Discussiones Mathematicae
General Algebra and Applications 44 (2024) 73–91
https://doi.org/10.7151/dmgaa.1450

ON SYMMETRIC GENERALIZED (θ, η)-BIDERIVATIONS OF PRIME RINGS

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Abstract

In this paper, we characterize the actions of symmetric generalized (θ, η)-biderivations and generalized left (θ, η)-biderivations on Lie ideals and ideals of a prime ring $\mathscr{A}$. It is shown that $\mathscr{L}$ (nonzero square-closed Lie ideal of $\mathscr{A}$) $\subseteq \mathscr{Z}(\mathscr{A})$, whenever traces of these derivations satisfy any of the following conditions:

(i) $([l_1, l_2])^\Delta = 0$,
(ii) $(l_1 l_2)^\Delta \in \mathscr{Z}(\mathscr{A})$,
(iii) $([l_1, l_2])^\Delta = (l_1)^\theta \circ (l_2)^\Delta$,
(iv) $(l_1)^\Delta (l_2)^\Delta + (l_1)^\eta (l_2)^\theta \in \mathscr{Z}(\mathscr{A})$,
(v) $a_1((l_1)^\Delta (l_2)^\Delta + (l_1 l_2)^\theta) = 0$,
(vi) $(l_1)^\Delta (l_2)^\theta + (l_1)^\theta (l_2)^\Delta = 0$,
(vii) $([l_1, l_2])^\Delta + [(l_1)^\Delta, l_2] \in \mathscr{Z}(\mathscr{A})$,
(viii) $(l_1 l_2)^\Delta \pm (l_1)^\theta (l_2)^\Delta + (l_1 l_2)^\theta \in \mathscr{Z}(\mathscr{A})$, $\forall\, l_1, l_2 \in \mathscr{L}$, where $0 \neq a_1 \in \mathscr{A}$ is a fixed element, Δ is a trace of these biadditive mappings and θ, η are automorphisms of $\mathscr{A}$.

Keywords: Lie ideals, prime rings, generalized (θ, η)-biderivations, generalized left (θ, η)-biderivations.

2020 Mathematics Subject Classification: 16N60, 16U80, 16W25.

1. Introduction

In recent years, various authors have examined the commutativity of prime and semiprime rings, in reference of derivations, generalized derivations and generalized (θ, η) derivations (cf. [1,2,5,6,9–15,17]). Generalized biderivations were first introduced by Brešar [7] and further studied by Muthana [16]. Thereafter, in [4] Ashraf and Rehman had explored the concept of generalized (θ,η)-biderivations of rings and proved a few results regarding these derivations which motivates us to study more about these derivations and also to characterize generalized left (θ,η)-biderivations of rings.

Throughout the paper, $\mathscr{A}$ represents an associative ring with center $\mathscr{Z}(\mathscr{A})$. Further, for $a_1, b_1 \in \mathscr{A}$, the symbol $[a_1, b_1]$ (resp. $a_1 \circ b_1$) will denote the commutator $a_1b_1 - b_1a_1$ (resp. $a_1b_1 + b_1a_1$). An additive subgroup $\mathscr{L}$ of $\mathscr{A}$ is called a Lie ideal of $\mathscr{A}$ if $[\mathscr{L}, \mathscr{A}] \subseteq \mathscr{L}$ and it is a square-closed Lie ideal if $l_1^2 \in \mathscr{L}$, $\forall$ $l_1 \in \mathscr{L}$. It is easy to verify that if $\mathscr{L}$ is a square-closed nonzero Lie ideal, then $2l_1l_2 \in \mathscr{L}$, $\forall$ $l_1, l_2 \in \mathscr{L}$. Following [19], if $\mathscr{L}$ is a square-closed Lie ideal of $\mathscr{A}$, then $2\mathscr{A}[\mathscr{L},\mathscr{L}] \subseteq \mathscr{L}$ and $2[\mathscr{L},\mathscr{L}]\mathscr{A} \subseteq \mathscr{L}$. Suppose that $\theta, \eta : \mathscr{A} \to \mathscr{A}$ are endomorphisms of $\mathscr{A}$. Then, an additive mapping $\mathscr{D}$ is called a (θ,η)-derivation if $(a_1b_1)^{\mathscr{D}} = (a_1)^{\mathscr{D}}(b_1)^{\theta} + (a_1)^{\eta}(b_1)^{\mathscr{D}}$, $\forall$ $a_1, b_1 \in \mathscr{A}$. By [3], an additive mapping $F : \mathscr{A} \to \mathscr{A}$, is said to be a generalized (θ,η)-derivation, if there exists a (θ,η)-derivation $\mathscr{D} : \mathscr{A} \to \mathscr{A}$ such that $(a_1b_1)^F = (a_1)^F(b_1)^{\theta} + (a_1)^{\eta}(b_1)^{\mathscr{D}}$, $\forall$ $a_1, b_1 \in \mathscr{A}$. In addition, a mapping $\Psi : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ is symmetric if $(a_1, b_1)^{\Psi} = (b_1, a_1)^{\Psi}$, $\forall$ $a_1, b_1 \in \mathscr{A}$. Also, a mapping $\Delta : \mathscr{A} \to \mathscr{A}$ defined by $(a_1)^{\Delta} = (a_1, a_1)^{\Psi}$ is called a trace of Ψ. It is obvious that in case $\Psi : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ is symmetric mapping which is also biadditive, the trace of Ψ satisfies the relation $(a_1 + b_1)^{\Delta} = (a_1)^{\Delta} + (b_1)^{\Delta} + 2(a_1, b_1)^{\Psi}$, $\forall$ $a_1, b_1 \in \mathscr{A}$.

By a symmetric (θ,η)-biderivation, we mean a symmetric biadditive mapping $\mathscr{D} : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ such that $(a_1b_1, c_1)^{\mathscr{D}} = (a_1, c_1)^{\mathscr{D}}(b_1)^{\theta} + (a_1)^{\eta}(b_1, c_1)^{\mathscr{D}}$, $\forall$ $a_1, b_1, c_1 \in \mathscr{A}$ and a symmetric biadditive mapping $\Psi : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ is said to be a symmetric generalized (θ,η)-biderivation, if there exists a symmetric (θ,η)-biderivation $\mathscr{D} : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ such that $(a_1b_1, c_1)^{\Psi} = (a_1, c_1)^{\Psi}(b_1)^{\theta} + (a_1)^{\eta}(b_1, c_1)^{\mathscr{D}}$, $\forall$ $a_1, b_1, c_1 \in \mathscr{A}$. By [18], a symmetric left biderivation is a map $\mathscr{D}$ such that $(a_1b_1, c_1)^{\mathscr{D}} = a_1(b_1, c_1)^{\mathscr{D}} + b_1(a_1, c_1)^{\mathscr{D}}$, $\forall$ $a_1, b_1, c_1 \in \mathscr{A}$, where $\mathscr{D} : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ is a symmetric biadditive map. Similarly, a symmetric biadditive mapping $\mathscr{D} : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ is called a symmetric left (θ,η)-biderivation if $(a_1b_1, c_1)^{\mathscr{D}} = (a_1)^{\theta}(b_1, c_1)^{\mathscr{D}} + (b_1)^{\eta}(a_1, c_1)^{\mathscr{D}}$, $\forall$ $a_1, b_1, c_1 \in \mathscr{A}$. Also, a symmetric biadditive mapping $\Psi : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ is called a symmetric generalized left (θ,η)-biderivation, if there exists a symmetric left (θ,η)-biderivation $\mathscr{D} : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ such that $(a_1b_1, c_1)^{\Psi} = (a_1)^{\theta}(b_1, c_1)^{\Psi} + (b_1)^{\eta}(a_1, c_1)^{\mathscr{D}}$, $\forall$ $a_1, b_1, c_1 \in \mathscr{A}$.

In [21], Rehman and Huang had studied generalized (θ,η)-biderivations which satisfy some algebraic restrictions and assessed the commutativity of rings. This encouraged us to explore a few results from [18] and [22] for generalized (θ,η)-biderivations and generalized left (θ,η)-biderivations. In [22], Sandhu and Ali examined the action of generalized (θ,η)-derivations on Lie ideals of prime rings and established several algebraic identities. We establish some of these results in the framework of generalized (θ,η)-biderivations in Section 3 and analyse the action of these derivations on Lie ideals of rings. In Section 4, the notion of generalized left (θ,η)-biderivations is characterized. Furthermore, we extend some results of [18] for generalized left (θ,η)-biderivations.

2. Preliminary results

In this section, we discuss some key results which are frequently used in proving the main theorems of this paper. The proof of the upcoming lemmas are quite easy so we omit the proofs.

Lemma 1. *If $\mathscr{A}$ is a ring and $a_1, b_1, c_1 \in \mathscr{A}$, then the following statements hold:*

(i) $[a_1, b_1c_1] = b_1[a_1, c_1] + [a_1, b_1]c_1$;

(ii) $[a_1b_1, c_1] = a_1[b_1, c_1] + [a_1, c_1]b_1$;

(iii) $[a_1, b_1 + c_1] = [a_1, b_1] + [a_1, c_1]$;

(iv) $[a_1 + b_1, c_1] = [a_1, c_1] + [b_1, c_1]$;

(v) $[a_1b_1, a_1] = a_1[b_1, a_1]$;

(vi) $[a_1, a_1b_1] = a_1[a_1, b_1]$;

(vii) $[a_1, b_1a_1] = [a_1, b_1]a_1$;

(viii) $[b_1a_1, a_1] = [b_1, a_1]a_1$; (ix_1) $a_1\circ(b_1c_1) = (a_1\circ b_1)c_1 - b_1[a_1, c_1] = b_1(a_1\circ c_1) + [a_1, b_1]c_1$; (x_1) $(a_1b_1)\circ c_1 = a_1(b_1\circ c_1) - [a_1, c_1]b_1 = (a_1\circ c_1)b_1 + a_1[b_1, c_1]$.

Lemma 2. *If $\mathscr{L}$ is a nonzero Lie ideal of a ring $\mathscr{A}$ and f is an automorphism of $\mathscr{A}$ then $f(\mathscr{L})$ is a nonzero Lie ideal of $\mathscr{A}$. Moreover, if $\mathscr{L}$ is non-central, then $f(\mathscr{L})$ is non-central.*

Now onwards, $\mathscr{A}$ is a prime ring with $char(\mathscr{A}) \neq 2$ and $\mathscr{L}$ is a nonzero Lie ideal of $\mathscr{A}$ unless otherwise stated.

