

## ON SYMMETRIC GENERALIZED $(\theta, \eta)$ -BIDERIVATIONS OF PRIME RINGS

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### Abstract

In this paper, we characterize the actions of symmetric generalized  $(\theta, \eta)$ -biderivations and generalized left  $(\theta, \eta)$ -biderivations on Lie ideals and ideals of a prime ring  $\mathcal{A}$ . It is shown that  $\mathcal{L}$  (nonzero square-closed Lie ideal of  $\mathcal{A}$ )  $\subseteq \mathcal{Z}(\mathcal{A})$ , whenever traces of these derivations satisfy any of the following conditions:

- (i)  $([l_1, l_2])^\Delta = 0$ ,
- (ii)  $(l_1 l_2)^\Delta \in \mathcal{Z}(\mathcal{A})$ ,
- (iii)  $([l_1, l_2])^\Delta = (l_1)^\theta \circ (l_2)^\Delta$ ,
- (iv)  $(l_1)^\Delta (l_2)^\Delta + (l_1)^\eta (l_2)^\theta \in \mathcal{Z}(\mathcal{A})$ ,
- (v)  $a_1((l_1)^\Delta (l_2)^\Delta + (l_1 l_2)^\theta) = 0$ ,
- (vi)  $(l_1)^\Delta (l_2)^\theta + (l_1)^\theta (l_2)^\Delta = 0$ ,
- (vii)  $([l_1, l_2])^\Delta + [(l_1)^\Delta, l_2] \in \mathcal{Z}(\mathcal{A})$ ,
- (viii)  $(l_1 l_2)^\Delta \pm (l_1)^\theta (l_2)^\Delta + (l_1 l_2)^\theta \in \mathcal{Z}(\mathcal{A})$ ,  $\forall l_1, l_2 \in \mathcal{L}$ , where  $0 \neq a_1 \in \mathcal{A}$  is a fixed element,  $\Delta$  is a trace of these biadditive mappings and  $\theta, \eta$  are automorphisms of  $\mathcal{A}$ .

**Keywords:** Lie ideals, prime rings, generalized  $(\theta, \eta)$ -biderivations, generalized left  $(\theta, \eta)$ -biderivations.

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### 1. INTRODUCTION

In recent years, various authors have examined the commutativity of prime and semiprime rings, in reference of derivations, generalized derivations and generalized  $(\theta, \eta)$  derivations (cf. [1, 2, 5, 6, 9–15, 17]). Generalized biderivations were first

introduced by Brešar [7] and further studied by Muthana [16]. Thereafter, in [4] Ashraf and Rehman had explored the concept of generalized  $(\theta, \eta)$ -biderivations of rings and proved a few results regarding these derivations which motivates us to study more about these derivations and also to characterize generalized left  $(\theta, \eta)$ -biderivations of rings.

Throughout the paper,  $\mathcal{A}$  represents an associative ring with center  $\mathcal{Z}(\mathcal{A})$ . Further, for  $a_1, b_1 \in \mathcal{A}$ , the symbol  $[a_1, b_1]$  (resp.  $a_1 \circ b_1$ ) will denote the commutator  $a_1b_1 - b_1a_1$  (resp.  $a_1b_1 + b_1a_1$ ). An additive subgroup  $\mathcal{L}$  of  $\mathcal{A}$  is called a Lie ideal of  $\mathcal{A}$  if  $[\mathcal{L}, \mathcal{A}] \subseteq \mathcal{L}$  and it is a square-closed Lie ideal if  $l_1^2 \in \mathcal{L}, \forall l_1 \in \mathcal{L}$ . It is easy to verify that if  $\mathcal{L}$  is a square-closed nonzero Lie ideal, then  $2l_1l_2 \in \mathcal{L}, \forall l_1, l_2 \in \mathcal{L}$ . Following [19], if  $\mathcal{L}$  is a square-closed Lie ideal of  $\mathcal{A}$ , then  $2\mathcal{A}[\mathcal{L}, \mathcal{L}] \subseteq \mathcal{L}$  and  $2[\mathcal{L}, \mathcal{L}]\mathcal{A} \subseteq \mathcal{L}$ . Suppose that  $\theta, \eta : \mathcal{A} \rightarrow \mathcal{A}$  are endomorphisms of  $\mathcal{A}$ . Then, an additive mapping  $\mathcal{D}$  is called a  $(\theta, \eta)$ -derivation if  $(a_1b_1)^{\mathcal{D}} = (a_1)^{\mathcal{D}}(b_1)^{\theta} + (a_1)^{\eta}(b_1)^{\mathcal{D}}, \forall a_1, b_1 \in \mathcal{A}$ . By [3], an additive mapping  $F : \mathcal{A} \rightarrow \mathcal{A}$ , is said to be a generalized  $(\theta, \eta)$ -derivation, if there exists a  $(\theta, \eta)$ -derivation  $\mathcal{D} : \mathcal{A} \rightarrow \mathcal{A}$  such that  $(a_1b_1)^F = (a_1)^F(b_1)^{\theta} + (a_1)^{\eta}(b_1)^{\mathcal{D}}, \forall a_1, b_1 \in \mathcal{A}$ . In addition, a mapping  $\Psi : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$  is symmetric if  $(a_1, b_1)^{\Psi} = (b_1, a_1)^{\Psi}, \forall a_1, b_1 \in \mathcal{A}$ . Also, a mapping  $\Delta : \mathcal{A} \rightarrow \mathcal{A}$  defined by  $(a_1)^{\Delta} = (a_1, a_1)^{\Psi}$  is called a trace of  $\Psi$ . It is obvious that in case  $\Psi : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$  is symmetric mapping which is also biadditive, the trace of  $\Psi$  satisfies the relation  $(a_1 + b_1)^{\Delta} = (a_1)^{\Delta} + (b_1)^{\Delta} + 2(a_1, b_1)^{\Psi}, \forall a_1, b_1 \in \mathcal{A}$ .

By a symmetric  $(\theta, \eta)$ -biderivation, we mean a symmetric biadditive mapping  $\mathcal{D} : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$  such that  $(a_1b_1, c_1)^{\mathcal{D}} = (a_1, c_1)^{\mathcal{D}}(b_1)^{\theta} + (a_1)^{\eta}(b_1, c_1)^{\mathcal{D}}, \forall a_1, b_1, c_1 \in \mathcal{A}$  and a symmetric biadditive mapping  $\Psi : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$  is said to be a symmetric generalized  $(\theta, \eta)$ -biderivation, if there exists a symmetric  $(\theta, \eta)$ -biderivation  $\mathcal{D} : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$  such that  $(a_1b_1, c_1)^{\Psi} = (a_1, c_1)^{\Psi}(b_1)^{\theta} + (a_1)^{\eta}(b_1, c_1)^{\mathcal{D}}, \forall a_1, b_1, c_1 \in \mathcal{A}$ . By [18], a symmetric left biderivation is a map  $\mathcal{D}$  such that  $(a_1b_1, c_1)^{\mathcal{D}} = a_1(b_1, c_1)^{\mathcal{D}} + b_1(a_1, c_1)^{\mathcal{D}}, \forall a_1, b_1, c_1 \in \mathcal{A}$ , where  $\mathcal{D} : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$  is a symmetric biadditive map. Similarly, a symmetric biadditive mapping  $\mathcal{D} : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$  is called a symmetric left  $(\theta, \eta)$ -biderivation if  $(a_1b_1, c_1)^{\mathcal{D}} = (a_1)^{\theta}(b_1, c_1)^{\mathcal{D}} + (b_1)^{\eta}(a_1, c_1)^{\mathcal{D}}, \forall a_1, b_1, c_1 \in \mathcal{A}$ . Also, a symmetric biadditive mapping  $\Psi : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$  is called a symmetric generalized left  $(\theta, \eta)$ -biderivation, if there exists a symmetric left  $(\theta, \eta)$ -biderivation  $\mathcal{D} : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$  such that  $(a_1b_1, c_1)^{\Psi} = (a_1)^{\theta}(b_1, c_1)^{\mathcal{D}} + (b_1)^{\eta}(a_1, c_1)^{\mathcal{D}}, \forall a_1, b_1, c_1 \in \mathcal{A}$ .

In [21], Rehman and Huang had studied generalized  $(\theta, \eta)$ -biderivations which satisfy some algebraic restrictions and assessed the commutativity of rings. This encouraged us to explore a few results from [18] and [22] for generalized  $(\theta, \eta)$ -biderivations and generalized left  $(\theta, \eta)$ -biderivations. In [22], Sandhu and Ali examined the action of generalized  $(\theta, \eta)$ -derivations on Lie ideals of prime rings and established several algebraic identities. We establish some of these results in the framework of generalized  $(\theta, \eta)$ -biderivations in Section 3 and analyse the

action of these derivations on Lie ideals of rings. In Section 4, the notion of generalized left  $(\theta, \eta)$ -biderivations is characterized. Furthermore, we extend some results of [18] for generalized left  $(\theta, \eta)$ -biderivations.

