only if $|\operatorname{im}\alpha| \leq |\operatorname{im}\beta|, |Y\alpha| \leq |Y\beta|$, and $|\operatorname{im}\alpha \setminus Y| \leq |\operatorname{im}\beta \setminus Y|$.*

***Proof.*** Assume that $\alpha = \gamma\beta\mu$ for some $\gamma, \mu \in \overline{PT}(X,Y)$. Then $|\operatorname{im}\alpha| = |(\operatorname{dom}\alpha)\alpha| = |(\operatorname{dom}(\gamma\beta\mu))\gamma\beta\mu| \leq |(\operatorname{dom}(\gamma\beta))\gamma\beta\mu| = |(\operatorname{im}(\gamma\beta))\mu| \leq |\operatorname{im}(\gamma\beta)| \leq |\operatorname{im}\beta|$, $|Y\alpha| = |(\operatorname{dom}\alpha \cap Y)\alpha| = |(\operatorname{dom}(\gamma\beta\mu) \cap Y)\gamma\beta\mu| \leq |(\operatorname{dom}(\gamma\beta) \cap Y)\gamma\beta\mu| = |(Y\gamma\beta)\mu| \leq |Y\gamma\beta| \leq |Y\beta|$, and $|\operatorname{im}\alpha \setminus Y| = |\operatorname{im}(\gamma\beta\mu) \setminus Y| \leq |\operatorname{im}(\beta\mu) \setminus Y| = |(\operatorname{dom}(\beta\mu))\beta\mu \setminus Y| \leq |(\operatorname{dom}(\beta\mu))\beta \setminus Y| \leq |(\operatorname{dom}\beta)\beta \setminus Y| = |\operatorname{im}\beta \setminus Y|$.

Conversely, assume that $|\operatorname{im}\alpha| \leq |\operatorname{im}\beta|, |Y\alpha| \leq |Y\beta|$, and $|\operatorname{im}\alpha \setminus Y| \leq |\operatorname{im}\beta \setminus Y|$. Write

$$\alpha = \begin{pmatrix} A_i & B_j & C_k \\ a_i & b_j & c_k \end{pmatrix},$$

where $A_i \cap Y \neq \emptyset; B_j, C_k \subseteq X \setminus Y; \{a_i\}, \{b_j\} \subseteq Y$; and $\{c_k\} \subseteq X \setminus Y$. By our assumptions, we can write

$$\beta = \begin{pmatrix} U_i & U_l & V_m & W_n & W_k \\ u_i & u_l & v_m & w_n & w_k \end{pmatrix},$$

where $U_i \cap Y \neq \emptyset \neq U_l \cap Y; V_m, W_n, W_k \subseteq X \setminus Y; \{u_i\}, \{u_l\}, \{v_m\} \subseteq Y; \{w_n\}, \{w_k\} \subseteq X \setminus Y$; and $|I| + |J| + |K| \leq |I| + |L| + |M| + |N| + |K|$.

*Case* 1. $|J| \leq |L| + |M| + |N|$. Let $L \cup M \cup N = P \dot{\cup} Q$, such that $|P| = |J|$. Then, we can express $\{U_l\} \cup \{V_m\} \cup \{W_n\} = \{R_p\} \cup \{S_q\}$ and rewrite $\beta$ as

$$\beta = \begin{pmatrix} U_i & R_p & S_q & W_k \\ u_i & r_p & s_q & w_k \end{pmatrix}.$$

Since $|J| = |P|$, there exists a bijective function $\varphi : J \to P$. For each $i, j$, and $k$, choose $y_i \in U_i \cap Y$, $x_{j\varphi} \in R_{j\varphi}$, and $z_k \in W_k$, respectively. Now, define

$$\gamma = \begin{pmatrix} A_i & B_j & C_k \\ y_i & x_{j\varphi} & z_k \end{pmatrix} \text{ and } \mu = \begin{pmatrix} u_i & r_{j\varphi} & w_k \\ a_i & b_j & c_k \end{pmatrix}.$$

Hence, $\gamma, \mu \in \overline{PT}(X, Y)$ and $\alpha = \gamma\beta\mu$.