Lemma 3 [6, Lemma 4]**.** *If $\mathscr{L} \nsubseteq \mathscr{Z}(\mathscr{A})$ and $x_1, y_1 \in \mathscr{A}$ such that $x_1\mathscr{L}y_1 = (0)$, then either $x_1 = 0$ or $y_1 = 0$.*

Lemma 4 [21, Proposition 1]**.** *Suppose that there exists a symmetric (θ,η)-biderivation $\mathscr{D}$ of $\mathscr{A}$ with trace Δ and θ,η are automorphisms such that $(\mathscr{L})^{\Delta} = (0)$, then either $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$ or $\mathscr{D} = 0$.*

Lemma 5 [20, Lemma 2.6]**.** *If $[\mathscr{L},\mathscr{L}] = (0)$, then $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$.*

Lemma 6 [22, Lemma 2.6]**.** *Every square-closed Lie ideal $\mathscr{L} \nsubseteq \mathscr{Z}(\mathscr{A})$ contains a nonzero ideal $\mathscr{J} = 2\mathscr{A}[\mathscr{L},\mathscr{L}]\mathscr{A}$ of $\mathscr{A}$.*

By using Lemma 2 and 3, one can easily prove

Lemma 7. *Let $\mathscr{L} \nsubseteq \mathscr{Z}(\mathscr{A})$ and f be an automorphism of $\mathscr{A}$. If $x_1, y_1 \in \mathscr{A}$ such that $x_1 f(\mathscr{L}) y_1 = (0)$, then either $x_1 = 0$ or $y_1 = 0$.*

The next proposition is an extension to Lemma 2.6 of [20].

Proposition 8. *If η is an automorphism of $\mathscr{A}$ such that $[[(x_1)^\eta, (y_1)^\eta], [(l_1)^\eta, (l_2)^\eta]] = 0$, $\forall\, l_1, l_2, x_1, y_1 \in \mathscr{L}$, then $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$, where $\mathscr{L}$ is a square-closed Lie ideal of $\mathscr{A}$.*

Proof. By given hypothesis, we have

$$0 = [[(x_1)^\eta, (y_1)^\eta], [(l_1)^\eta, (l_2)^\eta]] = ([[x_1, y_1], [l_1, l_2]])^\eta \tag{2.1}$$

$\forall\, l_1, l_2, x_1, y_1 \in \mathscr{L}$. Since η is an automorphism, so equation (2.1) infers that $[[x_1, y_1], [l_1, l_2]] = 0$, $\forall\, l_1, l_2, x_1, y_1 \in \mathscr{L}$. If possible, let $\mathscr{L} \nsubseteq \mathscr{Z}(\mathscr{A})$. Then by replacing l_2 by $2l_1l_2$ in the last equation and using the fact $\mathrm{char}(\mathscr{A}) \neq 2$, we obtain

$$[[x_1, y_1], l_1][l_1, l_2] = 0.$$

Putting $2tl_2$ instead of l_2 in the above expression and applying $\mathrm{char}(\mathscr{A}) \neq 2$, we get

$$[[x_1, y_1], l_1]t[l_1, l_2] = 0$$

$\forall\, l_1, l_2, t, x_1, y_1 \in \mathscr{L}$, as $\mathrm{char}(\mathscr{A}) \neq 2$. By Lemma 3 the above equation infers that for each $l_1 \in \mathscr{L}$, either $[[x_1, y_1], l_1] = 0$, $\forall\, x_1, y_1 \in \mathscr{L}$ or $[l_1, l_2] = 0$, $\forall\, l_2 \in \mathscr{L}$. Let $A = \{l_1 \in \mathscr{L} : [[\mathscr{L}, \mathscr{L}], l_1] = (0)\}$ and $B = \{l_1 \in \mathscr{L} : [l_1, \mathscr{L}] = (0)\}$. Clearly, A and B are additive subgroups of $\mathscr{L}$ and $\mathscr{L} = A \cup B$. By Brauer's trick, either $\mathscr{L} = A$ or $\mathscr{L} = B$. Suppose that $\mathscr{L} = A$, then $[[x_1, y_1], l_1] = 0$, $\forall\, l_1, x_1, y_1 \in \mathscr{L}$. Now, replacing y_1 by $2y_1x_1$ and using $\mathrm{char}(\mathscr{A}) \neq 2$, we conclude that

$$[x_1, y_1][x_1, l_1] = 0. \tag{2.2}$$

Putting $2l_1y_1$ instead of l_1 in equation (2.2) and applying again the fact that $\mathrm{char}(\mathscr{A}) \neq 2$, we are left with $[x_1, y_1]l_1[x_1, y_1] = 0$, $\forall\, l_1, x_1, y_1 \in \mathscr{L}$ and by Lemma 3, $[\mathscr{L}, \mathscr{L}] = (0)$. By Lemma 5, $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$, which is a contradiction. On other hand, if $\mathscr{L} = B$, then $[\mathscr{L}, \mathscr{L}] = (0)$ and by Lemma 5, $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$, again a contradiction. Therefore, $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$. ∎

Corollary 9. *If η is an automorphism of $\mathscr{A}$ and $\mathscr{L}$ is a square-closed Lie ideal of $\mathscr{A}$ such that $[(\mathscr{L})^\eta, (\mathscr{L})^\eta] = (0)$, then $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$.*

Lemma 10. *If $\mathscr{J}$ is a nonzero ideal of $\mathscr{A}$ such that $[\mathscr{J}, \mathscr{J}] = (0)$, then $\mathscr{J} \subseteq \mathscr{Z}(\mathscr{A})$. Moreover, $\mathscr{A}$ is commutative.*

Proof. Straightforward. ∎

The proof of the following lemma is quite easy, so we omit the proof.

Lemma 11. *If $a_1 \in \mathscr{Z}(\mathscr{A})$ and $b_1 \in \mathscr{A}$ such that $a_1 b_1 \in \mathscr{Z}(\mathscr{A})$, then either $b_1 \in \mathscr{Z}(\mathscr{A})$ or $a_1 = 0$.*

Lemma 12 [22, Lemma 2.7]**.** *Let η and θ be automorphisms of $\mathscr{A}$ such that $[(\mathscr{L})^\eta, (\mathscr{L})^\theta] = (0)$. Then $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$.*

Proposition 13. *Let $\mathscr{J} \neq (0)$ be an ideal of $\mathscr{A}$ and $\mathscr{D}$ be a symmetric (θ, η)-biderivation with θ, η two automorphisms such that $([l_1, l_2], \mathscr{A})^{\mathscr{D}} = (0)$, $\forall\ l_1, l_2 \in \mathscr{J}$. Then, either $\mathscr{J} \subseteq \mathscr{Z}(\mathscr{A})$ or $\mathscr{D} = 0$.*

Proof. In view of the given hypothesis, we have

$$([l_1, l_2], \mathscr{A})^{\mathscr{D}} = (0), \forall\ l_1, l_2 \in \mathscr{J}.$$

Replacing l_2 by $l_2 l_1$ in the above equation and using it, we get $([l_1, l_2])^\eta (l_1, r)^{\mathscr{D}} = 0$, $\forall\ l_1, l_2 \in \mathscr{J}, r \in \mathscr{A}$. Further, taking rs in place of r, we are left with $([l_1, l_2])^\eta \mathscr{A} (l_1, s)^{\mathscr{D}} = (0)$, $\forall\ l_1, l_2 \in \mathscr{J}, s \in \mathscr{A}$ and by using the primeness of $\mathscr{A}$, this concludes that for each $l_1 \in \mathscr{J}$, either $([l_1, \mathscr{J}])^\eta = (0)$ or $(l_1, \mathscr{A})^{\mathscr{D}} = (0)$. This implies that either $([\mathscr{J}, \mathscr{J}])^\eta = (0)$ or $(\mathscr{J}, \mathscr{A})^{\mathscr{D}} = (0)$. As η is an automorphism, so the former case forces $[\mathscr{J}, \mathscr{J}] = (0)$ and by Lemma 10, we can deduce that $\mathscr{J} \subseteq \mathscr{Z}(\mathscr{A})$. In latter case, we have $(l_1, r)^{\mathscr{D}} = 0$, $\forall\ l_1 \in \mathscr{J}, r \in \mathscr{A}$. By putting $l_1 s$ instead of l_1, this gives that $(l_1)^\eta (s, r)^{\mathscr{D}} = 0$, $\forall\ l_1 \in \mathscr{J}, r, s \in \mathscr{A}$. Now replacing l_1 by $l_1 p$, we get $(l_1)^\eta (p)^\eta (s, r)^{\mathscr{D}} = 0$, $\forall\ l_1 \in \mathscr{J}, p, r, s \in \mathscr{A}$. Since $\mathscr{J}$ is nonzero and η is an automorphism, therefore the primeness of $\mathscr{A}$ implies that $\mathscr{D} = 0$. This completes the proof. ∎

In the forthcoming sections, $\mathscr{L}$ is a square-closed Lie ideal and θ, η are automorphisms of $\mathscr{A}$.

3. Symmetric generalized (θ, η)-biderivations

In this section, the action of generalized (θ, η)-biderivation on ideals and Lie ideals of rings is characterized. Also, we explore some results of [22] for generalized (θ, η)-biderivations of rings. In this section, Ψ represents a symmetric generalized (θ, η)-biderivation of $\mathscr{A}$ associated with a symmetric (θ, η)-biderivation $\mathscr{D}$ and Δ is a trace of Ψ.