## 2. PRELIMINARY RESULTS

In this section, we discuss some key results which are frequently used in proving the main theorems of this paper. The proof of the upcoming lemmas are quite easy so we omit the proofs.

**Lemma 1.** *If  $\mathcal{A}$  is a ring and  $a_1, b_1, c_1 \in \mathcal{A}$ , then the following statements hold:*

- (i)  $[a_1, b_1 c_1] = b_1 [a_1, c_1] + [a_1, b_1] c_1$ ;
- (ii)  $[a_1 b_1, c_1] = a_1 [b_1, c_1] + [a_1, c_1] b_1$ ;
- (iii)  $[a_1, b_1 + c_1] = [a_1, b_1] + [a_1, c_1]$ ;
- (iv)  $[a_1 + b_1, c_1] = [a_1, c_1] + [b_1, c_1]$ ;
- (v)  $[a_1 b_1, a_1] = a_1 [b_1, a_1]$ ;
- (vi)  $[a_1, a_1 b_1] = a_1 [a_1, b_1]$ ;
- (vii)  $[a_1, b_1 a_1] = [a_1, b_1] a_1$ ;
- (viii)  $[b_1 a_1, a_1] = [b_1, a_1] a_1$ ;  $(ix_1) a_1 \circ (b_1 c_1) = (a_1 \circ b_1) c_1 - b_1 [a_1, c_1] = b_1 (a_1 \circ c_1) + [a_1, b_1] c_1$ ;  $(x_1) (a_1 b_1) \circ c_1 = a_1 (b_1 \circ c_1) - [a_1, c_1] b_1 = (a_1 \circ c_1) b_1 + a_1 [b_1, c_1]$ .

**Lemma 2.** *If  $\mathcal{L}$  is a nonzero Lie ideal of a ring  $\mathcal{A}$  and  $f$  is an automorphism of  $\mathcal{A}$  then  $f(\mathcal{L})$  is a nonzero Lie ideal of  $\mathcal{A}$ . Moreover, if  $\mathcal{L}$  is non-central, then  $f(\mathcal{L})$  is non-central.*

Now onwards,  $\mathcal{A}$  is a prime ring with  $\text{char}(\mathcal{A}) \neq 2$  and  $\mathcal{L}$  is a nonzero Lie ideal of  $\mathcal{A}$  unless otherwise stated.

**Lemma 3** [6, Lemma 4]. *If  $\mathcal{L} \not\subseteq \mathcal{Z}(\mathcal{A})$  and  $x_1, y_1 \in \mathcal{A}$  such that  $x_1 \mathcal{L} y_1 = (0)$ , then either  $x_1 = 0$  or  $y_1 = 0$ .*

**Lemma 4** [21, Proposition 1]. *Suppose that there exists a symmetric  $(\theta, \eta)$ -biderivation  $\mathcal{D}$  of  $\mathcal{A}$  with trace  $\Delta$  and  $\theta, \eta$  are automorphisms such that  $(\mathcal{L})^\Delta = (0)$ , then either  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$  or  $\mathcal{D} = 0$ .*

**Lemma 5** [20, Lemma 2.6]. *If  $[\mathcal{L}, \mathcal{L}] = (0)$ , then  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$ .*

**Lemma 6** [22, Lemma 2.6]. *Every square-closed Lie ideal  $\mathcal{L} \not\subseteq \mathcal{Z}(\mathcal{A})$  contains a nonzero ideal  $\mathcal{I} = 2\mathcal{A}[\mathcal{L}, \mathcal{L}]\mathcal{A}$  of  $\mathcal{A}$ .*

By using Lemma 2 and 3, one can easily prove

**Lemma 7.** *Let  $\mathcal{L} \not\subseteq \mathcal{Z}(\mathcal{A})$  and  $f$  be an automorphism of  $\mathcal{A}$ . If  $x_1, y_1 \in \mathcal{A}$  such that  $x_1 f(\mathcal{L}) y_1 = (0)$ , then either  $x_1 = 0$  or  $y_1 = 0$ .*

The next proposition is an extension to Lemma 2.6 of [20].

**Proposition 8.** *If  $\eta$  is an automorphism of  $\mathcal{A}$  such that  $[[x_1]^\eta, (y_1)^\eta], [(l_1)^\eta, (l_2)^\eta] = 0, \forall l_1, l_2, x_1, y_1 \in \mathcal{L}$ , then  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$ , where  $\mathcal{L}$  is a square-closed Lie ideal of  $\mathcal{A}$ .*

**Proof.** By given hypothesis, we have

$$(2.1) \quad 0 = [[(x_1)^\eta, (y_1)^\eta], [(l_1)^\eta, (l_2)^\eta]] = ([[x_1, y_1], [l_1, l_2]])^\eta$$

$\forall l_1, l_2, x_1, y_1 \in \mathcal{L}$ . Since  $\eta$  is an automorphism, so equation (2.1) infers that  $[[x_1, y_1], [l_1, l_2]] = 0, \forall l_1, l_2, x_1, y_1 \in \mathcal{L}$ . If possible, let  $\mathcal{L} \not\subseteq \mathcal{Z}(\mathcal{A})$ . Then by replacing  $l_2$  by  $2l_1 l_2$  in the last equation and using the fact  $\text{char}(\mathcal{A}) \neq 2$ , we obtain

$$[[x_1, y_1], l_1][l_1, l_2] = 0.$$

Putting  $2tl_2$  instead of  $l_2$  in the above expression and applying  $\text{char}(\mathcal{A}) \neq 2$ , we get

$$[[x_1, y_1], l_1]t[l_1, l_2] = 0$$

$\forall l_1, l_2, t, x_1, y_1 \in \mathcal{L}$ , as  $\text{char}(\mathcal{A}) \neq 2$ . By Lemma 3 the above equation infers that for each  $l_1 \in \mathcal{L}$ , either  $[[x_1, y_1], l_1] = 0, \forall x_1, y_1 \in \mathcal{L}$  or  $[l_1, l_2] = 0, \forall l_2 \in \mathcal{L}$ . Let  $A = \{l_1 \in \mathcal{L} : [[\mathcal{L}, \mathcal{L}], l_1] = (0)\}$  and  $B = \{l_1 \in \mathcal{L} : [l_1, \mathcal{L}] = (0)\}$ . Clearly,  $A$  and  $B$  are additive subgroups of  $\mathcal{L}$  and  $\mathcal{L} = A \cup B$ . By Brauer's trick, either  $\mathcal{L} = A$  or  $\mathcal{L} = B$ . Suppose that  $\mathcal{L} = A$ , then  $[[x_1, y_1], l_1] = 0, \forall l_1, x_1, y_1 \in \mathcal{L}$ . Now, replacing  $y_1$  by  $2y_1 x_1$  and using  $\text{char}(\mathcal{A}) \neq 2$ , we conclude that

$$(2.2) \quad [x_1, y_1][x_1, l_1] = 0.$$

Putting  $2l_1 y_1$  instead of  $l_1$  in equation (2.2) and applying again the fact that  $\text{char}(\mathcal{A}) \neq 2$ , we are left with  $[x_1, y_1]l_1[x_1, y_1] = 0, \forall l_1, x_1, y_1 \in \mathcal{L}$  and by Lemma 3,  $[\mathcal{L}, \mathcal{L}] = (0)$ . By Lemma 5,  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$ , which is a contradiction. On other hand, if  $\mathcal{L} = B$ , then  $[\mathcal{L}, \mathcal{L}] = (0)$  and by Lemma 5,  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$ , again a contradiction. Therefore,  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$ .  $\blacksquare$

**Corollary 9.** *If  $\eta$  is an automorphism of  $\mathcal{A}$  and  $\mathcal{L}$  is a square-closed Lie ideal of  $\mathcal{A}$  such that  $[(\mathcal{L})^\eta, (\mathcal{L})^\eta] = (0)$ , then  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$ .*

**Lemma 10.** *If  $\mathcal{J}$  is a nonzero ideal of  $\mathcal{A}$  such that  $[\mathcal{J}, \mathcal{J}] = (0)$ , then  $\mathcal{J} \subseteq \mathcal{Z}(\mathcal{A})$ . Moreover,  $\mathcal{A}$  is commutative.*

**Proof.** Straightforward.  $\blacksquare$

The proof of the following lemma is quite easy, so we omit the proof.