*Case* 2. $|J| > |L| + |M| + |N|$. Then, $\operatorname{im}\beta$ is an infinite set. This implies $|J| \le |I|$ or $|J| \le |K|$ are infinite cardinals.

*Subcase* 2.1. $|J| \le |I|$. Let $I = P\dot{\cup}Q$, such that $|P| = |I|$ and $|Q| = |J|$. Then, we can express $\{U_i\} = \{R_p\} \cup \{S_q\}$ in which $R_p \cap Y \neq \emptyset$ and rewrite $\beta$ as

$$\beta = \begin{pmatrix} R_p & S_q & U_l & V_m & W_n & W_k \\ r_p & s_q & u_l & v_m & w_n & w_k \end{pmatrix}.$$

Since $|P| = |I|$ and $|Q| = |J|$, there exist bijective functions $\varphi : I \to P$ and $\psi : J \to Q$. For each $i, j$, and $k$, choose $y_{i\varphi} \in R_{i\varphi} \cap Y, x_{j\psi} \in S_{j\psi}$ and $z_k \in W_k$, respectively. Define

$$\gamma = \begin{pmatrix} A_i & B_j & C_k \\ y_{i\varphi} & x_{j\psi} & z_k \end{pmatrix} \text{ and } \mu = \begin{pmatrix} r_{i\varphi} & s_{j\psi} & w_k \\ a_i & b_j & c_k \end{pmatrix}.$$

Hence, $\gamma, \mu \in \overline{PT}(X, Y)$ and $\alpha = \gamma\beta\mu$.

*Subcase* 2.2. $|J| \le |K|$. Let $K = G\dot{\cup}H$, such that $|G| = |J|$ and $|H| = |K|$. Then, we can express $\{W_k\} = \{D_g\} \cup \{E_h\}$ and rewrite $\beta$ as

$$\beta = \begin{pmatrix} U_i & U_l & V_m & W_n & D_g & E_h \\ u_i & u_l & v_m & w_n & d_g & e_h \end{pmatrix}.$$

Since $|G| = |J|$ and $|H| = |K|$, there exist bijective functions $\sigma : J \to G$ and $\theta : K \to H$. For each $i, j$, and $k$, choose $y_i \in U_i \cap Y, x_{j\sigma} \in D_{g\sigma}$ and $z_{k\theta} \in E_{k\theta}$, respectively. Define

$$\gamma = \begin{pmatrix} A_i & B_j & C_k \\ y_i & x_{j\sigma} & z_{k\theta} \end{pmatrix} \text{ and } \mu = \begin{pmatrix} u_i & d_{j\sigma} & e_{k\theta} \\ a_i & b_j & c_k \end{pmatrix}.$$

Hence, $\gamma, \mu \in \overline{PT}(X, Y)$ and $\alpha = \gamma\beta\mu$. ∎

Since $\operatorname{im}\alpha^2 \subseteq \operatorname{im}\alpha, Y\alpha^2 \subseteq Y\alpha$ and $\operatorname{im}\alpha^2 \setminus Y \subseteq \operatorname{im}\alpha \setminus Y$, we obtain the following criterion.

**Theorem 2.** *Let $\alpha \in \overline{PT}(X, Y)$. Then, $\alpha$ is intra-regular if and only if $|\operatorname{im}\alpha| = |\operatorname{im}\alpha^2|, |Y\alpha| = |Y\alpha^2|$ and $|\operatorname{im}\alpha \setminus Y| = |\operatorname{im}\alpha^2 \setminus Y|$.*

In order to re-writte the above criterion in terms of $\alpha$, where $\operatorname{im}\alpha$ is finite, the following three lemmas are needed.