Theorem 14. *Let $\mathscr{J}$ be a nonzero ideal of $\mathscr{A}$ such that $([l_1, l_2])^\Delta = 0$, $\forall\ l_1, l_2 \in \mathscr{J}$. Then either $\mathscr{J} \subseteq \mathscr{Z}(\mathscr{A})$ ($\mathscr{A}$ is commutative in this case) or $\mathscr{D} = 0$, and $\Psi = 0$.*

Proof. By hypothesis, we get

$$0 = ([l_1, l_2])^{\Delta} = ([l_1, l_2], [l_1, l_2])^{\Psi} \tag{3.1}$$

$\forall\ l_1, l_2 \in \mathscr{J}$. Putting $l_2 + r_1$ in place of l_2 in equation (3.1), we obtain that $0 = 2([l_1, l_2], [l_1, r_1])^{\Psi}$, $\forall\ l_1, l_2, r_1 \in \mathscr{J}$. As char$(\mathscr{A}) \neq 2$, so

$$([l_1, l_2], [l_1, r_1])^{\Psi} = 0. \tag{3.2}$$

Replacing r_1 by $r_1 i$ in the last expression, we get

$$\begin{aligned} 0 &= ([l_1, l_2], [l_1, r_1]i + r_1[l_1, i])^{\Psi} \\ &= ([l_1, l_2], [l_1, r_1])^{\Psi}(i)^{\theta} + ([l_1, r_1])^{\eta}([l_1, l_2], i)^{\mathscr{D}} + ([l_1, l_2], r_1)^{\Psi}([l_1, i])^{\theta} \\ &\quad + (r_1)^{\eta}([l_1, l_2], [l_1, i])^{\mathscr{D}} \\ &= ([l_1, r_1])^{\eta}([l_1, l_2], i)^{\mathscr{D}} + ([l_1, l_2], r_1)^{\Psi}([l_1, i])^{\theta} + (r_1)^{\eta}([l_1, l_2], [l_1, i])^{\mathscr{D}}. \end{aligned}$$

That is

$$([l_1, r_1])^{\eta}([l_1, l_2], i)^{\mathscr{D}} + ([l_1, l_2], r_1)^{\Psi}([l_1, i])^{\theta} + (r_1)^{\eta}([l_1, l_2], [l_1, i])^{\mathscr{D}} = 0 \tag{3.3}$$

$\forall\ l_1, l_2, r_1, i \in \mathscr{J}$. Replacing i by l_1 in the above equation, we obtain

$$([l_1, r_1])^{\eta}([l_1, l_2], l_1)^{\mathscr{D}} = 0. \tag{3.4}$$

Putting $l_1 + t$ in place of l_1, we get $([l_1, r_1])^{\eta}(([l_1, l_2], t)^{\mathscr{D}} + ([t, l_2], l_1)^{\mathscr{D}} + ([t, l_2], t)^{\mathscr{D}}) + ([t, r_1])^{\eta}(([l_1, l_2], l_1)^{\mathscr{D}} + ([l_1, l_2], t)^{\mathscr{D}} + ([t, l_2], l_1)^{\mathscr{D}}) = 0\ \forall\ l_1, l_2, r_1, t \in \mathscr{J}$.

Replacing l_1 by $-l_1$, we have $([l_1, r_1])^{\eta}(([l_1, l_2], t)^{\mathscr{D}} + ([t, l_2], l_1)^{\mathscr{D}}) + ([t, r_1])^{\eta}$ $([l_1, l_2], l_1)^{\mathscr{D}} = ([l_1, r_1])^{\eta}([t, l_2], t)^{\mathscr{D}} + ([t, r_1])^{\eta}(([l_1, l_2], t)^{\mathscr{D}} + ([t, l_2], l_1)^{\mathscr{D}})$ and using this in the above relation, we get $([l_1, r_1])^{\eta}(([l_1, l_2], t)^{\mathscr{D}} + ([t, l_2], l_1)^{\mathscr{D}}) + ([t, r_1])^{\eta}$ $([l_1, l_2], l_1)^{\mathscr{D}} = 0$, as $char(\mathscr{A}) \neq 2$. On taking tr_1 in place of r_1, we have

$$([l_1, t])^{\eta}(r_1)^{\eta}(([l_1, l_2], t)^{\mathscr{D}} + ([t, l_2], l_1)^{\mathscr{D}} = 0$$

$\forall\ l_1, l_2, r_1, t \in \mathscr{J}$. Taking $t = l_2$, we are left with

$$([l_1, l_2])^{\eta}(r_1)^{\eta}([l_1, l_2], l_2)^{\mathscr{D}} = 0$$

$\forall\ l_1, l_2, r_1 \in \mathscr{J}$. Since $\mathscr{A}$ is prime and η is an automorphism, therefore either $[l_1, l_2] = 0$ or $([l_1, l_2], l_2)^{\mathscr{D}} = 0$, $\forall\ l_1, l_2 \in \mathscr{J}$. This infers that

$$([l_1, l_2], l_2)^{\mathscr{D}} = 0. \tag{3.5}$$

On putting $l_2 = l_2 + t$, we deduce that

$$[l_1, l_2], t)^{\mathscr{D}} + ([l_1, t], l_2)^{\mathscr{D}} = 0$$

$\forall\ l_1, l_2, t \in \mathscr{J}$. Now, putting tl_1 instead of t in, we conclude that

$$(t)^\eta([l_1,l_2],l_1)^{\mathscr{D}} + ([l_1,t])^\eta(l_1,l_2)^{\mathscr{D}} = 0.$$

By using equation (3.5), the above equation leads to $([l_1,t])^\eta(l_1,l_2)^{\mathscr{D}} = 0$. Further, by taking $t = ti$, we obtain $([l_1,t])^\eta(i)^\eta(l_1,l_2)^{\mathscr{D}} = 0$, $\forall\ l_1,l_2,t,i \in \mathscr{J}$. As η is an automorphism and $\mathscr{A}$ is prime, so for each $l_1 \in \mathscr{J}$, either $(0) = [(l_1)^\eta, (\mathscr{J})^\eta]$ or $(l_1,\mathscr{J})^{\mathscr{D}} = (0)$. Therefore, for each $l_1 \in \mathscr{J}$, either $(0) = [l_1,\mathscr{J}]$ or $(l_1,\mathscr{J})^{\mathscr{D}} = (0)$. Let $A = \{l_1 \in \mathscr{J} : [l_1,\mathscr{J}] = (0)\}$ and $B = \{l_1 \in \mathscr{J} : (l_1,\mathscr{J})^{\mathscr{D}} = (0)\}$. Clearly, A and B are additive subgroups of $\mathscr{J}$ and $\mathscr{J} = A \cup B$. By Brauer's trick, either $\mathscr{J} = A$ or $\mathscr{J} = B$. If $\mathscr{J} = A$, then by Lemma 10, $\mathscr{J} \subseteq \mathscr{Z}(\mathscr{A})$. On other hand, if $\mathscr{J} = B$, then $(l_1,l_2)^{\mathscr{D}} = 0$, $\forall\ l_1,l_2 \in \mathscr{J}$. Now, replacing l_2 by l_2r, we have $(l_2)^\eta(l_1,r)^{\mathscr{D}} = 0$, $\forall\ l_1,l_2 \in \mathscr{J}, r \in \mathscr{A}$. This implies that

$$(\mathscr{J},\mathscr{A})^{\mathscr{D}} = (0)$$

and by Proposition 13, we have either $\mathscr{J} \subseteq \mathscr{Z}(\mathscr{A})$ or $\mathscr{D} = 0$. By using $\mathscr{D} = 0$ in (3.3), we get

$$([l_1,l_2],r_1)^\Psi([l_1,i])^\theta = 0$$

$\forall\ l_1,l_2,r_1 \in \mathscr{J}$ and by replacing r_1 by rr_1, we have $([l_1,l_2],r)^\Psi(r_1)^\theta([l_1,i])^\theta = 0$, $\forall\ l_1,l_2,r_1 \in \mathscr{J}, r \in \mathscr{A}$. As $\mathscr{A}$ is prime and θ is an automorphism of $\mathscr{A}$, so the last equation implies that for each $l_1 \in \mathscr{J}$ either $([l_1,\mathscr{J}],\mathscr{A})^\Psi = (0)$ or $([l_1,\mathscr{J}])^\theta = (0)$. This concludes that either $([\mathscr{J},\mathscr{J}],\mathscr{A})^\Psi = (0)$ or $([\mathscr{J},\mathscr{J}])^\theta = (0)$. If $([\mathscr{J},\mathscr{J}])^\theta = (0)$, then $[\mathscr{J},\mathscr{J}] = (0)$, as θ is an automorphism. By Lemma 10, the previous equation gives that $\mathscr{J} \subseteq \mathscr{Z}(\mathscr{A})$ and $\mathscr{A}$ is commutative. Now consider $([l_1,l_2],r)^\Psi = 0$, then by taking $l_2 = sl_2$ and using $\mathscr{D} = 0$, we obtain

$$(s,r)^\Psi([l_1,l_2])^\theta + ([l_1,s],r)^\Psi(l_2)^\theta = 0 \tag{3.6}$$

$\forall\ l_1,l_2 \in \mathscr{J}, r,s \in \mathscr{A}$. Now, replacing l_2 by r_1l_2 for $r_1 \in \mathscr{J}$ in (3.6) and using it to get

$$(s,r)^\Psi(r_1)^\theta([l_1,l_2])^\theta = 0$$

$\forall\ l_1,l_2,r_1 \in \mathscr{J}, r,s \in \mathscr{A}$. Again by using the primeness of $\mathscr{A}$ and the fact that θ is an automorphism of $\mathscr{A}$, the above equation implies that, either $\Psi = 0$ or $[\mathscr{J},\mathscr{J}] = (0)$. In view of Lemma 10, $[\mathscr{J},\mathscr{J}] = (0)$ infers that $\mathscr{J} \subseteq \mathscr{Z}(\mathscr{A})$ ($\mathscr{A}$ is commutative). With this our proof is completed. ■

Theorem 15. *If* $([l_1,l_2])^\Delta = 0$, $\forall\ l_1,l_2 \in \mathscr{L}$, *then either* $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$ *or* $\mathscr{D} = 0$, *and* $\Psi = 0$.