**Lemma 11.** *If  $a_1 \in \mathcal{Z}(\mathcal{A})$  and  $b_1 \in \mathcal{A}$  such that  $a_1 b_1 \in \mathcal{Z}(\mathcal{A})$ , then either  $b_1 \in \mathcal{Z}(\mathcal{A})$  or  $a_1 = 0$ .*

**Lemma 12** [22, Lemma 2.7]. *Let  $\eta$  and  $\theta$  be automorphisms of  $\mathcal{A}$  such that  $[(\mathcal{L})^\eta, (\mathcal{L})^\theta] = (0)$ . Then  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$ .*

**Proposition 13.** *Let  $\mathcal{J} \neq (0)$  be an ideal of  $\mathcal{A}$  and  $\mathcal{D}$  be a symmetric  $(\theta, \eta)$ -biderivation with  $\theta, \eta$  two automorphisms such that  $([l_1, l_2], \mathcal{A})^\mathcal{D} = (0)$ ,  $\forall l_1, l_2 \in \mathcal{J}$ . Then, either  $\mathcal{J} \subseteq \mathcal{Z}(\mathcal{A})$  or  $\mathcal{D} = 0$ .*

**Proof.** In view of the given hypothesis, we have

$$([l_1, l_2], \mathcal{A})^\mathcal{D} = (0), \forall l_1, l_2 \in \mathcal{J}.$$

Replacing  $l_2$  by  $l_2 l_1$  in the above equation and using it, we get  $([l_1, l_2])^\eta (l_1, r)^\mathcal{D} = 0$ ,  $\forall l_1, l_2 \in \mathcal{J}, r \in \mathcal{A}$ . Further, taking  $rs$  in place of  $r$ , we are left with  $([l_1, l_2])^\eta \mathcal{A} (l_1, s)^\mathcal{D} = (0)$ ,  $\forall l_1, l_2 \in \mathcal{J}, s \in \mathcal{A}$  and by using the primeness of  $\mathcal{A}$ , this concludes that for each  $l_1 \in \mathcal{J}$ , either  $([l_1, \mathcal{J}])^\eta = (0)$  or  $(l_1, \mathcal{A})^\mathcal{D} = (0)$ . This implies that either  $([\mathcal{J}, \mathcal{J}])^\eta = (0)$  or  $(\mathcal{J}, \mathcal{A})^\mathcal{D} = (0)$ . As  $\eta$  is an automorphism, so the former case forces  $[\mathcal{J}, \mathcal{J}] = (0)$  and by Lemma 10, we can deduce that  $\mathcal{J} \subseteq \mathcal{Z}(\mathcal{A})$ . In latter case, we have  $(l_1, r)^\mathcal{D} = 0$ ,  $\forall l_1 \in \mathcal{J}, r \in \mathcal{A}$ . By putting  $l_1 s$  instead of  $l_1$ , this gives that  $(l_1)^\eta (s, r)^\mathcal{D} = 0$ ,  $\forall l_1 \in \mathcal{J}, r, s \in \mathcal{A}$ . Now replacing  $l_1$  by  $l_1 p$ , we get  $(l_1)^\eta (p)^\eta (s, r)^\mathcal{D} = 0$ ,  $\forall l_1 \in \mathcal{J}, p, r, s \in \mathcal{A}$ . Since  $\mathcal{J}$  is nonzero and  $\eta$  is an automorphism, therefore the primeness of  $\mathcal{A}$  implies that  $\mathcal{D} = 0$ . This completes the proof. ■

In the forthcoming sections,  $\mathcal{L}$  is a square-closed Lie ideal and  $\theta, \eta$  are automorphisms of  $\mathcal{A}$ .

### 3. SYMMETRIC GENERALIZED $(\theta, \eta)$ -BIDERIVATIONS

In this section, the action of generalized  $(\theta, \eta)$ -biderivation on ideals and Lie ideals of rings is characterized. Also, we explore some results of [22] for generalized  $(\theta, \eta)$ -biderivations of rings. In this section,  $\Psi$  represents a symmetric generalized  $(\theta, \eta)$ -biderivation of  $\mathcal{A}$  associated with a symmetric  $(\theta, \eta)$ -biderivation  $\mathcal{D}$  and  $\Delta$  is a trace of  $\Psi$ .

**Theorem 14.** *Let  $\mathcal{J}$  be a nonzero ideal of  $\mathcal{A}$  such that  $([l_1, l_2])^\Delta = 0$ ,  $\forall l_1, l_2 \in \mathcal{J}$ . Then either  $\mathcal{J} \subseteq \mathcal{Z}(\mathcal{A})$  ( $\mathcal{A}$  is commutative in this case) or  $\mathcal{D} = 0$ , and  $\Psi = 0$ .*

**Proof.** By hypothesis, we get

$$(3.1) \quad 0 = ([l_1, l_2])^\Delta = ([l_1, l_2], [l_1, l_2])^\Psi$$

$\forall l_1, l_2 \in \mathcal{J}$ . Putting  $l_2 + r_1$  in place of  $l_2$  in equation (3.1), we obtain that  $0 = 2([l_1, l_2], [l_1, r_1])^\Psi$ ,  $\forall l_1, l_2, r_1 \in \mathcal{J}$ . As  $\text{char}(\mathcal{A}) \neq 2$ , so

$$(3.2) \quad ([l_1, l_2], [l_1, r_1])^\Psi = 0.$$

Replacing  $r_1$  by  $r_1 i$  in the last expression, we get

$$\begin{aligned} 0 &= ([l_1, l_2], [l_1, r_1]i + r_1[l_1, i])^\Psi \\ &= ([l_1, l_2], [l_1, r_1])^\Psi(i)^\theta + ([l_1, r_1])^\eta([l_1, l_2], i)^\mathcal{S} + ([l_1, l_2], r_1)^\Psi([l_1, i])^\theta \\ &\quad + (r_1)^\eta([l_1, l_2], [l_1, i])^\mathcal{S} \\ &= ([l_1, r_1])^\eta([l_1, l_2], i)^\mathcal{S} + ([l_1, l_2], r_1)^\Psi([l_1, i])^\theta + (r_1)^\eta([l_1, l_2], [l_1, i])^\mathcal{S}. \end{aligned}$$

That is

$$(3.3) \quad ([l_1, r_1])^\eta([l_1, l_2], i)^\mathcal{S} + ([l_1, l_2], r_1)^\Psi([l_1, i])^\theta + (r_1)^\eta([l_1, l_2], [l_1, i])^\mathcal{S} = 0$$

$\forall l_1, l_2, r_1, i \in \mathcal{J}$ . Replacing  $i$  by  $l_1$  in the above equation, we obtain

$$(3.4) \quad ([l_1, r_1])^\eta([l_1, l_2], l_1)^\mathcal{S} = 0.$$

Putting  $l_1 + t$  in place of  $l_1$ , we get  $([l_1, r_1])^\eta([l_1, l_2], t)^\mathcal{S} + ([t, l_2], l_1)^\mathcal{S} + ([t, l_2], t)^\mathcal{S} + ([t, r_1])^\eta([l_1, l_2], l_1)^\mathcal{S} + ([l_1, l_2], t)^\mathcal{S} + ([t, l_2], l_1)^\mathcal{S} = 0 \forall l_1, l_2, r_1, t \in \mathcal{J}$ .

Replacing  $l_1$  by  $-l_1$ , we have  $([l_1, r_1])^\eta([l_1, l_2], t)^\mathcal{S} + ([t, l_2], l_1)^\mathcal{S} + ([t, r_1])^\eta([l_1, l_2], l_1)^\mathcal{S} = ([l_1, r_1])^\eta([t, l_2], t)^\mathcal{S} + ([t, r_1])^\eta([l_1, l_2], t)^\mathcal{S} + ([t, l_2], l_1)^\mathcal{S}$  and using this in the above relation, we get  $([l_1, r_1])^\eta([l_1, l_2], t)^\mathcal{S} + ([t, l_2], l_1)^\mathcal{S} + ([t, r_1])^\eta([l_1, l_2], l_1)^\mathcal{S} = 0$ , as  $\text{char}(\mathcal{A}) \neq 2$ . On taking  $tr_1$  in place of  $r_1$ , we have

$$([l_1, t])^\eta(r_1)^\eta([l_1, l_2], t)^\mathcal{S} + ([t, l_2], l_1)^\mathcal{S} = 0$$

$\forall l_1, l_2, r_1, t \in \mathcal{J}$ . Taking  $t = l_2$ , we are left with

$$([l_1, l_2])^\eta(r_1)^\eta([l_1, l_2], l_2)^\mathcal{S} = 0$$

$\forall l_1, l_2, r_1 \in \mathcal{J}$ . Since  $\mathcal{A}$  is prime and  $\eta$  is an automorphism, therefore either  $[l_1, l_2] = 0$  or  $([l_1, l_2], l_2)^\mathcal{S} = 0$ ,  $\forall l_1, l_2 \in \mathcal{J}$ . This infers that

$$(3.5) \quad ([l_1, l_2], l_2)^\mathcal{S} = 0.$$

On putting  $l_2 = l_2 + t$ , we deduce that

$$[l_1, l_2], t)^\mathcal{S} + ([l_1, t], l_2)^\mathcal{S} = 0$$

$\forall l_1, l_2, t \in \mathcal{J}$ . Now, putting  $tl_1$  instead of  $t$  in, we conclude that

$$(t)^\eta([l_1, l_2], l_1)^\mathcal{D} + ([l_1, t])^\eta(l_1, l_2)^\mathcal{D} = 0.$$