**Lemma 3.** *Let* $\alpha \in \overline{PT}(X, Y)$ *be such that* $\alpha^2 \neq \emptyset$ *and* $\operatorname{im}\alpha$ *is finite. Then,* $|\operatorname{im}\alpha| = |\operatorname{im}\alpha^2|$ *if and only if* $\operatorname{im}\alpha \subseteq \operatorname{dom}\alpha$ *and* $\alpha|_{\operatorname{im}\alpha}$ *is injective.*

***Proof.*** Assume $|\operatorname{im}\alpha| = |\operatorname{im}\alpha^2|$. To show $\operatorname{im}\alpha \subseteq \operatorname{dom}\alpha$, we suppose, to the contrary, that there exists $x \in \operatorname{im}\alpha \setminus \operatorname{dom}\alpha$. Then, $|\operatorname{im}\alpha^2| = |(\operatorname{dom}\alpha^2)\alpha^2| \leq |(\operatorname{dom}\alpha)\alpha^2| = |(\operatorname{im}\alpha)\alpha| < |\operatorname{im}\alpha|$ which is a contradiction. Thus, $\operatorname{im}\alpha \subseteq \operatorname{dom}\alpha$. To show $\alpha|_{\operatorname{im}\alpha}$ is injective, assume the contrary that there exist distinct $x_1, x_2 \in \operatorname{im}\alpha$, such that $x_1\alpha = x_2\alpha$. Since $x_1, x_2 \in \operatorname{im}\alpha$, there exist $x_1', x_2' \in \operatorname{dom}\alpha$ such that $x_1'\alpha = x_1$ and $x_2'\alpha = x_2$. Since $\alpha$ is a function and $x_1 \neq x_2$, we obtain $x_1' \neq x_2'$. However, $x_1'\alpha^2 = x_1\alpha = x_2\alpha = x_2'\alpha^2$, which leads $|\operatorname{im}\alpha^2| < |\operatorname{im}\alpha|$, a contradiction.

Conversely, assume that $\operatorname{im}\alpha \subseteq \operatorname{dom}\alpha$ and $\alpha|_{\operatorname{im}\alpha}$ is injective. Let $\operatorname{im}\alpha = \{x_1, \ldots, x_n\} \subseteq \operatorname{dom}\alpha$. Since $\alpha|_{\operatorname{im}\alpha}$ is injective,

$$|\operatorname{im}\alpha| = |\{x_1, \ldots, x_n\}| = |\{x_1\alpha, \ldots, x_n\alpha\}| = |\operatorname{im}\alpha^2|. \qquad \blacksquare$$

**Lemma 4.** *Let* $\alpha \in \overline{PT}(X, Y)$ *be such that* $\alpha^2 \neq \emptyset$ *and* $\operatorname{im}\alpha$ *is finite. Then,* $|Y\alpha| = |Y\alpha^2|$ *if and only if* $Y\alpha \subseteq \operatorname{dom}\alpha$ *and* $\alpha|_{Y\alpha}$ *is injective.*

***Proof.*** Assume that $|Y\alpha| = |Y\alpha^2|$. To show $Y\alpha \subseteq \operatorname{dom}\alpha$, we suppose, to the contrary, that there exists $x \in Y\alpha \setminus \operatorname{dom}\alpha$. Then, $|Y\alpha^2| = |(Y\alpha)\alpha| < |Y\alpha|$ which is a contradiction. To show $\alpha|_{Y\alpha}$ is injective, we assume the contrary that there exist distinct $y_1, y_2 \in Y\alpha$, such that $y_1\alpha = y_2\alpha$. Since $y_1, y_2 \in Y\alpha$, there exist $y_1', y_2' \in \operatorname{dom}\alpha \cap Y$, such that $y_1'\alpha = y_1$ and $y_2'\alpha = y_2$. Since $\alpha$ is a function and $y_1 \neq y_2$, we obtain that $y_1' \neq y_2'$. Thus, $y_1'\alpha^2 = y_1\alpha = y_2\alpha = y_2'\alpha^2$, which leads $|Y\alpha^2| < |Y\alpha|$, a contradiction.

Conversely, assume that $Y\alpha \subseteq \operatorname{dom}\alpha$ and $\alpha|_{Y\alpha}$ is injective. Let $Y\alpha = \{y_1, \ldots, y_n\} \subseteq \operatorname{dom}\alpha$. Since $\alpha|_{Y\alpha}$ is injective,

$$|Y\alpha| = |\{y_1, \ldots, y_n\}| = |\{y_1\alpha, \ldots, y_n\alpha\}| = |Y\alpha^2|. \qquad \blacksquare$$

**Lemma 5.** *Let* $\alpha \in \overline{PT}(X, Y)$ *be such that* $\alpha^2 \neq \emptyset$ *and* $\operatorname{im}\alpha$ *is finite. Then,* $|\operatorname{im}\alpha \setminus Y| = |\operatorname{im}\alpha^2 \setminus Y|$ *if and only if* $\operatorname{im}\alpha \setminus Y \subseteq \operatorname{dom}\alpha$*,* $\alpha|_{\operatorname{im}\alpha \setminus Y}$ *is injective, and* $\operatorname{im}(\alpha|_{\operatorname{im}\alpha\setminus Y}) \subseteq X \setminus Y$.