Proof. By the given hypothesis, we have $([l_1,l_2])^\Delta = 0$, $\forall\ l_1,l_2 \in \mathscr{L}$. If possible, let $\mathscr{L} \nsubseteq \mathscr{Z}(\mathscr{A})$. Then, by Lemma 6, there exists a nonzero ideal $\mathscr{J} = 2\mathscr{A}[\mathscr{L},\mathscr{L}]\mathscr{A} \subseteq \mathscr{L}$. Therefore, we have

$$([l_1,l_2])^\Delta = 0, \forall\ l_1,l_2 \in \mathscr{J}.$$

By Theorem 14, either $\mathscr{J} \subseteq \mathscr{Z}(\mathscr{A})$, or $\mathscr{D} = 0$ and $\Psi = 0$. Now, consider the case $\mathscr{J} \subseteq \mathscr{Z}(\mathscr{A})$, that is $2p[l_1, l_2]r \in \mathscr{Z}(\mathscr{A})$, $\forall\ l_1, l_2 \in \mathscr{L}, p, r \in \mathscr{A}$. By replacing r by rl, we have $2p[l_1, l_2]rl \in \mathscr{Z}(\mathscr{A})$, $\forall\ l_1, l_2, l \in \mathscr{L}, p, r \in \mathscr{A}$. By Lemma 11, either $2p[l_1, l_2]r = 0$ or $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$. Now, consider $2p[l_1, l_2]r = 0$, $\forall$ $l_1, l_2 \in \mathscr{L}, p, r \in \mathscr{A}$. As $\text{char}(\mathscr{A}) \neq 2$ and $\mathscr{A}$ is a prime ring, so the last relation implies that $[\mathscr{L}, \mathscr{L}] = (0)$. By applying Lemma 5, $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$. Thus, in each case, we have $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$, which is absurd. Hence, $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$ and this finishes the proof. ∎

The following theorem is an extension of [22, Theorem 3.7].

Theorem 16. *If $\mathscr{D}$ is nonzero and $(x_1y_1)^\Delta \in \mathscr{Z}(\mathscr{A})$, $\forall\ x_1, y_1 \in \mathscr{L}$, then $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$.*

Proof. Suppose that $\mathscr{D}$ is nonzero and

$$\big[(x_1y_1)^\Delta, \mathscr{A}\big] = (0) \tag{3.7}$$

$\forall\ x_1, y_1 \in \mathscr{L}$, where Δ is a trace of Ψ. If possible, let $\mathscr{L} \nsubseteq \mathscr{Z}(\mathscr{A})$. Now, replacing y_1 by $y_1 + z_1$ in (3.7) and using this, we obtain $2[(x_1y_1, x_1z_1)^\Psi, r] = 0$, $\forall\ x_1, y_1, z_1 \in \mathscr{L}, r \in \mathscr{A}$. As $\text{char}(\mathscr{A}) \neq 2$, so the last relation leads to

$$\big[(x_1y_1, x_1z_1)^\Psi, r\big] = 0. \tag{3.8}$$

Consider $2y_1j$ instead of y_1 in equation (3.8) and using the fact that $\text{char}(\mathscr{A}) \neq 2$, we get $(x_1y_1, x_1z_1)^\Psi[(j)^\theta, r] + [(x_1y_1)^\eta(j, x_1z_1)^\mathscr{D}, r] = 0$, $\forall\ x_1, y_1, z_1, j \in \mathscr{L}, r \in \mathscr{A}$. By replacing r by $r(j)^\theta$, the above equation implies that

$$\mathscr{A}\big[(x_1y_1)^\eta(j, x_1z_1)^\mathscr{D}, (j)^\theta\big] = (0).$$

This implies that $[(x_1y_1)^\eta(j, x_1z_1)^\mathscr{D}, (j)^\theta]\mathscr{A}[(x_1y_1)^\eta(j, x_1z_1)^\mathscr{D}, (j)^\theta] = (0)$ and by using the primeness of $\mathscr{A}$ it is obtained that

$$\big[(x_1y_1)^\eta(j, x_1z_1)^\mathscr{D}, (j)^\theta\big] = 0 \tag{3.9}$$

$\forall\ j, x_1, y_1, z_1 \in \mathscr{L}$. Thus, $(x_1y_1)^\eta[(j, x_1z_1)^\mathscr{D}, (j)^\theta] + [(x_1y_1)^\eta, (j)^\theta](j, x_1z_1)^\mathscr{D} = 0$ and putting $y_1 = 2x_1y_1$, we conclude that

$$\big[(x_1)^\eta, (j)^\theta\big](x_1)^\eta(y_1)^\eta(j, x_1z_1)^\mathscr{D} = 0$$

$\forall\ j, x_1, y_1, z_1 \in \mathscr{L}$, as $\text{char}(\mathscr{A}) \neq 2$. Taking $2z_1i$ in place of z_1 in the above equation, we get $[(x_1)^\eta, (j)^\theta](x_1)^\eta(y_1)^\eta(x_1)^\eta(z_1)^\eta(j, i)^\mathscr{D} = 0$, $\forall\ i, j, x_1, y_1, z_1 \in \mathscr{L}$. Then by using Lemma 7 in the preceding equation, we obtain for each $j \in \mathscr{L}$, either $[(x_1)^\eta, (j)^\theta](x_1)^\eta(y_1)^\eta(x_1)^\eta = 0$, $\forall\ x_1, y_1 \in \mathscr{L}$ or $(j, \mathscr{L})^\mathscr{D} = (0)$. Applying Brauer's trick, we have either $[(x_1)^\eta, (\mathscr{L})^\theta](x_1)^\eta(y_1)^\eta(x_1)^\eta = (0)$, $\forall\ x_1, y_1 \in \mathscr{L}$

or $(\mathscr{L}, \mathscr{L})^{\mathscr{D}} = (0)$. If $[(x_1)^\eta, (j)^\theta](x_1)^\eta (y_1)^\eta (x_1)^\eta = (0)$, $\forall\, j, x_1, y_1 \in \mathscr{L}$, then by Lemma 7, we get that for each $x_1 \in \mathscr{L}$, either $(x_1)^\eta = 0$ or $[(x_1)^\eta, (j)^\theta](x_1)^\eta = 0$, $\forall\, j \in \mathscr{L}$. In any case it follows that

$$\big[(x_1)^\eta, (j)^\theta\big](x_1)^\eta = 0. \tag{3.10}$$

Then by taking $j = 2jz_1$ in (3.10) and using the fact that $\mathrm{char}(\mathscr{A}) \neq 2$, we get

$$\big[(x_1)^\eta, (j)^\theta\big](z_1)^\theta (x_1)^\eta = 0 \tag{3.11}$$

$\forall\, j, x_1, z_1 \in \mathscr{L}$. On multiplying (3.10) from the right hand side by $(z_1)^\theta$, we find

$$\big[(x_1)^\eta, (j)^\theta\big](x_1)^\eta (z_1)^\theta = 0. \tag{3.12}$$

Subtracting (3.11) from (3.12), we have $[(x_1)^\eta, (j)^\theta][(x_1)^\eta, (z_1)^\theta] = 0$, $\forall\, j, x_1, z_1 \in \mathscr{L}$ and by replacing z_1 by $2z_1 j$, it gives $[(x_1)^\eta, (j)^\theta](z_1)^\theta[(x_1)^\eta, (j)^\theta] = 0$, $\forall\, j, x_1, z_1 \in \mathscr{L}$. Again by Lemma 7, $[(\mathscr{L})^\eta, (\mathscr{L})^\theta] = (0)$ and by Lemma 12, $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$, a contradiction.

On other hand, if we consider $(\mathscr{L}, \mathscr{L})^{\mathscr{D}} = (0)$. Then, by Lemma 4, we have $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$, a contradiction. Both of these cases lead to a contradiction. Hence, $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$. ∎

Corollary 17. *If $\mathscr{D}$ is nonzero and $(l_1)^\Delta \in \mathscr{Z}(\mathscr{A})$, $\forall\, l_1 \in \mathscr{L}$, then $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$.*

Theorem 18. *Let $([l_1, l_2])^\Delta = (l_1)^\theta \circ (l_2)^\Delta$, $\forall\, l_1, l_2 \in \mathscr{L}$. Then either $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$ or $\mathscr{D} = 0$, and $\Psi = 0$.*

Proof. The hypothesis gives that

$$([l_1, l_2])^\Delta = (l_1)^\theta \circ (l_2)^\Delta \tag{3.13}$$

$\forall\, l_1, l_2 \in \mathscr{L}$. Putting $l_1 + r_1$ instead of l_1 in (3.13), we get $([l_1, l_2])^\Delta + ([r_1, l_2])^\Delta + 2([l_1, l_2], [r_1, l_2])^\Psi = (l_1)^\theta \circ (l_2)^\Delta + (r_1)^\theta \circ (l_2)^\Delta$, $\forall\, l_1, l_2, r_1 \in \mathscr{L}$. By using (3.13), the last expression infers that

$$2([l_1, l_2], [r_1, l_2])^\Psi = 0.$$

As $\mathrm{char}(\mathscr{A}) \neq 2$, so the above equation implies $([l_1, l_2], [r_1, l_2])^\Psi = 0$. In particular, for $r_1 = l_1$, we obtain $0 = ([l_1, l_2], [l_1, l_2])^\Psi$. This implies $([\mathscr{L}, \mathscr{L}])^\Delta = (0)$. Therefore, by Theorem 15, either $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$ or $\mathscr{D} = 0$, and $\Psi = 0$. ∎

By using a similar technique with the necessary variations, one can easily prove the following result.

Theorem 19. *If $(l_1 \circ l_2)^\Delta = [(l_1)^\theta, (l_2)^\Delta]$, $\forall\, l_1, l_2 \in \mathscr{L}$, then either $\mathscr{L} \in \mathscr{Z}(\mathscr{A})$ or $\mathscr{D} = 0$, and $\Psi = 0$.*

Theorem 20. *If any one of the following holds true:*
(i) $[(l_1)^\Delta(l_2)^\Delta + (l_1)^\eta(l_2)^\theta, \mathscr{A}] = (0)$,
(ii) $[(l_1)^\Delta(l_2)^\Delta - (l_1)^\eta(l_2)^\theta, \mathscr{A}] = (0)$, $\forall\ l_1, l_2 \in \mathscr{L}$, *then* $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$.