By using equation (3.5), the above equation leads to  $([l_1, t])^\eta(l_1, l_2)^\mathcal{D} = 0$ . Further, by taking  $t = ti$ , we obtain  $([l_1, t])^\eta(i)^\eta(l_1, l_2)^\mathcal{D} = 0, \forall l_1, l_2, t, i \in \mathcal{J}$ . As  $\eta$  is an automorphism and  $\mathcal{A}$  is prime, so for each  $l_1 \in \mathcal{J}$ , either  $(0) = [(l_1)^\eta, (\mathcal{J})^\eta]$  or  $(l_1, \mathcal{J})^\mathcal{D} = (0)$ . Therefore, for each  $l_1 \in \mathcal{J}$ , either  $(0) = [l_1, \mathcal{J}]$  or  $(l_1, \mathcal{J})^\mathcal{D} = (0)$ . Let  $A = \{l_1 \in \mathcal{J} : [l_1, \mathcal{J}] = (0)\}$  and  $B = \{l_1 \in \mathcal{J} : (l_1, \mathcal{J})^\mathcal{D} = (0)\}$ . Clearly,  $A$  and  $B$  are additive subgroups of  $\mathcal{J}$  and  $\mathcal{J} = A \cup B$ . By Brauer's trick, either  $\mathcal{J} = A$  or  $\mathcal{J} = B$ . If  $\mathcal{J} = A$ , then by Lemma 10,  $\mathcal{J} \subseteq \mathcal{Z}(\mathcal{A})$ . On other hand, if  $\mathcal{J} = B$ , then  $(l_1, l_2)^\mathcal{D} = 0, \forall l_1, l_2 \in \mathcal{J}$ . Now, replacing  $l_2$  by  $l_2r$ , we have  $(l_2)^\eta(l_1, r)^\mathcal{D} = 0, \forall l_1, l_2 \in \mathcal{J}, r \in \mathcal{A}$ . This implies that

$$(\mathcal{J}, \mathcal{A})^\mathcal{D} = (0)$$

and by Proposition 13, we have either  $\mathcal{J} \subseteq \mathcal{Z}(\mathcal{A})$  or  $\mathcal{D} = 0$ . By using  $\mathcal{D} = 0$  in (3.3), we get

$$([l_1, l_2], r_1)^\Psi([l_1, i])^\theta = 0$$

$\forall l_1, l_2, r_1 \in \mathcal{J}$  and by replacing  $r_1$  by  $rr_1$ , we have  $([l_1, l_2], r)^\Psi(r_1)^\theta([l_1, i])^\theta = 0, \forall l_1, l_2, r_1 \in \mathcal{J}, r \in \mathcal{A}$ . As  $\mathcal{A}$  is prime and  $\theta$  is an automorphism of  $\mathcal{A}$ , so the last equation implies that for each  $l_1 \in \mathcal{J}$  either  $([l_1, \mathcal{J}], \mathcal{A})^\Psi = (0)$  or  $([l_1, \mathcal{J}])^\theta = (0)$ . This concludes that either  $([\mathcal{J}, \mathcal{J}], \mathcal{A})^\Psi = (0)$  or  $([\mathcal{J}, \mathcal{J}])^\theta = (0)$ . If  $([\mathcal{J}, \mathcal{J}])^\theta = (0)$ , then  $[\mathcal{J}, \mathcal{J}] = (0)$ , as  $\theta$  is an automorphism. By Lemma 10, the previous equation gives that  $\mathcal{J} \subseteq \mathcal{Z}(\mathcal{A})$  and  $\mathcal{A}$  is commutative. Now consider  $([l_1, l_2], r)^\Psi = 0$ , then by taking  $l_2 = sl_2$  and using  $\mathcal{D} = 0$ , we obtain

$$(3.6) \quad (s, r)^\Psi([l_1, l_2])^\theta + ([l_1, s], r)^\Psi(l_2)^\theta = 0$$

$\forall l_1, l_2 \in \mathcal{J}, r, s \in \mathcal{A}$ . Now, replacing  $l_2$  by  $r_1l_2$  for  $r_1 \in \mathcal{J}$  in (3.6) and using it to get

$$(s, r)^\Psi(r_1)^\theta([l_1, l_2])^\theta = 0$$

$\forall l_1, l_2, r_1 \in \mathcal{J}, r, s \in \mathcal{A}$ . Again by using the primeness of  $\mathcal{A}$  and the fact that  $\theta$  is an automorphism of  $\mathcal{A}$ , the above equation implies that, either  $\Psi = 0$  or  $[\mathcal{J}, \mathcal{J}] = (0)$ . In view of Lemma 10,  $[\mathcal{J}, \mathcal{J}] = (0)$  infers that  $\mathcal{J} \subseteq \mathcal{Z}(\mathcal{A})$  ( $\mathcal{A}$  is commutative). With this our proof is completed. ■

**Theorem 15.** *If  $([l_1, l_2])^\Delta = 0, \forall l_1, l_2 \in \mathcal{L}$ , then either  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$  or  $\mathcal{D} = 0$ , and  $\Psi = 0$ .*

**Proof.** By the given hypothesis, we have  $([l_1, l_2])^\Delta = 0, \forall l_1, l_2 \in \mathcal{L}$ . If possible, let  $\mathcal{L} \not\subseteq \mathcal{Z}(\mathcal{A})$ . Then, by Lemma 6, there exists a nonzero ideal  $\mathcal{J} = 2\mathcal{A}[\mathcal{L}, \mathcal{L}]\mathcal{A} \subseteq \mathcal{L}$ . Therefore, we have

$$([l_1, l_2])^\Delta = 0, \forall l_1, l_2 \in \mathcal{J}.$$

By Theorem 14, either  $\mathcal{J} \subseteq \mathcal{Z}(\mathcal{A})$ , or  $\mathcal{D} = 0$  and  $\Psi = 0$ . Now, consider the case  $\mathcal{J} \subseteq \mathcal{Z}(\mathcal{A})$ , that is  $2p[l_1, l_2]r \in \mathcal{Z}(\mathcal{A})$ ,  $\forall l_1, l_2 \in \mathcal{L}, p, r \in \mathcal{A}$ . By replacing  $r$  by  $rl$ , we have  $2p[l_1, l_2]rl \in \mathcal{Z}(\mathcal{A})$ ,  $\forall l_1, l_2, l \in \mathcal{L}, p, r \in \mathcal{A}$ . By Lemma 11, either  $2p[l_1, l_2]r = 0$  or  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$ . Now, consider  $2p[l_1, l_2]r = 0$ ,  $\forall l_1, l_2 \in \mathcal{L}, p, r \in \mathcal{A}$ . As  $\text{char}(\mathcal{A}) \neq 2$  and  $\mathcal{A}$  is a prime ring, so the last relation implies that  $[\mathcal{L}, \mathcal{L}] = (0)$ . By applying Lemma 5,  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$ . Thus, in each case, we have  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$ , which is absurd. Hence,  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$  and this finishes the proof.  $\blacksquare$

The following theorem is an extension of [22, Theorem 3.7].

**Theorem 16.** *If  $\mathcal{D}$  is nonzero and  $(x_1y_1)^\Delta \in \mathcal{Z}(\mathcal{A})$ ,  $\forall x_1, y_1 \in \mathcal{L}$ , then  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$ .*

*Proof.* Suppose that  $\mathcal{D}$  is nonzero and

$$(3.7) \quad [(x_1y_1)^\Delta, \mathcal{A}] = (0)$$

$\forall x_1, y_1 \in \mathcal{L}$ , where  $\Delta$  is a trace of  $\Psi$ . If possible, let  $\mathcal{L} \not\subseteq \mathcal{Z}(\mathcal{A})$ . Now, replacing  $y_1$  by  $y_1 + z_1$  in (3.7) and using this, we obtain  $2[(x_1y_1, x_1z_1)^\Psi, r] = 0$ ,  $\forall x_1, y_1, z_1 \in \mathcal{L}, r \in \mathcal{A}$ . As  $\text{char}(\mathcal{A}) \neq 2$ , so the last relation leads to

$$(3.8) \quad [(x_1y_1, x_1z_1)^\Psi, r] = 0.$$

Consider  $2y_1j$  instead of  $y_1$  in equation (3.8) and using the fact that  $\text{char}(\mathcal{A}) \neq 2$ , we get  $(x_1y_1, x_1z_1)^\Psi[(j)^\theta, r] + [(x_1y_1)^\eta(j, x_1z_1)^\mathcal{D}, r] = 0$ ,  $\forall x_1, y_1, z_1, j \in \mathcal{L}, r \in \mathcal{A}$ . By replacing  $r$  by  $r(j)^\theta$ , the above equation implies that