***Proof.*** Assume that $|\operatorname{im}\alpha \setminus Y| = |\operatorname{im}\alpha^2 \setminus Y|$. To show $\operatorname{im}\alpha \setminus Y \subseteq \operatorname{dom}\alpha$, we suppose, to the contrary, that there exists $x \in (\operatorname{im}\alpha \setminus Y) \setminus \operatorname{dom}\alpha$. Then, $|\operatorname{im}\alpha^2 \setminus Y| \leq |(\operatorname{im}\alpha \setminus Y)\alpha| < |\operatorname{im}\alpha \setminus Y|$, which is a contradiction. To show $\alpha|_{\operatorname{im}\alpha\setminus Y}$ is injective, assume the contrary that there exist distinct $x_1, x_2 \in \operatorname{im}\alpha \setminus Y$ such that $x_1\alpha = x_2\alpha$. Since $x_1, x_2 \in \operatorname{im}\alpha \setminus Y$, there exist $x_1', x_2' \in \operatorname{dom}\alpha \setminus Y$ such that $x_1'\alpha = x_1$ and $x_2'\alpha = x_2$. Since $\alpha$ is a function and $x_1 \neq x_2$, then $x_1' \neq x_2'$. Thus, $x_1'\alpha^2 = x_1\alpha = x_2\alpha = x_2'\alpha^2$, which leads $|\operatorname{im}\alpha^2 \setminus Y| \leq |(\operatorname{im}\alpha \setminus Y)\alpha| < |\operatorname{im}\alpha \setminus Y|$, a contradiction. To show $\operatorname{im}(\alpha|_{\operatorname{im}\alpha\setminus Y}) \subseteq X \setminus Y$, we letting $z \in \operatorname{im}(\alpha|_{\operatorname{im}\alpha\setminus Y}) \cap Y$.

Hence, there exists $x \in (\operatorname{im}\alpha \setminus Y) \cap \operatorname{dom}\alpha$ such that $x\alpha = z$. This yields $|\operatorname{im}\alpha^2 \setminus Y| < |(\operatorname{im}\alpha \setminus Y)\alpha| \leq |\operatorname{im}\alpha \setminus Y|$, which is a contradiction.

Conversely, assume that all three aforementioned conditions hold. Let $\operatorname{im}\alpha \setminus Y = \{x_1, \ldots, x_n\} \subseteq \operatorname{dom}\alpha$. Since $\alpha|_{\operatorname{im}\alpha\setminus Y}$ is injective and $\operatorname{im}(\alpha|_{\operatorname{im}\alpha\setminus Y}) \subseteq X \setminus Y$, we obtain $|\operatorname{im}\alpha \setminus Y| = |\{x_1, \ldots, x_n\}| = |\{x_1\alpha, \ldots, x_n\alpha\}| = |\operatorname{im}\alpha^2 \setminus Y|$. ■

Combining Theorem 2 with Lemmas 3–5, we obtain the following theorem.

**Theorem 6.** *Let $\alpha \in \overline{PT}(X,Y)$ be such that $\operatorname{im}\alpha$ is finite. Then, $\alpha$ is intra-regular if and only if all of the following statements hold:*

1. $\operatorname{im}\alpha \subseteq \operatorname{dom}\alpha$;
2. $\alpha|_{\operatorname{im}\alpha}$ *is injective;*
3. $\operatorname{im}(\alpha|_{\operatorname{im}\alpha\setminus Y}) \subseteq X \setminus Y$.

We now provide properties of intra-regular elements with respect to the finiteness of their images.