Proof. (i) By the given hypothesis, we have

$$\big[(l_1)^\Delta(l_2)^\Delta + (l_1)^\eta(l_2)^\theta, \mathscr{A}\big] = (0) \tag{3.14}$$

$\forall\ l_1, l_2 \in \mathscr{L}$. Suppose that $\mathscr{L} \nsubseteq \mathscr{Z}(\mathscr{A})$. Replacing l_2 by $l_2 + i$ in (3.14), we get $2[(l_1)^\Delta(l_2, i)^\Psi, \mathscr{A}] = (0)$, $\forall\ l_1, l_2, i \in \mathscr{L}$. Since $\mathrm{char}(\mathscr{A}) \neq 2$, so the last relation infers that

$$\big[(l_1)^\Delta(l_2, i)^\Psi, \mathscr{A}\big] = (0).$$

Replacing i by l_2 in the above equation, we get

$$\big[(l_1)^\Delta(l_2)^\Delta, \mathscr{A}\big] = (0)$$

$\forall\ l_1, l_2 \in \mathscr{L}$. On combining (3.14) and the above equation, we have $[(l_1)^\eta(l_2)^\theta, \mathscr{A}] = (0)$, $\forall\ l_1, l_2 \in \mathscr{L}$. This implies that

$$(l_1)^\eta\big[(l_2)^\theta, r\big] + [(l_1)^\eta, r](l_2)^\theta = 0. \tag{3.15}$$

Taking $l_1 = 2l_1r_1$ in (3.15) and using the fact that $\mathrm{char}(\mathscr{A}) \neq 2$, we obtain $[(l_1)^\eta, r](r_1)^\eta(l_2)^\theta = 0$, $\forall\ l_1, l_2, r_1 \in \mathscr{L}, r \in \mathscr{A}$. By applying Lemma 7, we get $[(l_1)^\eta, r] = 0$. On replacing r with $(l_2)^\eta$, last expression infers that $[(\mathscr{L})^\eta, (\mathscr{L})^\eta] = (0)$. Thus, by Corollary 9, $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$. This is a contradiction to our supposition. Hence, $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$.

After applying the similar technique with necessary modifications, we can prove (ii). ∎

Consequently, we have

Corollary 21. *If any one of the following holds true:*
(i) $\big[(l_1)^\Delta(l_2)^\Delta + (l_1)^\eta(l_2)^\theta, \mathscr{A}\big] = (0)$,
(ii) $\big[(l_1)^\Delta(l_2)^\Delta - (l_1)^\eta(l_2)^\theta, \mathscr{A}\big] = (0)$,
$\forall\ l_1, l_2 \in \mathscr{A}$, *then* $\mathscr{A}$ *is commutative.*

Note that Q_{mr} stands for the right Utumi quotient ring (also called the maximal right ring of quotients) of $\mathscr{A}$. Then the center of Q_{mr} is called the extended centroid of $\mathscr{A}$ and is denoted by C.

The next result extends [22, Theorem 3.15].

Theorem 22. *If* $0 \neq a_1 \in \mathscr{A}$ *such that* $a_1((l_1)^\Delta(l_2)^\Delta + (l_1l_2)^\theta) = 0$, $\forall\ l_1, l_2 \in \mathscr{L}$, *then* $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$ *or there exists* $\lambda \in C$ *such that* $(x_1)^\Delta = \lambda(x_1)^\theta$, $\forall\ x_1 \in \mathscr{A}$ *and* $\lambda^2 = -1$.

Proof. By the given hypothesis,

$$a_1\big((l_1)^{\Delta}(l_2)^{\Delta} + (l_1 l_2)^{\theta}\big) = 0 \tag{3.16}$$

$\forall\ l_1, l_2 \in \mathscr{L}$. Let us assume that $\mathscr{L} \nsubseteq \mathscr{Z}(\mathscr{A})$. Then, replacing l_2 by $l_2 + z_1$ in equation (3.16), we get $2a_1(l_1)^{\Delta}(l_2, z_1)^{\Psi} = 0$, $\forall\ l_1, l_2, z_1 \in \mathscr{L}$. As $\mathrm{char}(\mathscr{A}) \neq 2$, so

$$a_1(l_1)^{\Delta}(l_2, z_1)^{\Psi} = 0. \tag{3.17}$$

Taking $2z_1 i$ instead of z_1 in (3.17) and by $\mathrm{char}(\mathscr{A}) \neq 2$, we get

$$a_1(l_1)^{\Delta}(z_1)^{\eta}(l_2, i)^{\mathscr{D}} = 0$$

$\forall\ i, l_1, l_2, z_1 \in \mathscr{L}$. By Lemma 7, either $a_1(\mathscr{L})^{\Delta} = (0)$ or $(\mathscr{L}, \mathscr{L})^{\mathscr{D}} = (0)$. If $a_1(\mathscr{L})^{\Delta} = (0)$, with this equation (3.16) implies that $a_1(l_1)^{\theta}(l_2)^{\theta} = 0$, $\forall\ l_1, l_2 \in \mathscr{L}$. By Lemma 7, $a_1 = 0$, which is not possible. Thus, $(\mathscr{L}, \mathscr{L})^{\mathscr{D}} = (0)$ and by Lemma 4, either $\mathscr{D} = 0$ or $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$ and this concludes that $\mathscr{D} = 0$, as we have assumed that $\mathscr{L} \nsubseteq \mathscr{Z}(\mathscr{A})$. Hence $\mathscr{D} = 0$. By replacing l_2 by $2rs[l_2, r_1]$ in (3.17) for $r, s \in \mathscr{A}$ and using $\mathscr{D} = 0$, we find $a_1(l_1)^{\Delta}(r, z_1)^{\Psi}\mathscr{A}[(l_2)^{\theta}, (r_1)^{\theta}] = (0)$. The primeness of $\mathscr{A}$ infers that, either $a_1(\mathscr{L})^{\Delta}(\mathscr{A}, \mathscr{L})^{\Psi} = (0)$ or $[(\mathscr{L})^{\theta}, (\mathscr{L})^{\theta}] = (0)$. As we have assumed that $\mathscr{L} \nsubseteq \mathscr{Z}(\mathscr{A})$, so by Corollary 9, we observe that the latter case is not possible. Therefore, $a_1(l_1)^{\Delta}(r, z_1)^{\Psi} = 0$ and by taking $l_1 = l_1 + l_2$, this gives

$$a_1(l_1, l_2)^{\Psi}(r, z_1)^{\Psi} = 0 \tag{3.18}$$

$\forall\ l_1, l_2, z_1 \in \mathscr{L}, r \in \mathscr{A}$. On putting $2l_2 r_1$ in place of l_2 in (3.18) and using $\mathscr{D} = 0$, we obtain

$$a_1(l_1, l_2)^{\Psi}(r_1)^{\theta}(r, z_1)^{\Psi} = 0 \tag{3.19}$$

$\forall\ l_1, l_2, z_1, r_1 \in \mathscr{L}, r \in r$. After multiplying (3.18) by $(r_1)^{\theta}$ from right hand side, we have

$$a_1(l_1, l_2)^{\Psi}(r, z_1)^{\Psi}(r_1)^{\theta} = 0. \tag{3.20}$$

On subtracting equation (3.19) from (3.20), we conclude that

$$a_1(l_1, l_2)^{\Psi}\big[(r, z_1)^{\Psi}, (r_1)^{\theta}\big] = 0$$

and by putting $r_1 = 2r_1 z_1$, we are left with $a_1(l_1, l_2)^{\Psi}(r_1)^{\theta}[(r, z_1)^{\Psi}, (z_1)^{\theta}] = 0$, $\forall$ $l_1, l_2, r_1, z_1 \in \mathscr{L}, r \in \mathscr{A}$. By Lemma 7, either $a_1(\mathscr{L}, \mathscr{L})^{\Psi} = (0)$ or $[(\mathscr{A}, z_1)^{\Psi}, (z_1)^{\theta}]$ $= (0)$, $\forall\ z_1 \in \mathscr{L}$. If $a_1(\mathscr{L}, \mathscr{L})^{\Psi} = (0)$, then $a_1(l_1)^{\Delta} = 0$, by using this in given hypothesis, we have $a_1(l_1)^\theta(l_2)^\theta = 0$, $\forall\, l_1, l_2 \in \mathscr{L}$ and by Lemma 7, $a_1 = 0$, which is a contradiction. Therefore,

$$\big[(r, z_1)^\Psi, (z_1)^\theta\big] = 0 \tag{3.21}$$

$\forall\, z_1 \in \mathscr{L}, r \in \mathscr{A}$. Replacing z_1 by $z_1 + x_1$ in (3.21), we obtain $[(r, z_1)^\Psi, (x_1)^\theta] + [(r, x_1)^\Psi, (z_1)^\theta] = 0$, $\forall\, x_1, z_1 \in \mathscr{L}, r \in \mathscr{A}$. Putting $z_1 = 2z_1y_1$ and using $\mathscr{D} = 0$, we have

$$(r, z_1)^\Psi\big[(y_1)^\theta, (x_1)^\theta\big] + (z_1)^\theta\big[(r, x_1)^\Psi, (y_1)^\theta\big] = 0 \tag{3.22}$$

$\forall\, x_1, y_1, z_1 \in \mathscr{L}, r \in \mathscr{A}$. Replacing z_1 by $2sp[z_1, x_1]$ in (3.22) and using $\mathscr{D} = 0$, we get $0 = 2(r, s)^\Psi(p[z_1, x_1])^\theta[(y_1)^\theta, (x_1)^\theta] + (s)^\theta(2p[z_1, x_1])^\theta[(r, x_1)^\Psi, \theta(y_1)] = 0$, $\forall\, x_1, y_1, z_1 \in \mathscr{L}, p, r, s \in \mathscr{A}$. As $2p[z_1, x_1] \in \mathscr{L}$, so by using (3.22), $(2p[z_1, x_1])^\theta$ $[(r, x_1)^\Psi, (y_1)^\theta] = -(r, 2p[z_1, x_1])^\Psi[(y_1)^\theta, (x_1)^\theta]$ and using this in last equation, we conclude that

$$\begin{aligned} 0 &= ((r, s)^\Psi(p[z_1, x_1])^\theta - (s)^\theta(r, p[z_1, x_1])^\Psi)\big[(y_1)^\theta, (x_1)^\theta\big] \\ &= ((r, s)^\Psi(p)^\theta - (s)^\theta(r, p)^\Psi)\big[(z_1)^\theta, (x_1)^\theta\big]\big[(y_1)^\theta, (x_1)^\theta\big] \end{aligned}$$

$\forall\, x_1, y_1, z_1 \in \mathscr{L}, p, r, s \in \mathscr{A}$, as $\mathscr{D} = 0$. By taking mp instead of p and using $\mathscr{D} = 0$, this concludes that

$$\big((r, s)^\Psi(m)^\theta - (s)^\theta(r, m)^\Psi\big)\mathscr{A}\big[(z_1)^\theta, (x_1)^\theta\big]\big[(y_1)^\theta, (x_1)^\theta\big] = (0)$$