$$\mathcal{A}[(x_1y_1)^\eta(j, x_1z_1)^\mathcal{D}, (j)^\theta] = (0).$$

This implies that  $[(x_1y_1)^\eta(j, x_1z_1)^\mathcal{D}, (j)^\theta]\mathcal{A}[(x_1y_1)^\eta(j, x_1z_1)^\mathcal{D}, (j)^\theta] = (0)$  and by using the primeness of  $\mathcal{A}$  it is obtained that

$$(3.9) \quad [(x_1y_1)^\eta(j, x_1z_1)^\mathcal{D}, (j)^\theta] = 0$$

$\forall j, x_1, y_1, z_1 \in \mathcal{L}$ . Thus,  $(x_1y_1)^\eta[(j, x_1z_1)^\mathcal{D}, (j)^\theta] + [(x_1y_1)^\eta, (j)^\theta](j, x_1z_1)^\mathcal{D} = 0$  and putting  $y_1 = 2x_1y_1$ , we conclude that

$$[(x_1)^\eta, (j)^\theta](x_1)^\eta(y_1)^\eta(j, x_1z_1)^\mathcal{D} = 0$$

$\forall j, x_1, y_1, z_1 \in \mathcal{L}$ , as  $\text{char}(\mathcal{A}) \neq 2$ . Taking  $2z_1i$  in place of  $z_1$  in the above equation, we get  $[(x_1)^\eta, (j)^\theta](x_1)^\eta(y_1)^\eta(x_1)^\eta(z_1)^\eta(j, i)^\mathcal{D} = 0$ ,  $\forall i, j, x_1, y_1, z_1 \in \mathcal{L}$ . Then by using Lemma 7 in the preceding equation, we obtain for each  $j \in \mathcal{L}$ , either  $[(x_1)^\eta, (j)^\theta](x_1)^\eta(y_1)^\eta(x_1)^\eta = 0$ ,  $\forall x_1, y_1 \in \mathcal{L}$  or  $(j, \mathcal{L})^\mathcal{D} = (0)$ . Applying Brauer's trick, we have either  $[(x_1)^\eta, (\mathcal{L})^\theta](x_1)^\eta(y_1)^\eta(x_1)^\eta = (0)$ ,  $\forall x_1, y_1 \in \mathcal{L}$



or  $(\mathcal{L}, \mathcal{L})^{\mathcal{D}} = (0)$ . If  $[(x_1)^\eta, (j)^\theta](x_1)^\eta(y_1)^\eta(x_1)^\eta = (0)$ ,  $\forall j, x_1, y_1 \in \mathcal{L}$ , then by Lemma 7, we get that for each  $x_1 \in \mathcal{L}$ , either  $(x_1)^\eta = 0$  or  $[(x_1)^\eta, (j)^\theta](x_1)^\eta = 0$ ,  $\forall j \in \mathcal{L}$ . In any case it follows that

$$(3.10) \quad [(x_1)^\eta, (j)^\theta](x_1)^\eta = 0.$$

Then by taking  $j = 2jz_1$  in (3.10) and using the fact that  $\text{char}(\mathcal{A}) \neq 2$ , we get

$$(3.11) \quad [(x_1)^\eta, (j)^\theta](z_1)^\theta(x_1)^\eta = 0$$

$\forall j, x_1, z_1 \in \mathcal{L}$ . On multiplying (3.10) from the right hand side by  $(z_1)^\theta$ , we find

$$(3.12) \quad [(x_1)^\eta, (j)^\theta](x_1)^\eta(z_1)^\theta = 0.$$

Subtracting (3.11) from (3.12), we have  $[(x_1)^\eta, (j)^\theta][(x_1)^\eta, (z_1)^\theta] = 0$ ,  $\forall j, x_1, z_1 \in \mathcal{L}$  and by replacing  $z_1$  by  $2z_1j$ , it gives  $[(x_1)^\eta, (j)^\theta](z_1)^\theta[(x_1)^\eta, (j)^\theta] = 0$ ,  $\forall j, x_1, z_1 \in \mathcal{L}$ . Again by Lemma 7,  $[(\mathcal{L})^\eta, (\mathcal{L})^\theta] = (0)$  and by Lemma 12,  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$ , a contradiction.

On other hand, if we consider  $(\mathcal{L}, \mathcal{L})^{\mathcal{D}} = (0)$ . Then, by Lemma 4, we have  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$ , a contradiction. Both of these cases lead to a contradiction. Hence,  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$ .  $\blacksquare$

**Corollary 17.** *If  $\mathcal{D}$  is nonzero and  $(l_1)^\Delta \in \mathcal{Z}(\mathcal{A})$ ,  $\forall l_1 \in \mathcal{L}$ , then  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$ .*

**Theorem 18.** *Let  $([l_1, l_2])^\Delta = (l_1)^\theta \circ (l_2)^\Delta$ ,  $\forall l_1, l_2 \in \mathcal{L}$ . Then either  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$  or  $\mathcal{D} = 0$ , and  $\Psi = 0$ .*

**Proof.** The hypothesis gives that

$$(3.13) \quad ([l_1, l_2])^\Delta = (l_1)^\theta \circ (l_2)^\Delta$$

$\forall l_1, l_2 \in \mathcal{L}$ . Putting  $l_1 + r_1$  instead of  $l_1$  in (3.13), we get  $([l_1, l_2])^\Delta + ([r_1, l_2])^\Delta + 2([l_1, l_2], [r_1, l_2])^\Psi = (l_1)^\theta \circ (l_2)^\Delta + (r_1)^\theta \circ (l_2)^\Delta$ ,  $\forall l_1, l_2, r_1 \in \mathcal{L}$ . By using (3.13), the last expression infers that

$$2([l_1, l_2], [r_1, l_2])^\Psi = 0.$$

As  $\text{char}(\mathcal{A}) \neq 2$ , so the above equation implies  $([l_1, l_2], [r_1, l_2])^\Psi = 0$ . In particular, for  $r_1 = l_1$ , we obtain  $0 = ([l_1, l_2], [l_1, l_2])^\Psi$ . This implies  $([\mathcal{L}, \mathcal{L}])^\Delta = (0)$ . Therefore, by Theorem 15, either  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$  or  $\mathcal{D} = 0$ , and  $\Psi = 0$ .  $\blacksquare$

By using a similar technique with the necessary variations, one can easily prove the following result.

**Theorem 19.** *If  $(l_1 \circ l_2)^\Delta = [(l_1)^\theta, (l_2)^\Delta]$ ,  $\forall l_1, l_2 \in \mathcal{L}$ , then either  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$  or  $\mathcal{D} = 0$ , and  $\Psi = 0$ .*

**Theorem 20.** *If any one of the following holds true:*

- (i)  $[(l_1)^\Delta(l_2)^\Delta + (l_1)^\eta(l_2)^\theta, \mathcal{A}] = (0)$ ,
- (ii)  $[(l_1)^\Delta(l_2)^\Delta - (l_1)^\eta(l_2)^\theta, \mathcal{A}] = (0)$ ,  $\forall l_1, l_2 \in \mathcal{L}$ , then  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$ .

**Proof.** (i) By the given hypothesis, we have

$$(3.14) \quad [(l_1)^\Delta(l_2)^\Delta + (l_1)^\eta(l_2)^\theta, \mathcal{A}] = (0)$$

$\forall l_1, l_2 \in \mathcal{L}$ . Suppose that  $\mathcal{L} \not\subseteq \mathcal{Z}(\mathcal{A})$ . Replacing  $l_2$  by  $l_2 + i$  in (3.14), we get  $2[(l_1)^\Delta(l_2, i)^\Psi, \mathcal{A}] = (0)$ ,  $\forall l_1, l_2, i \in \mathcal{L}$ . Since  $\text{char}(\mathcal{A}) \neq 2$ , so the last relation infers that

$$[(l_1)^\Delta(l_2, i)^\Psi, \mathcal{A}] = (0).$$

Replacing  $i$  by  $l_2$  in the above equation, we get

$$[(l_1)^\Delta(l_2)^\Delta, \mathcal{A}] = (0)$$

$\forall l_1, l_2 \in \mathcal{L}$ . On combining (3.14) and the above equation, we have  $[(l_1)^\eta(l_2)^\theta, \mathcal{A}] = (0)$ ,  $\forall l_1, l_2 \in \mathcal{L}$ . This implies that

$$(3.15) \quad (l_1)^\eta[(l_2)^\theta, r] + [(l_1)^\eta, r](l_2)^\theta = 0.$$

Taking  $l_1 = 2l_1r_1$  in (3.15) and using the fact that  $\text{char}(\mathcal{A}) \neq 2$ , we obtain  $[(l_1)^\eta, r](r_1)^\eta(l_2)^\theta = 0$ ,  $\forall l_1, l_2, r_1 \in \mathcal{L}, r \in \mathcal{A}$ . By applying Lemma 7, we get  $[(l_1)^\eta, r] = 0$ . On replacing  $r$  with  $(l_2)^\eta$ , last expression infers that  $[(\mathcal{L})^\eta, (\mathcal{L})^\eta] = (0)$ . Thus, by Corollary 9,  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$ . This is a contradiction to our supposition. Hence,  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$ .