**Lemma 7.** *Let $\alpha \in \overline{PT}(X,Y)$ be such that $\operatorname{im}\alpha$ is finite. If $\alpha$ is intra-regular, then the following statements hold:*

1. $(X \setminus Y)\alpha \subseteq (X \setminus Y) \cup Y\alpha$.
2. $|y\alpha^{-1} \cap Y\alpha| = 1$ *for all* $y \in Y\alpha$.
3. $|x\alpha^{-1} \cap (\operatorname{im}\alpha \setminus Y\alpha)| = 1$ *for all* $x \in \operatorname{im}\alpha \setminus Y\alpha$.

***Proof.*** Assume that $\alpha$ is intra-regular. According to Theorem 6, we can express this $\alpha$ as

$$\alpha = \begin{pmatrix} A_1 & \cdots & A_r & B_1 & \cdots & B_s & C_1 & \cdots & C_t \\ a_1 & \cdots & a_r & b_1 & \cdots & b_s & c_1 & \cdots & c_t \end{pmatrix},$$

where $A_i \cap Y \neq \emptyset, B_j, C_k \subseteq X \setminus Y$ and $a_i, b_j \in Y, c_k \in X \setminus Y$ for all $i \in I = \{1, \ldots, r\}, j \in J = \{1, \ldots, s\}$ and $k \in K = \{1, \ldots, t\}$. To show $J = \emptyset$, we suppose, to the contrary, that there exists $j_0 \in J$. Then $b_{j_0} \in \operatorname{im}\alpha \subseteq \operatorname{dom}\alpha$. Since $b_{j_0} \in Y$, we obtain $b_{j_0} \in A_{i_0}$ for some $i_0 \in I$. Hence, $B_{j_0}\alpha^2 = a_{i_0}$. Since $\operatorname{im}(\alpha|_{\operatorname{im}\alpha\setminus Y}) \subseteq X \setminus Y$, we obtain

$$|\operatorname{im}\alpha^2| = |\operatorname{im}\alpha^2 \cap Y| + |\operatorname{im}\alpha^2 \setminus Y| \leq r + (s-1) + t < r + s + t = |\operatorname{im}\alpha|$$

which is a contradiction by Theorem 2. Consequently,

$$(*) \qquad \alpha = \begin{pmatrix} A_1 & \cdots & A_r & C_1 & \cdots & C_t \\ a_1 & \cdots & a_r & c_1 & \cdots & c_t \end{pmatrix}.$$

1. According to $(*)$, we have $(X \setminus Y)\alpha \subseteq \operatorname{im}\alpha \subseteq Y\alpha \cup X \setminus Y$ which yields $(X \setminus Y)\alpha \subseteq (X \setminus Y) \cup Y\alpha$.

2. Since $\operatorname{im}\alpha \subseteq \operatorname{dom}\alpha$, we obtain $a_1,\ldots,a_r$ belong to $A_i$ for some $i \in I$. Since $\alpha|_{\operatorname{im}\alpha}$ is injective, we obtain $|A_i \cap \{a_1,\ldots,a_r\}| = 1$ for all $i \in I$, that is, $|y\alpha^{-1} \cap Y\alpha| = 1$ for all $y \in Y\alpha$.

3. Since $\operatorname{im}\alpha \subseteq \operatorname{dom}\alpha$ and $\alpha|_{\operatorname{im}\alpha}$ is injective, we obtain $c_1,\ldots,c_t$ belong to $C_k$ for some $k \in K$ and $|C_k \cap \{c_1,\ldots,c_t\}| = 1$ for all $k \in K$, that is, $|x\alpha^{-1} \cap (\operatorname{im}\alpha \setminus Y\alpha)| = 1$ for all $x \in \operatorname{im}\alpha \setminus Y\alpha$. ∎

For the rest of this section, $X$ is assumed to be is a finite set with $n$ elements and $\emptyset \neq Y \subseteq X$ has $m$ elements. The following lemmas will serve as our starting point as we count the intra-regular elements in $\overline{PT}(X,Y)$.