$\forall\, x_1, y_1, z_1 \in \mathscr{L}, m, r, s \in \mathscr{A}$. As $\mathscr{A}$ is prime, so the last equation infers that either $(r, s)^\Psi(m)^\theta - (s)^\theta(r, m)^\Psi = 0$, $\forall\, m, r, s \in \mathscr{A}$ or $[(z_1)^\theta, (x_1)^\theta][(y_1)^\theta, (x_1)^\theta] = 0$, $\forall\, x_1, y_1, z_1 \in \mathscr{L}$. If

$$\big[(z_1)^\theta, (x_1)^\theta\big]\big[(y_1)^\theta, (x_1)^\theta\big] = 0$$

then by replacing z_1 by $2y_1z_1$, we find

$$\big[(y_1)^\theta, (x_1)^\theta\big](\mathscr{L})^\theta\big[(y_1)^\theta, (x_1)^\theta\big] = (0)$$

$\forall\, x_1, y_1 \in \mathscr{L}$ and by using Lemma 7 and Corollary 9, $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$, a contradiction. Thus,

$$(r, s)^\Psi(m)^\theta - (s)^\theta(r, m)^\Psi = 0$$

$\forall\, m, r, s \in \mathscr{A}$. Further, for each $r \in \mathscr{A}$, we define a function $f_r : \mathscr{A} \to \mathscr{A}$ by $(x_1)^{f_r} = (x_1, r)^\Psi = (r, x_1)^\Psi$. Then the previous equation implies that for each $r \in \mathscr{A}$

$$(s)^{f_r}(m)^\theta = (s)^\theta(m)^{f_r} \tag{3.23}$$

$\forall\, s, m \in \mathscr{A}$. On replacing s by st in (3.23) and using $\mathscr{D} = 0$, we have $(s)^{f_r}(t)^\theta(m)^\theta = (s)^\theta(t)^\theta(m)^{f_r}$, $\forall\, s, t, m \in \mathscr{A}$. As θ is an automorphism, so last equation infers that

$$(s)^{f_r}p(m)^\theta = (s)^\theta p(m)^{f_r}$$

$\forall\, m, p, s \in \mathscr{A}$. In view of [8, Lemma], there exists some $\lambda \in C$ such that $(s)^{f_r} = (s, r)^\Psi = \lambda(s)^\theta$, $\forall\, s \in \mathscr{A}$. In this way we find $(s, r)^\Psi = \lambda(s)^\theta$, $\forall\, s, r \in \mathscr{A}$. In particular for $s = r$, we have

$$(r, r)^\Psi = (r)^\Delta = \lambda(r)^\theta \tag{3.24}$$

$\forall\, r \in \mathscr{A}$. Then from the initial hypothesis, we get $a_1(\lambda^2 + 1)(l_1 l_2)^\theta = 0$, $\forall\, l_1, l_2 \in \mathscr{L}$. This infers that $\lambda^2 = -1$. ■

In similar way, one can prove the following result:

Theorem 23. *If $0 \neq a_1 \in \mathscr{A}$ such that $a_1((l_1)^\Delta(l_2)^\Delta - (l_1 l_2)^\theta) = 0$, $\forall\, l_1, l_2 \in \mathscr{L}$, then $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$ or there exists $\lambda \in C$ such that $(x_1)^\Delta = \lambda(x_1)^\theta$, $\forall\, x_1 \in \mathscr{A}$ and $\lambda^2 = 1$.*

4. Symmetric generalized left (θ, η)-biderivations

In this section, the behaviour of generalized left (θ, η)-biderivations on Lie ideals of rings is examined and we also extend some well known results of [18] in the framework of generalized left (θ, η)-biderivations. We now proceed with the following result which is an extension of ([18, Lemma 2]).

In this section, Ψ represents a symmetric generalized left (θ, η)-biderivation of $\mathscr{A}$ associated with a symmetric left (θ, η)-biderivation $\mathscr{D}$ and Δ is a trace of Ψ, ω is a trace of $\mathscr{D}$.

Proposition 24. *If $(\mathscr{L})^\omega = (0)$, then $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$ or $\mathscr{D} = 0$.*

Proof. Let $\mathscr{L} \nsubseteq \mathscr{Z}(\mathscr{A})$ and the given hypothesis $(l_1)^\omega = 0$, $\forall\, l_1 \in \mathscr{L}$. Now, replacing l_1 by $l_1 + l_2$ and using the fact $\mathrm{char}(\mathscr{A}) \neq 2$, we obtain

$$(l_1, l_2)^\mathscr{D} = 0. \tag{4.1}$$

$\forall\, l_1, l_2 \in \mathscr{L}$. Putting $l_1 = 2r[i, j]$, we get

$$([i, j])^\eta(r, l_2)^\mathscr{D} = 0 \tag{4.2}$$

$\forall\, i, j, l_2 \in \mathscr{L}, r \in \mathscr{A}$. Putting $2r[x_1, j]s$ instead of r, the above equation infers that $([i, j])^\eta(r)^\theta(2[x_1, j]s, l_2)^\mathscr{D} + 2([i, j])^\eta([x_1, j])^\eta(s)^\eta(r, l_2)^\mathscr{D} = 0$, $\forall\, i, j, l_2, x_1 \in$

$\mathscr{L}, r, s \in \mathscr{A}$. Since $2[x_1, j]s \in \mathscr{L}$, so by using (4.1) and $\text{char}(\mathscr{A}) \neq 2$, the previous equation implies that

$$([i, j])^\eta([x_1, j])^\eta \mathscr{A}(r, l_2)^{\mathscr{D}} = (0). \tag{4.3}$$

As $\mathscr{A}$ is prime, so equation (4.3) concludes that either $([i, j])^\eta([x_1, j])^\eta = 0$, $\forall$ $i, j, x_1 \in \mathscr{L}$ or $(\mathscr{A}, l_2)^{\mathscr{D}} = (0)$, $\forall$ $l_2 \in \mathscr{L}$.
The former case implies $([i, j])^\eta([x_1, j])^\eta = 0$, then by taking $2x_1 i$ instead of i and using $\text{char}(\mathscr{A}) \neq 2$, we have

$$[(x_1)^\eta, (j)^\eta])(i)^\eta([(x_1)^\eta, (j)^\eta]) = 0$$

$\forall$ $i, j, x_1 \in \mathscr{L}$. By Lemma 7 and Corollary 9, the preceding equation forces $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$, which is not possible. In latter case, we have

$$(r, l_2)^{\mathscr{D}} = 0 \tag{4.4}$$

$\forall$ $l_2 \in \mathscr{L}, r \in \mathscr{A}$. Further, replacing l_2 by $2[l_1, l_2]ps$, we have

$$2[(l_1)^\theta, (l_2)^\theta](p)^\theta(r, s)^{\mathscr{D}} + (s)^\eta(r, 2[l_1, l_2]s)^{\mathscr{D}} = 0$$

$\forall$ $l_1, l_2 \in \mathscr{L}, p, r, s \in \mathscr{A}$. As $2[l_1, l_2]s \in \mathscr{L}$, therefore by using (4.4) the last relation yields $[(l_1)^\theta, (l_2)^\theta]\mathscr{A}(r, s)^{\mathscr{D}} = (0)$, $\forall$ $l_1, l_2 \in \mathscr{L}, r, s \in \mathscr{A}$ and the primeness of $\mathscr{A}$ implies either $[(\mathscr{L})^\theta, (\mathscr{L})^\theta] = (0)$ or $\mathscr{D} = 0$. In view of Corollary 9, the former gives that $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$, a contradiction. Hence $\mathscr{D} = 0$. ■

Corollary 25. *If $(\mathscr{A}, \mathscr{L})^{\mathscr{D}} = (0)$, then $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$ or $\mathscr{D} = 0$.*

Theorem 26. *If $\mathscr{D}$ is nonzero and any one of the following holds true:*

(i) $(l_1)^\Delta(l_2)^\theta + (l_1)^\theta(l_2)^\Delta = 0$,
(ii) $(l_1)^\Delta(l_2)^\theta - (l_1)^\theta(l_2)^\Delta = 0$ $\forall$ $l_1, l_2 \in \mathscr{L}$, *then* $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$.

Proof. (i) Suppose that

$$(l_1)^\Delta(l_2)^\theta + (l_1)^\theta(l_2)^\Delta = 0 \tag{4.5}$$

$\forall$ $l_1, l_2 \in \mathscr{L}$. On replacing l_1 by $l_1 + r_1$ in (4.5) and using $\text{char}(\mathscr{A}) \neq 2$, we find $(l_1, r_1)^\Psi(l_2)^\theta = 0$, $\forall$ $l_1, l_2, r_1 \in \mathscr{L}$. Taking $2r[l_2, x_1]$ instead of l_2, the previous expression gives $2(l_1, r_1)^\Psi(r)^\theta([l_2, x_1])^\theta = 0$, $\forall$ $l_1, l_2, r_1, x_1 \in \mathscr{L}, r \in \mathscr{A}$. Since $\text{char}(\mathscr{A}) \neq 2$, so $(l_1, r_1)^\Psi\mathscr{A}([l_2, x_1])^\theta = (0)$. The primeness of $\mathscr{A}$ yields this either $(\mathscr{L}, \mathscr{L})^\Psi = (0)$ or $[(\mathscr{L})^\theta, (\mathscr{L})^\theta] = (0)$. By Corollary 9, the latter case infers that $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$. From the former case we have

$$(l_1, r_1)^\Psi = 0 \tag{4.6}$$

$\forall\ l_1, r_1 \in \mathscr{L}$. Replacing l_1 by $2r[l_1, i]s$, we get

$$(r)^{\theta}(2[l_1,i]s, r_1)^{\Psi} + 2([l_1,i])^{\eta}(s)^{\eta}(r,r_1)^{\mathscr{D}} = 0$$

$\forall\ l_1, i, r_1 \in \mathscr{L}, r, s \in \mathscr{A}$. As $2[l_1,i]s \in \mathscr{L}$, so by using (4.6), the preceding equation gives $([l_1,i])^{\eta}\mathscr{A}(r,r_1)^{\mathscr{D}} = (0)$, $\forall\ l_1, i, r_1 \in \mathscr{L}, r \in \mathscr{A}$. The primeness of $\mathscr{A}$ implies that either $[(\mathscr{L})^{\eta}, (\mathscr{L})^{\eta}] = (0)$ or $(\mathscr{A},\mathscr{L})^{\mathscr{D}} = (0)$. If $[(\mathscr{L})^{\eta}, (\mathscr{L})^{\eta}] = (0)$, then by Corollary 9, $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$. Now, consider the case $(\mathscr{A},\mathscr{L})^{\mathscr{D}} = (0)$. Then, by the previous corollary, $(\mathscr{A},\mathscr{L})^{\mathscr{D}} = (0)$ infers that $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$, since $\mathscr{D}$ is nonzero.