After applying the similar technique with necessary modifications, we can prove (ii). ■

Consequently, we have

**Corollary 21.** *If any one of the following holds true:*

- (i)  $[(l_1)^\Delta(l_2)^\Delta + (l_1)^\eta(l_2)^\theta, \mathcal{A}] = (0)$ ,
- (ii)  $[(l_1)^\Delta(l_2)^\Delta - (l_1)^\eta(l_2)^\theta, \mathcal{A}] = (0)$ ,

$\forall l_1, l_2 \in \mathcal{A}$ , then  $\mathcal{A}$  is commutative.

Note that  $Q_{mr}$  stands for the right Utumi quotient ring (also called the maximal right ring of quotients) of  $\mathcal{A}$ . Then the center of  $Q_{mr}$  is called the extended centroid of  $\mathcal{A}$  and is denoted by  $C$ .

The next result extends [22, Theorem 3.15].

**Theorem 22.** *If  $0 \neq a_1 \in \mathcal{A}$  such that  $a_1((l_1)^\Delta(l_2)^\Delta + (l_1l_2)^\theta) = 0$ ,  $\forall l_1, l_2 \in \mathcal{L}$ , then  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$  or there exists  $\lambda \in C$  such that  $(x_1)^\Delta = \lambda(x_1)^\theta$ ,  $\forall x_1 \in \mathcal{A}$  and  $\lambda^2 = -1$ .*

**Proof.** By the given hypothesis,

$$(3.16) \quad a_1((l_1)^\Delta(l_2)^\Delta + (l_1l_2)^\theta) = 0$$

$\forall l_1, l_2 \in \mathcal{L}$ . Let us assume that  $\mathcal{L} \not\subseteq \mathcal{Z}(\mathcal{A})$ . Then, replacing  $l_2$  by  $l_2 + z_1$  in equation (3.16), we get  $2a_1(l_1)^\Delta(l_2, z_1)^\Psi = 0, \forall l_1, l_2, z_1 \in \mathcal{L}$ . As  $\text{char}(\mathcal{A}) \neq 2$ , so

$$(3.17) \quad a_1(l_1)^\Delta(l_2, z_1)^\Psi = 0.$$

Taking  $2z_1i$  instead of  $z_1$  in (3.17) and by  $\text{char}(\mathcal{A}) \neq 2$ , we get

$$a_1(l_1)^\Delta(z_1)^\eta(l_2, i)^\mathcal{D} = 0$$

$\forall i, l_1, l_2, z_1 \in \mathcal{L}$ . By Lemma 7, either  $a_1(\mathcal{L})^\Delta = (0)$  or  $(\mathcal{L}, \mathcal{L})^\mathcal{D} = (0)$ . If  $a_1(\mathcal{L})^\Delta = (0)$ , with this equation (3.16) implies that  $a_1(l_1)^\theta(l_2)^\theta = 0, \forall l_1, l_2 \in \mathcal{L}$ . By Lemma 7,  $a_1 = 0$ , which is not possible. Thus,  $(\mathcal{L}, \mathcal{L})^\mathcal{D} = (0)$  and by Lemma 4, either  $\mathcal{D} = 0$  or  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$  and this concludes that  $\mathcal{D} = 0$ , as we have assumed that  $\mathcal{L} \not\subseteq \mathcal{Z}(\mathcal{A})$ . Hence  $\mathcal{D} = 0$ . By replacing  $l_2$  by  $2rs[l_2, r_1]$  in (3.17) for  $r, s \in \mathcal{A}$  and using  $\mathcal{D} = 0$ , we find  $a_1(l_1)^\Delta(r, z_1)^\Psi \mathcal{A}[(l_2)^\theta, (r_1)^\theta] = (0)$ . The primeness of  $\mathcal{A}$  infers that, either  $a_1(\mathcal{L})^\Delta(\mathcal{A}, \mathcal{L})^\Psi = (0)$  or  $[(\mathcal{L})^\theta, (\mathcal{L})^\theta] = (0)$ . As we have assumed that  $\mathcal{L} \not\subseteq \mathcal{Z}(\mathcal{A})$ , so by Corollary 9, we observe that the latter case is not possible. Therefore,  $a_1(l_1)^\Delta(r, z_1)^\Psi = 0$  and by taking  $l_1 = l_1 + l_2$ , this gives

$$(3.18) \quad a_1(l_1, l_2)^\Psi(r, z_1)^\Psi = 0$$

$\forall l_1, l_2, z_1 \in \mathcal{L}, r \in \mathcal{A}$ . On putting  $2l_2r_1$  in place of  $l_2$  in (3.18) and using  $\mathcal{D} = 0$ , we obtain

$$(3.19) \quad a_1(l_1, l_2)^\Psi(r_1)^\theta(r, z_1)^\Psi = 0$$

$\forall l_1, l_2, z_1, r_1 \in \mathcal{L}, r \in \mathcal{A}$ . After multiplying (3.18) by  $(r_1)^\theta$  from right hand side, we have

$$(3.20) \quad a_1(l_1, l_2)^\Psi(r, z_1)^\Psi(r_1)^\theta = 0.$$

On subtracting equation (3.19) from (3.20), we conclude that

$$a_1(l_1, l_2)^\Psi[(r, z_1)^\Psi, (r_1)^\theta] = 0$$

and by putting  $r_1 = 2r_1z_1$ , we are left with  $a_1(l_1, l_2)^\Psi(r_1)^\theta[(r, z_1)^\Psi, (z_1)^\theta] = 0, \forall l_1, l_2, r_1, z_1 \in \mathcal{L}, r \in \mathcal{A}$ . By Lemma 7, either  $a_1(\mathcal{L}, \mathcal{L})^\Psi = (0)$  or  $[(\mathcal{A}, z_1)^\Psi, (z_1)^\theta] = (0), \forall z_1 \in \mathcal{L}$ . If  $a_1(\mathcal{L}, \mathcal{L})^\Psi = (0)$ , then  $a_1(l_1)^\Delta = 0$ , by using this in given

hypothesis, we have  $a_1(l_1)^\theta(l_2)^\theta = 0, \forall l_1, l_2 \in \mathcal{L}$  and by Lemma 7,  $a_1 = 0$ , which is a contradiction. Therefore,

$$(3.21) \quad [(r, z_1)^\Psi, (z_1)^\theta] = 0$$

$\forall z_1 \in \mathcal{L}, r \in \mathcal{A}$ . Replacing  $z_1$  by  $z_1 + x_1$  in (3.21), we obtain  $[(r, z_1)^\Psi, (x_1)^\theta] + [(r, x_1)^\Psi, (z_1)^\theta] = 0, \forall x_1, z_1 \in \mathcal{L}, r \in \mathcal{A}$ . Putting  $z_1 = 2z_1y_1$  and using  $\mathcal{D} = 0$ , we have

$$(3.22) \quad (r, z_1)^\Psi[(y_1)^\theta, (x_1)^\theta] + (z_1)^\theta[(r, x_1)^\Psi, (y_1)^\theta] = 0$$

$\forall x_1, y_1, z_1 \in \mathcal{L}, r \in \mathcal{A}$ . Replacing  $z_1$  by  $2sp[z_1, x_1]$  in (3.22) and using  $\mathcal{D} = 0$ , we get  $0 = 2(r, s)^\Psi(p[z_1, x_1])^\theta[(y_1)^\theta, (x_1)^\theta] + (s)^\theta(2p[z_1, x_1])^\theta[(r, x_1)^\Psi, \theta(y_1)] = 0, \forall x_1, y_1, z_1 \in \mathcal{L}, p, r, s \in \mathcal{A}$ . As  $2p[z_1, x_1] \in \mathcal{L}$ , so by using (3.22),  $(2p[z_1, x_1])^\theta[(r, x_1)^\Psi, (y_1)^\theta] = -(r, 2p[z_1, x_1])^\Psi[(y_1)^\theta, (x_1)^\theta]$  and using this in last equation, we conclude that

$$\begin{aligned} 0 &= ((r, s)^\Psi(p[z_1, x_1])^\theta - (s)^\theta(r, p[z_1, x_1])^\Psi)[(y_1)^\theta, (x_1)^\theta] \\ &= ((r, s)^\Psi(p)^\theta - (s)^\theta(r, p)^\Psi)[(z_1)^\theta, (x_1)^\theta][(y_1)^\theta, (x_1)^\theta] \end{aligned}$$