**Lemma 8.** *Let $|X| = n, |Y| = m$ and $\alpha \in \overline{PT}(X,Y)$ be intra-regular. If $\operatorname{im}\alpha \cap Y \neq \emptyset$, then $\alpha|_Y : (\operatorname{dom}\alpha \cap Y) \to Y$ has $\sum_{r=1}^{m}\sum_{s=0}^{m-r}\binom{m}{r}\binom{m-r}{s}r!r^s$ different forms.*

***Proof.*** Let $\emptyset \neq \operatorname{im}\alpha \cap Y = Y' = \{a_1,\ldots,a_r\}$, where $1 \le r \le m$. From $(*)$, we obtain $Y\alpha = \operatorname{im}\alpha \cap Y = Y'$. We can write

$$\alpha|_Y = \begin{pmatrix} A_1 & \cdots & A_r \\ a_1 & \cdots & a_r \end{pmatrix},$$

where $\bigcup_{i=1}^{r} A_i = \operatorname{dom}\alpha \cap Y$. By Theorem 6 and Lemma 7, we have $\operatorname{im}\alpha \subseteq \operatorname{dom}\alpha$, $\alpha|_{\operatorname{im}\alpha}$ is injective and $|A_i \cap \{a_1,\ldots,a_r\}| = 1$ for all $i = 1,\ldots,r$. Since the number of permutations on $\{a_1,\ldots,a_r\}$ is $r!$ and the number of ways of choosing $Y'$ with $|Y'| = r$ is equal to $\binom{m}{r}$, we obtain that $\alpha|_{Y'} : Y' \to Y'$ has $\binom{m}{r}r!$ different forms. Let $Y^* = (\operatorname{dom}\alpha \cap Y) \setminus Y'$ be such that $|Y^*| = s$. Then, the number of ways of choosing $Y^*$ is equal to $\binom{m-r}{s}$ and there are $r$ options for where to place each elements of $Y^*$ in $A_1,\ldots,A_r$. Moreover, since $1 \le r \le m$, we conclude that $\alpha|_Y$ has $\sum_{r=1}^{m}\sum_{s=0}^{m-r}\binom{m}{r}\binom{m-r}{s}r!r^s$ different forms. ∎

**Lemma 9.** *Let $|X| = n, |Y| = m$, $\alpha \in \overline{PT}(X,Y)$ be intra-regular and $X' = \operatorname{im}\alpha \setminus Y\alpha$. Then $\alpha|_{X'} : X' \to X'$ has $\sum_{t=0}^{n-m}\binom{n-m}{t}t!$ different forms.*

***Proof.*** Suppose that $|X'| = t$ where $0 \le t \le n-m$. By Lemma 7(3), $\alpha|_{X'}$ is a permutation on $X'$. Hence, $\alpha|_{X'}$ has $t!$ different forms. Since the number of ways of choosing $X'$ is equal to $\binom{n-m}{t}$, we obtain that $\alpha|_{X'} : X' \to X'$ has $\sum_{t=0}^{n-m}\binom{n-m}{t}t!$ different forms. ∎

**Theorem 10.** *The number of intra-regular elements in $\overline{PT}(X,Y)$ is*

$$\sum_{r=1}^{m}\sum_{s=0}^{m-r}\sum_{t=0}^{n-m}\sum_{j=0}^{n-m-t}\binom{m}{r}\binom{m-r}{s}\binom{n-m}{t}\binom{n-m-t}{j}r!t!r^s(r+t)^j$$

$$+\sum_{t=1}^{n-m}\sum_{j=0}^{n-m-t}\binom{n-m}{t}\binom{n-m-t}{j}t!t^j + 1,$$

*where* $|X| = n$ *and* $|Y| = m$.

***Proof.*** Let $\alpha \in \overline{PT}(X,Y)$ be an intra-regular element. Suppose that $Y' = Y\alpha = \{a_1, \ldots, a_r\}$ and $X' = \operatorname{im}\alpha \setminus Y\alpha = \{c_1, \ldots, c_t\}$.