On applying the similar technique with necessary modifications, we obtain the same conclusion for (ii). This completes the proof. ■

Immediately, we obtain the next result which gives the commutativity of $\mathscr{A}$.

Corollary 27. *If $\mathscr{D}$ is nonzero and any one of the following holds true:*

(i) $(l_1)^{\Delta}(l_2)^{\theta} + (l_1)^{\theta}(l_2)^{\Delta} = 0,$

(ii) $(l_1)^{\Delta}(l_2)^{\theta} - (l_1)^{\theta}(l_2)^{\Delta} = 0\ \forall\ l_1, l_2 \in \mathscr{A}$*, then $\mathscr{A}$ is commutative.*

Proposition 28. *If $\mathscr{D}$ is nonzero and $(l_1)^{\Delta} \in \mathscr{Z}(\mathscr{A})$, $\forall\ l_1 \in \mathscr{L}$, then $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$.*

Proof. If possible, assume that $\mathscr{L} \nsubseteq \mathscr{Z}(\mathscr{A})$. By the given hypothesis, $\mathscr{D}$ is nonzero and $[(l_1)^{\Delta}, \mathscr{A}] = (0)$, $\forall\ l_1 \in \mathscr{L}$. Taking $l_1 + l_2$ instead of l_1, we obtain

$$\big[(l_1,l_2)^{\Psi}, r\big] = 0 \tag{4.7}$$

$\forall\ l_1, l_2 \in \mathscr{L}, r \in \mathscr{A}$. Taking $l_2 = 2r_1l_2$ in (4.7), we get

$$\big[(r_1)^{\theta}, r\big](l_1,l_2)^{\Psi} + \big[(l_2)^{\eta}(l_1,r_1)^{\mathscr{D}}, r\big] = 0$$

$\forall\ l_1, l_2, r_1 \in \mathscr{L}, r \in \mathscr{A}$. By taking $(r_1)^{\theta}r$ in place of r, the above equation yields $[(l_2)^{\eta}(l_1,r_1)^{\mathscr{D}}, (r_1)^{\theta}]\mathscr{A} = (0)$. Since $\mathscr{A}$ is prime, therefore

$$0 = \big[(l_2)^{\eta}(l_1,r_1)^{\mathscr{D}}, (r_1)^{\theta}\big] = (l_2)^{\eta}\big[(l_1,r_1)^{\mathscr{D}}, (r_1)^{\theta}\big] + \big[(l_2)^{\eta}, (r_1)^{\theta}\big](l_1,r_1)^{\mathscr{D}} \tag{4.8}$$

$\forall\ l_1, l_2, r_1 \in \mathscr{L}$. Putting $2x_1l_2$ in place of l_2 in (4.8) and using char$(\mathscr{A}) \neq 2$, we find

$$\big[(x_1)^{\eta}, (r_1)^{\theta}\big](l_2)^{\eta}(l_1,r_1)^{\mathscr{D}} = 0$$

$\forall\ l_1, l_2, r_1, x_1 \in \mathscr{L}$. By using Lemma 7, we get that for each $r_1 \in \mathscr{L}$, either $[(\mathscr{L})^{\eta}, (r_1)^{\theta}] = (0)$ or $(\mathscr{L}, r_1)^{\mathscr{D}} = (0)$. Therefore, $\mathscr{L}$ is a union of the subgroups $A = \{r_1 \in \mathscr{L} : [(\mathscr{L})^{\eta}, (r_1)^{\theta}] = (0)\}$ and $B = \{r_1 \in \mathscr{L} : (\mathscr{L}, r_1)^{\mathscr{D}} = (0)\}$.

Since a group cannot be the union of its proper subgroups, so we are forced to conclude that either $\mathscr{L} = A$ or $\mathscr{L} = B$. If $\mathscr{L} = A$, then $[(\mathscr{L})^{\eta}, (\mathscr{L})^{\theta}] = (0)$ and by Lemma 12, $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$, a contradiction to our assumption. Therefore, we are left with $\mathscr{L} = B$, i.e. $(\mathscr{L},\mathscr{L})^{\mathscr{D}} = (0)$. By Proposition 24, we get that $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$, a contradiction. Hence, $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$. ■

The following theorem is a generalization of [18, Theorem 7].

Theorem 29. *Let $\mathscr{D}$ be nonzero and $([l_1,l_2])^{\Delta}+[(l_1)^{\Delta},l_2] \in \mathscr{Z}(\mathscr{A})$, $\forall\, l_1,l_2 \in \mathscr{L}$. Then $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$.*

Proof. By the given hypothesis, we get

$$\big[([l_1,l_2])^{\Delta} + [(l_1)^{\Delta},l_2], r\big] = 0 \tag{4.9}$$

$\forall\, l_1,l_2 \in \mathscr{L}, r \in \mathscr{A}$. On replacing l_2 by $l_2 + r_1$, the last equation gives that $[([l_1,l_2],[l_1,r_1])^{\Psi}, r] = 0$, $\forall\, l_1,l_2,r_1 \in \mathscr{L}, r \in \mathscr{A}$. In particular $r_1 = l_2$, we have $[([l_1,l_2])^{\Delta}, r] = 0$, $\forall\, l_1,l_2 \in \mathscr{L}, r \in \mathscr{A}$. With this, (4.9) implies that

$$\big[[(l_1)^{\Delta},l_2],r\big] = 0 \tag{4.10}$$

$\forall\, l_1,l_2 \in \mathscr{L}, r \in \mathscr{A}$. Putting $2l_2r_1$ instead of l_2 in (4.10), we find that $[(l_1)^{\Delta},l_2]$ $[r_1,r] + [l_2,r][(l_1)^{\Delta},r_1] = 0$, $\forall\, l_1,l_2,r_1 \in \mathscr{L}, r \in \mathscr{A}$. On taking $r = r_1r$ and using (4.10), the previous equation implies that

$$[l_2,r_1]\mathscr{A}\big[(l_1)^{\Delta},r_1\big] = (0)$$

$\forall\, l_1,l_2,r_1 \in \mathscr{L}$. By the primeness of $\mathscr{A}$, the above expression infers that for each $r_1 \in \mathscr{L}$, either $[\mathscr{L},r_1] = (0)$ or $[(\mathscr{L})^{\Delta},r_1] = (0)$. This implies that either $[\mathscr{L},\mathscr{L}] = (0)$ or $[(\mathscr{L})^{\Delta},\mathscr{L}] = (0)$. In view of Lemma 5, the former case gives $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$ and by the latter case, we have $[(l_1)^{\Delta},l_2] = 0$, $\forall\, l_1,l_2 \in \mathscr{L}$. Further, putting $2rs[l_2,r_1]$ in place of l_2, we conclude that

$$\big[(l_1)^{\Delta},r\big]s[l_2,r_1] = 0 \tag{4.11}$$

$\forall\, l_1,l_2,r_1 \in \mathscr{L}\; r,s \in \mathscr{A}$. Since $\mathscr{A}$ is prime, so (4.11) implies that, either $(l_1)^{\Delta} \in \mathscr{Z}(\mathscr{A})$, $\forall\, l_1 \in \mathscr{L}$ or $[\mathscr{L},\mathscr{L}] = (0)$. If $(l_1)^{\Delta} \in \mathscr{Z}(\mathscr{A})$, $\forall\, l_1 \in \mathscr{L}$, then by Proposition 28, $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$. On the other hand, if $[\mathscr{L},\mathscr{L}] = (0)$, then by Lemma 5, $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$. Therefore, $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$. ■

Corollary 30. *If $\mathscr{D}$ is nonzero and $([l_1,l_2])^{\Delta} + [(l_1)^{\Delta},l_2] \in \mathscr{Z}(\mathscr{A})$, $\forall\, l_1,l_2 \in \mathscr{A}$, then $\mathscr{A}$ is commutative.*

Theorem 31. *If one of the following conditions hold:*

(i) $(l_1l_2)^{\Delta} + (l_1)^{\theta}(l_2)^{\Delta} + (l_1l_2)^{\theta} \in \mathscr{Z}(\mathscr{A})$

(ii) $(l_1l_2)^{\Delta} - (l_1)^{\theta}(l_2)^{\Delta} + (l_1l_2)^{\theta} \in \mathscr{Z}(\mathscr{A})$

$\forall\, l_1,l_2 \in \mathscr{L}$, *then* $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$ *or* $\mathscr{D} = 0$.