$\forall x_1, y_1, z_1 \in \mathcal{L}, p, r, s \in \mathcal{A}$ , as  $\mathcal{D} = 0$ . By taking  $mp$  instead of  $p$  and using  $\mathcal{D} = 0$ , this concludes that

$$((r, s)^\Psi(m)^\theta - (s)^\theta(r, m)^\Psi)\mathcal{A}[(z_1)^\theta, (x_1)^\theta][(y_1)^\theta, (x_1)^\theta] = (0)$$

$\forall x_1, y_1, z_1 \in \mathcal{L}, m, r, s \in \mathcal{A}$ . As  $\mathcal{A}$  is prime, so the last equation infers that either  $(r, s)^\Psi(m)^\theta - (s)^\theta(r, m)^\Psi = 0, \forall m, r, s \in \mathcal{A}$  or  $[(z_1)^\theta, (x_1)^\theta][(y_1)^\theta, (x_1)^\theta] = 0, \forall x_1, y_1, z_1 \in \mathcal{L}$ . If

$$[(z_1)^\theta, (x_1)^\theta][(y_1)^\theta, (x_1)^\theta] = 0$$

then by replacing  $z_1$  by  $2y_1z_1$ , we find

$$[(y_1)^\theta, (x_1)^\theta](\mathcal{L})^\theta[(y_1)^\theta, (x_1)^\theta] = (0)$$

$\forall x_1, y_1 \in \mathcal{L}$  and by using Lemma 7 and Corollary 9,  $\mathcal{L} \subseteq \mathcal{L}(\mathcal{A})$ , a contradiction. Thus,

$$(r, s)^\Psi(m)^\theta - (s)^\theta(r, m)^\Psi = 0$$

$\forall m, r, s \in \mathcal{A}$ . Further, for each  $r \in \mathcal{A}$ , we define a function  $f_r : \mathcal{A} \rightarrow \mathcal{A}$  by  $(x_1)^{f_r} = (x_1, r)^\Psi = (r, x_1)^\Psi$ . Then the previous equation implies that for each  $r \in \mathcal{A}$

$$(3.23) \quad (s)^{f_r}(m)^\theta = (s)^\theta(m)^{f_r}$$

$\forall s, m \in \mathcal{A}$ . On replacing  $s$  by  $st$  in (3.23) and using  $\mathcal{D} = 0$ , we have  $(s)^{f_r}(t)^\theta(m)^\theta = (s)^\theta(t)^\theta(m)^{f_r}$ ,  $\forall s, t, m \in \mathcal{A}$ . As  $\theta$  is an automorphism, so last equation infers that

$$(s)^{f_r}p(m)^\theta = (s)^\theta p(m)^{f_r}$$

$\forall m, p, s \in \mathcal{A}$ . In view of [8, Lemma], there exists some  $\lambda \in C$  such that  $(s)^{f_r} = (s, r)^\Psi = \lambda(s)^\theta$ ,  $\forall s \in \mathcal{A}$ . In this way we find  $(s, r)^\Psi = \lambda(s)^\theta$ ,  $\forall s, r \in \mathcal{A}$ . In particular for  $s = r$ , we have

$$(3.24) \quad (r, r)^\Psi = (r)^\Delta = \lambda(r)^\theta$$

$\forall r \in \mathcal{A}$ . Then from the initial hypothesis, we get  $a_1(\lambda^2 + 1)(l_1 l_2)^\theta = 0$ ,  $\forall l_1, l_2 \in \mathcal{L}$ . This infers that  $\lambda^2 = -1$ .  $\blacksquare$

In similar way, one can prove the following result:

**Theorem 23.** *If  $0 \neq a_1 \in \mathcal{A}$  such that  $a_1((l_1)^\Delta(l_2)^\Delta - (l_1 l_2)^\theta) = 0$ ,  $\forall l_1, l_2 \in \mathcal{L}$ , then  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$  or there exists  $\lambda \in C$  such that  $(x_1)^\Delta = \lambda(x_1)^\theta$ ,  $\forall x_1 \in \mathcal{A}$  and  $\lambda^2 = 1$ .*

#### 4. SYMMETRIC GENERALIZED LEFT $(\theta, \eta)$ -BIDERIVATIONS

In this section, the behaviour of generalized left  $(\theta, \eta)$ -biderivations on Lie ideals of rings is examined and we also extend some well known results of [18] in the framework of generalized left  $(\theta, \eta)$ -biderivations. We now proceed with the following result which is an extension of ([18, Lemma 2]).

In this section,  $\Psi$  represents a symmetric generalized left  $(\theta, \eta)$ -biderivation of  $\mathcal{A}$  associated with a symmetric left  $(\theta, \eta)$ -biderivation  $\mathcal{D}$  and  $\Delta$  is a trace of  $\Psi$ ,  $\omega$  is a trace of  $\mathcal{D}$ .

**Proposition 24.** *If  $(\mathcal{L})^\omega = (0)$ , then  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$  or  $\mathcal{D} = 0$ .*

**Proof.** Let  $\mathcal{L} \not\subseteq \mathcal{Z}(\mathcal{A})$  and the given hypothesis  $(l_1)^\omega = 0$ ,  $\forall l_1 \in \mathcal{L}$ . Now, replacing  $l_1$  by  $l_1 + l_2$  and using the fact  $\text{char}(\mathcal{A}) \neq 2$ , we obtain

$$(4.1) \quad (l_1, l_2)^\mathcal{D} = 0.$$

$\forall l_1, l_2 \in \mathcal{L}$ . Putting  $l_1 = 2r[i, j]$ , we get

$$(4.2) \quad ([i, j])^\eta(r, l_2)^\mathcal{D} = 0$$

$\forall i, j, l_2 \in \mathcal{L}, r \in \mathcal{A}$ . Putting  $2r[x_1, j]s$  instead of  $r$ , the above equation infers that  $([i, j])^\eta(r)^\theta(2[x_1, j]s, l_2)^\mathcal{D} + 2([i, j])^\eta([x_1, j])^\eta(s)^\eta(r, l_2)^\mathcal{D} = 0$ ,  $\forall i, j, l_2, x_1 \in$

$\mathcal{L}, r, s \in \mathcal{A}$ . Since  $2[x_1, j]s \in \mathcal{L}$ , so by using (4.1) and  $\text{char}(\mathcal{A}) \neq 2$ , the previous equation implies that

$$(4.3) \quad ([i, j])^\eta([x_1, j])^\eta \mathcal{A}(r, l_2)^\mathcal{D} = (0).$$

As  $\mathcal{A}$  is prime, so equation (4.3) concludes that either  $([i, j])^\eta([x_1, j])^\eta = 0, \forall i, j, x_1 \in \mathcal{L}$  or  $(\mathcal{A}, l_2)^\mathcal{D} = (0), \forall l_2 \in \mathcal{L}$ .

The former case implies  $([i, j])^\eta([x_1, j])^\eta = 0$ , then by taking  $2x_1i$  instead of  $i$  and using  $\text{char}(\mathcal{A}) \neq 2$ , we have

$$[(x_1)^\eta, (j)^\eta](i)^\eta([(x_1)^\eta, (j)^\eta]) = 0$$

$\forall i, j, x_1 \in \mathcal{L}$ . By Lemma 7 and Corollary 9, the preceding equation forces  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$ , which is not possible. In latter case, we have

$$(4.4) \quad (r, l_2)^\mathcal{D} = 0$$

$\forall l_2 \in \mathcal{L}, r \in \mathcal{A}$ . Further, replacing  $l_2$  by  $2[l_1, l_2]ps$ , we have

$$2[(l_1)^\theta, (l_2)^\theta](p)^\theta(r, s)^\mathcal{D} + (s)^\eta(r, 2[l_1, l_2]s)^\mathcal{D} = 0$$

$\forall l_1, l_2 \in \mathcal{L}, p, r, s \in \mathcal{A}$ . As  $2[l_1, l_2]s \in \mathcal{L}$ , therefore by using (4.4) the last relation yields  $[(l_1)^\theta, (l_2)^\theta] \mathcal{A}(r, s)^\mathcal{D} = (0), \forall l_1, l_2 \in \mathcal{L}, r, s \in \mathcal{A}$  and the primeness of  $\mathcal{A}$  implies either  $[(\mathcal{L})^\theta, (\mathcal{L})^\theta] = (0)$  or  $\mathcal{D} = 0$ . In view of Corollary 9, the former gives that  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$ , a contradiction. Hence  $\mathcal{D} = 0$ . ■

**Corollary 25.** *If  $(\mathcal{A}, \mathcal{L})^\mathcal{D} = (0)$ , then  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$  or  $\mathcal{D} = 0$ .*

**Theorem 26.** *If  $\mathcal{D}$  is nonzero and any one of the following holds true:*

- (i)  $(l_1)^\Delta(l_2)^\theta + (l_1)^\theta(l_2)^\Delta = 0$ ,
- (ii)  $(l_1)^\Delta(l_2)^\theta - (l_1)^\theta(l_2)^\Delta = 0 \forall l_1, l_2 \in \mathcal{L}$ , then  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$ .