*Case* 1. $Y' \neq \emptyset$. Lemma 7 allow us to express this $\alpha$ as

$$\alpha = \begin{pmatrix} A_1 & \cdots & A_r & C_1 & \cdots & C_t \\ a_1 & \cdots & a_r & c_1 & \cdots & c_t \end{pmatrix},$$

where $A_i \cap Y \neq \emptyset, C_k \subseteq X \setminus Y$ and $|A_i \cap \{a_1, \ldots, a_r\}| = 1, |C_k \cap \{c_1, \ldots, c_t\}| = 1$ for all $i = 1, \ldots, r$ and $k = 1, \ldots, t$. Let $Y^* = (\operatorname{dom}\alpha \cap Y) \setminus Y'$ be such that $|Y^*| = s$. By Lemmas 8 and 9, there are $\sum_{r=1}^{m} \sum_{s=0}^{m-r} \sum_{t=0}^{n-m} \binom{m}{r}\binom{m-r}{s}\binom{n-m}{t} r!t!r^s$ different forms of $\alpha|_{Y \cup X'}$. Let $X^* = \operatorname{dom}\alpha \setminus (Y \cup X')$ be such that $|X^*| = j$. Then, there are $r + t$ different ways to arrange each element of $X^*$ in $A_1, \ldots, A_r, C_1, \ldots, C_t$ and $X^*$ has $\binom{n-m-t}{j}$ forms, where $0 \leq j \leq n - m - t$. Therefore, the number of intra-regular elements in this case is

$$\sum_{r=1}^{m} \sum_{s=0}^{m-r} \sum_{t=0}^{n-m} \sum_{j=0}^{n-m-t} \binom{m}{r}\binom{m-r}{s}\binom{n-m}{t}\binom{n-m-t}{j} r!t!r^s(r+t)^j.$$

*Case* 2. $Y' = \emptyset$. If $X' = \emptyset$, then $\alpha = \emptyset$, which is intra-regular. If $X' \neq \emptyset$, then $\alpha|_{X'} : X' \to X'$ has $\sum_{t=1}^{n-m} \binom{n-m}{t} t!$ different forms by Lemma 9. Let $X^* = \operatorname{dom}\alpha \setminus X' \subseteq (X \setminus Y) \setminus X'$ be such that $|X^*| = j$. Then, there are $t$ different ways to arrange each element of $X^*$ in $C_1, \ldots, C_t$ and $X^*$ has $\binom{n-m-t}{j}$ forms, where $0 \leq j \leq n - m - t$. Therefore, the number of intra-regular elements in this case is

$$\sum_{t=1}^{n-m} \sum_{j=0}^{n-m-t} \binom{n-m}{t}\binom{n-m-t}{j} t!t^j.$$

As a result, $\overline{PT}(X,Y)$ has the following number of intra-regular elements

$$\sum_{r=1}^{m} \sum_{s=0}^{m-r} \sum_{t=0}^{n-m} \sum_{j=0}^{n-m-t} \binom{m}{r}\binom{m-r}{s}\binom{n-m}{t}\binom{n-m-t}{j} r!t!r^s(r+t)^j$$
$$+ \sum_{t=1}^{n-m} \sum_{j=0}^{n-m-t} \binom{n-m}{t}\binom{n-m-t}{j} t!t^j + 1.$$

This completes the proof. ∎

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## References


[1] W. Choomanee, P. Honyam and J. Sanwong, *Regularity in semigroups of transformations with invariant sets*, Int. J. Pure Appl. Math. **87(1)** (2013) 151–164. https://doi.org/10.12732/ijpam.v87i1.9

[2] A.H. Clifford and G.B. Preston, The Algebraic Theory of Semigroups, Vol. I, Mathematical Surveys, 7 (American Mathematical Society, Providence, R.I., 1961). https://doi.org/10.1090/surv/007.1

[3] A.H. Clifford and G.B. Preston, The Algebraic Theory of Semigroups, Vol. II, Mathematical Surveys, 7 (American Mathematical Society, Providence, R.I., 1967). https://doi.org/10.1090/surv/007.2

[4] C. Doss, Certain Equivalence Relations in Transformation Semigroups, Master's thesis, directed by D.D. Miller (University of Tennessee, 1955).

[5] P. Honyam and J. Sanwong, *Semigroups of transformations with invariant set*, J. Korean Math. Soc. **48(2)** (2011) 289–300. https://doi.org/10.4134/JKMS.2011.48.2.289

[6] J.M. Howie, Fundamentals of Semigroup Theory (London Mathematical Society Monographs, New Series, 12, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1995).

[7] K.D. Jr. Magill, *Subsemigroups of $S(X)$*, Math. Japonica **11** (1966) 109–115.



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