Proof. (i) In case $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$, then we are done. Assume that $\mathscr{L} \nsubseteq \mathscr{Z}(\mathscr{A})$ and by hypothesis, we have $[(l_1l_2)^{\Delta} + (l_1)^{\theta}(l_2)^{\Delta} + (l_1l_2)^{\theta}, \mathscr{A}] = (0)$, $\forall\ l_1, l_2 \in \mathscr{L}$. Now, replacing l_1 by $l_1 + z_1$ and using the fact that char$(\mathscr{A}) \neq 2$, we obtain that

$$\big[(l_1l_2, z_1l_2)^{\Psi}, r\big] = 0 \tag{4.12}$$

$\forall\ l_1, l_2, z_1 \in \mathscr{L}, r \in \mathscr{A}$. Taking $2jl_1$ in place of l_1 in (4.12) and again using char$(\mathscr{A}) \neq 2$, we get

$$\big[(j)^{\theta}, r\big](l_1l_2, z_1l_2)^{\Psi} + \big[(l_1l_2)^{\eta}(j, z_1l_2)^{\mathscr{D}}, r\big] = 0$$

$\forall\ l_1, l_2, z, j \in \mathscr{L}, r \in \mathscr{A}$. On putting $r = (j)^{\theta}r$ in the last equation, we find $[(l_1l_2)^{\eta}(j, z_1l_2)^{\mathscr{D}}, (j)^{\theta}]r = 0$. This implies that

$$\big[(l_1l_2)^{\eta}(j, z_1l_2)^{\mathscr{D}}, (j)^{\theta}\big]\mathscr{A}\big[(l_1l_2)^{\eta}(j, z_1l_2)^{\mathscr{D}}, (j)^{\theta}\big] = (0) \tag{4.13}$$

$\forall\ l_1, l_2, z_1, j \in \mathscr{L}, r \in \mathscr{A}$. Further the primeness of $\mathscr{A}$ implies that $0 = [(l_1l_2)^{\eta}(j, z_1l_2)^{\mathscr{D}}, (j)^{\theta}] = [(l_1l_2)^{\eta}, (j)^{\theta}](j, z_1l_2)^{\mathscr{D}} + (l_1l_2)^{\eta}[(j, z_1l_2)^{\mathscr{D}}, (j)^{\theta}]$. Replacing l_1 by $2l_1k$ and using char$(\mathscr{A}) \neq 2$, in the resulting equation, we have

$$\big[(l_1)^{\eta}, (j)^{\theta}\big](k)^{\eta}(l_2)^{\eta}(j, z_1l_2)^{\mathscr{D}} = 0$$

$\forall\ l_1, l_2, z_1, j, k \in \mathscr{L}$. Therefore, by Lemma 7, the previous equation infers that for each $j \in \mathscr{L}$, either $[(\mathscr{L})^{\eta}, (j)^{\theta}] = (0)$ or $(l_2)^{\eta}(j, z_1l_2)^{\mathscr{D}} = 0$, $\forall\ l_2, z_1 \in \mathscr{L}$. This implies that $[(\mathscr{L})^{\eta}, (\mathscr{L})^{\theta}] = (0)$ or $(l_2)^{\eta}(j, z_1l_2)^{\mathscr{D}} = 0$, $\forall\ j, l_2, z_1 \in \mathscr{L}$. By Lemma 12, the former case infers that $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$, which is contradiction to our assumption. Thus, we have $(l_2)^{\eta}(j, z_1l_2)^{\mathscr{D}} = 0$, $\forall\ j, l_2, z_1 \in \mathscr{L}$. Taking $l_2 + l_1$ in place of l_2, we get

$$(l_2)^{\eta}(j, z_1l_1)^{\mathscr{D}} + (l_1)^{\eta}(j, z_1l_2)^{\mathscr{D}} = 0 \tag{4.14}$$

$\forall\ j, l_1, l_2, z_1 \in \mathscr{L}$. Replacing l_2 by $2l_2k$ we have

$$(l_2)^{\eta}(k)^{\eta}(j, z_1l_1)^{\mathscr{D}} + (l_1)^{\eta}(z_1l_2)^{\theta}(j, k)^{\mathscr{D}} + (l_1)^{\eta}(k)^{\eta}(j, z_1l_2)^{\mathscr{D}} = 0$$

$\forall\ j, k, l_1, l_2, z_1 \in \mathscr{L}$. By (4.14),

$$(k)^{\eta}(j, z_1l_2)^{\mathscr{D}} = -(l_2)^{\eta}(j, z_1k)^{\mathscr{D}}, (k)^{\eta}(j, z_1l_1)^{\mathscr{D}} = -(l_1)^{\eta}(j, z_1k)^{\mathscr{D}}$$

and using these in last relation, we have

$$-((l_1)^{\eta} \circ (l_2)^{\eta})(j, z_1k)^{\mathscr{D}} + (l_1)^{\eta}(z_1)^{\theta}(l_2)^{\theta}(j, k)^{\mathscr{D}} = 0.$$

By putting $2jl_1$ in place of l_1, we have

$$[(j)^{\eta}, (l_2)^{\eta}](l_1)^{\eta}(j, z_1k)^{\mathscr{D}} = 0$$

$\forall\ j,k,l_1,l_2,z_1 \in \mathscr{L}$. Further, by Lemma 7, we obtain that for each $j \in \mathscr{L}$, either $[(j)^\eta, (\mathscr{L})^\eta] = (0)$ or $(j, z_1k)^\mathscr{D} = (0)$, $\forall\ k, z_1 \in \mathscr{L}$. This concludes that either $[(\mathscr{L})^\eta, (\mathscr{L})^\eta] = (0)$ or $(j, z_1k)^\mathscr{D} = 0$, $\forall\ j,k,z_1 \in \mathscr{L}$. By Corollary 9, the former case implies $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{A})$, a contradiction. Therefore, we have $(j, z_1k)^\mathscr{D} = 0$, $\forall$ $j,k,z_1 \in \mathscr{L}$ and on replacing z_1 by $2z_1l_2$, this infers that $(l_2)^\eta(k)^\eta(j,z_1)^\mathscr{D} = 0$, $\forall$ $j,k,l_2,z_1 \in \mathscr{L}$. Since η is an automorphism of $\mathscr{A}$, so by Lemma 7 the running equation gives $(\mathscr{L},\mathscr{L})^\mathscr{D} = (0)$. Moreover, by Proposition 24, $\mathscr{D} = 0$.

By using the same technique with necessary variations, we can obtain the same conclusion for the case (ii). ■

Corollary 32. *If* $(l_1l_2)^\Delta \pm (l_1)^\theta(l_2)^\Delta + (l_1l_2)^\theta \in \mathscr{Z}(\mathscr{A})$, $\forall\ l_1, l_2 \in \mathscr{A}$. *Then* $\mathscr{A}$ *is commutative or* $\mathscr{D} = 0$.

Acknowledgement

The second author gratefully acknowledges the financial assistance by UGC.

References

[1] A. Ali, D. Kumar and P. Miyan, *On generalized derivations and commutativity of prime and semiprime rings*, Hacet. J. Math. Stat. **40(3)** (2011) 367–374.

[2] F. Ali and M.A. Chaudhry, *On generalized* (α,β)*-derivations of semiprime rings*, Turk. J. Math. **35** (2011) 399–404.
https://doi.org/10.3906/mat-0906-60

[3] M. Ashraf, A. Ali and S. Ali, *On Lie ideals and generalized* (θ,ϕ)*-derivations in prime rings*, Comm. Algebra **32(8)** (2004) 2977–2985.
https://doi.org/10.1081/AGB-120039276

[4] M. Ashraf, N. Rehman, S. Ali and M.R. Mozumder, *On generalized* (σ,τ) *- biderivations in rings*, Asian European J. Math. **1** (2010) 1–14.

[5] D. Beniss, B. Fahid and A. Mamouni, *On Jordan ideals in prime rings with generalized derivations*, Comm. Korean Math. Soc. **32(3)** (2017) 495–502.
https://doi.org/10.4134/CKMS.c160146

[6] J. Bergen, I.N. Herstein and J.W. Kerr, *Lie ideals and derivations of prime rings*, J. Algebra **71** (1981) 259–267.
https://doi.org/10.1016/0021-8693(81)90120-4

[7] M. Brešar, *On generalized biderivations and related maps*, J. Algebra **172** (1995) 764–786.
https://doi.org/10.1006/jabr.1995.1069

[8] M. Brešar, *Semiderivations of prime rings*, Proc. Am. Math. Soc. **108(4)** (1990) 859–860.

[9] M.N. Daif and H.E. Bell, *Remarks on derivations on semiprime rings*, Int. J. Math. Math. Sci. **15** (1992) 205–206.

[10] B. Dhara and K.G. Pradhan, *A note on multiplicative (generalized)-derivations with annihilator conditions*, Georgian Math. J. **23(2)** (2016) 191–198.
https://doi.org/10.1515/gmj-2016-0020

[11] Ö. Gölbaşi and E. Koç, *Generalized derivation on Lie ideals in prime rings*, Turk. J. Math. **35** (2011) 23–28.
https://doi.org/10.3906/mat-0807-27

[12] Ö. Gölbaşi and E. Koç, *Notes on commutativity of prime rings with generalized derivation*, Comm. Fac. Sci. Univ. Ank. Series A1. **58(2)** (2009) 39–46.

[13] B. Hvala, *Generalized derivations in rings*, Comm. Algebra **26(4)** (1998) 1147–1166.
https://doi.org/10.1080/00927879808826190

[14] N. Jacobson, *Structure of rings*, Amer. Math. Soc. Coll. Pub. **37** (Amer. Math. Soc. Providence R.I., 1956).

[15] H. Marubayashi, M. Ashraf, N. Rehman and S. Ali, *On generalized (α, β)-derivations in prime rings*, Algebra Colloq. **17** (2010) 865–874.
https://doi.org/10.1142/S1005386710000805

[16] N.M. Muthana, *Left centralizer traces, generalized biderivations, left bi-multipliers and Jordan biderivations*, Aligarh Bull. Math. **26(2)** (2007) 33–45.

[17] M.A. Quadri, M.S. Khan and N. Rehman, *Generalized derivations and commutativity of prime rings*, Indian J. Pure Appl. Math. **34(9)** (2003) 1393–1396.

[18] C.J. Reddy, S.V. Kumar and K.M. Reddy, *Lie ideals with symmetric left biderivation in prime rings*, Italian J. Pure Appl. Math. **41** (2019) 158–166.

[19] N. Rehman and A.Z. Ansari, *Generalized left derivations acting as homomorphism and anti-homomorphism on Lie ideals of rings*, J. Egypt. Math. Soc. **22** (2014) 327–329.
https://doi.org/10.1016/j.joems.2013.12.015

[20] N. Rehman, *On commutativity of rings with generalized derivation*, Math. J. Okayama Univ. **44** (2002) 43–49.

[21] N. Rehman and S. Huang, *On Lie ideals and symmetric Generalized (α, β)- biderivation in Prime ring*, Miskolc Math. Notes **20** (2019) 1175–1183.
https://doi.org/10.18514/MMN.2019.2450

[22] G.S. Sandhu, S. Ali, A. Boua and D. Kumar, *On generalized (α, β)-derivations and Lie ideals of prime rings*, Rend. Circ. Mat. Pal. Series **2** (2021) 1401–1411.
https://doi.org/10.1007/s12215-021-00685-9

Received 28 June 2022
Revised 1 October 2022
Accepted 4 October 2022