**Proof.** (i) Suppose that

$$(4.5) \quad (l_1)^\Delta(l_2)^\theta + (l_1)^\theta(l_2)^\Delta = 0$$

$\forall l_1, l_2 \in \mathcal{L}$ . On replacing  $l_1$  by  $l_1 + r_1$  in (4.5) and using  $\text{char}(\mathcal{A}) \neq 2$ , we find  $(l_1, r_1)^\Psi(l_2)^\theta = 0, \forall l_1, l_2, r_1 \in \mathcal{L}$ . Taking  $2r[l_2, x_1]$  instead of  $l_2$ , the previous expression gives  $2(l_1, r_1)^\Psi(r)^\theta([l_2, x_1])^\theta = 0, \forall l_1, l_2, r_1, x_1 \in \mathcal{L}, r \in \mathcal{A}$ . Since  $\text{char}(\mathcal{A}) \neq 2$ , so  $(l_1, r_1)^\Psi \mathcal{A}([l_2, x_1])^\theta = (0)$ . The primeness of  $\mathcal{A}$  yields this either  $(\mathcal{L}, \mathcal{L})^\Psi = (0)$  or  $[(\mathcal{L})^\theta, (\mathcal{L})^\theta] = (0)$ . By Corollary 9, the latter case infers that  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$ . From the former case we have

$$(4.6) \quad (l_1, r_1)^\Psi = 0$$

$\forall l_1, r_1 \in \mathcal{L}$ . Replacing  $l_1$  by  $2r[l_1, i]s$ , we get

$$(r)^\theta(2[l_1, i]s, r_1)^\Psi + 2([l_1, i])^\eta(s)^\eta(r, r_1)^\mathcal{D} = 0$$

$\forall l_1, i, r_1 \in \mathcal{L}, r, s \in \mathcal{A}$ . As  $2[l_1, i]s \in \mathcal{L}$ , so by using (4.6), the preceding equation gives  $([l_1, i])^\eta \mathcal{A}(r, r_1)^\mathcal{D} = (0)$ ,  $\forall l_1, i, r_1 \in \mathcal{L}, r \in \mathcal{A}$ . The primeness of  $\mathcal{A}$  implies that either  $[(\mathcal{L})^\eta, (\mathcal{L})^\eta] = (0)$  or  $(\mathcal{A}, \mathcal{L})^\mathcal{D} = (0)$ . If  $[(\mathcal{L})^\eta, (\mathcal{L})^\eta] = (0)$ , then by Corollary 9,  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$ . Now, consider the case  $(\mathcal{A}, \mathcal{L})^\mathcal{D} = (0)$ . Then, by the previous corollary,  $(\mathcal{A}, \mathcal{L})^\mathcal{D} = (0)$  infers that  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$ , since  $\mathcal{D}$  is nonzero.

On applying the similar technique with necessary modifications, we obtain the same conclusion for (ii). This completes the proof.  $\blacksquare$

Immediately, we obtain the next result which gives the commutativity of  $\mathcal{A}$ .

**Corollary 27.** *If  $\mathcal{D}$  is nonzero and any one of the following holds true:*

- (i)  $(l_1)^\Delta(l_2)^\theta + (l_1)^\theta(l_2)^\Delta = 0$ ,
- (ii)  $(l_1)^\Delta(l_2)^\theta - (l_1)^\theta(l_2)^\Delta = 0 \forall l_1, l_2 \in \mathcal{A}$ , then  $\mathcal{A}$  is commutative.

**Proposition 28.** *If  $\mathcal{D}$  is nonzero and  $(l_1)^\Delta \in \mathcal{Z}(\mathcal{A})$ ,  $\forall l_1 \in \mathcal{L}$ , then  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$ .*

**Proof.** If possible, assume that  $\mathcal{L} \not\subseteq \mathcal{Z}(\mathcal{A})$ . By the given hypothesis,  $\mathcal{D}$  is nonzero and  $[(l_1)^\Delta, \mathcal{A}] = (0)$ ,  $\forall l_1 \in \mathcal{L}$ . Taking  $l_1 + l_2$  instead of  $l_1$ , we obtain

$$(4.7) \quad [(l_1, l_2)^\Psi, r] = 0$$

$\forall l_1, l_2 \in \mathcal{L}, r \in \mathcal{A}$ . Taking  $l_2 = 2r_1l_2$  in (4.7), we get

$$[(r_1)^\theta, r](l_1, l_2)^\Psi + [(l_2)^\eta(l_1, r_1)^\mathcal{D}, r] = 0$$

$\forall l_1, l_2, r_1 \in \mathcal{L}, r \in \mathcal{A}$ . By taking  $(r_1)^\theta r$  in place of  $r$ , the above equation yields  $[(l_2)^\eta(l_1, r_1)^\mathcal{D}, (r_1)^\theta] \mathcal{A} = (0)$ . Since  $\mathcal{A}$  is prime, therefore

$$(4.8) \quad 0 = [(l_2)^\eta(l_1, r_1)^\mathcal{D}, (r_1)^\theta] = (l_2)^\eta[(l_1, r_1)^\mathcal{D}, (r_1)^\theta] + [(l_2)^\eta, (r_1)^\theta](l_1, r_1)^\mathcal{D}$$

$\forall l_1, l_2, r_1 \in \mathcal{L}$ . Putting  $2x_1l_2$  in place of  $l_2$  in (4.8) and using  $\text{char}(\mathcal{A}) \neq 2$ , we find

$$[(x_1)^\eta, (r_1)^\theta](l_2)^\eta(l_1, r_1)^\mathcal{D} = 0$$

$\forall l_1, l_2, r_1, x_1 \in \mathcal{L}$ . By using Lemma 7, we get that for each  $r_1 \in \mathcal{L}$ , either  $[(\mathcal{L})^\eta, (r_1)^\theta] = (0)$  or  $(\mathcal{L}, r_1)^\mathcal{D} = (0)$ . Therefore,  $\mathcal{L}$  is a union of the subgroups  $A = \{r_1 \in \mathcal{L} : [(\mathcal{L})^\eta, (r_1)^\theta] = (0)\}$  and  $B = \{r_1 \in \mathcal{L} : (\mathcal{L}, r_1)^\mathcal{D} = (0)\}$ .

Since a group cannot be the union of its proper subgroups, so we are forced to conclude that either  $\mathcal{L} = A$  or  $\mathcal{L} = B$ . If  $\mathcal{L} = A$ , then  $[(\mathcal{L})^\eta, (\mathcal{L})^\theta] = (0)$  and by Lemma 12,  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$ , a contradiction to our assumption. Therefore, we are left with  $\mathcal{L} = B$ , i.e.  $(\mathcal{L}, \mathcal{L})^\mathcal{D} = (0)$ . By Proposition 24, we get that  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$ , a contradiction. Hence,  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$ .  $\blacksquare$

The following theorem is a generalization of [18, Theorem 7].

**Theorem 29.** *Let  $\mathcal{D}$  be nonzero and  $[(l_1, l_2)]^\Delta + [(l_1)^\Delta, l_2] \in \mathcal{Z}(\mathcal{A})$ ,  $\forall l_1, l_2 \in \mathcal{L}$ . Then  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$ .*

**Proof.** By the given hypothesis, we get

$$(4.9) \quad [[(l_1, l_2)]^\Delta + [(l_1)^\Delta, l_2], r] = 0$$

$\forall l_1, l_2 \in \mathcal{L}, r \in \mathcal{A}$ . On replacing  $l_2$  by  $l_2 + r_1$ , the last equation gives that  $[[(l_1, l_2), [l_1, r_1]]^\Psi, r] = 0$ ,  $\forall l_1, l_2, r_1 \in \mathcal{L}, r \in \mathcal{A}$ . In particular  $r_1 = l_2$ , we have  $[[(l_1, l_2)]^\Delta, r] = 0$ ,  $\forall l_1, l_2 \in \mathcal{L}, r \in \mathcal{A}$ . With this, (4.9) implies that

$$(4.10) \quad [[(l_1)^\Delta, l_2], r] = 0$$

$\forall l_1, l_2 \in \mathcal{L}, r \in \mathcal{A}$ . Putting  $2l_2r_1$  instead of  $l_2$  in (4.10), we find that  $[(l_1)^\Delta, l_2][r_1, r] + [l_2, r][(l_1)^\Delta, r_1] = 0$ ,  $\forall l_1, l_2, r_1 \in \mathcal{L}, r \in \mathcal{A}$ . On taking  $r = r_1r$  and using (4.10), the previous equation implies that

$$[l_2, r_1]_{\mathcal{A}}[(l_1)^\Delta, r_1] = (0)$$

$\forall l_1, l_2, r_1 \in \mathcal{L}$ . By the primeness of  $\mathcal{A}$ , the above expression infers that for each  $r_1 \in \mathcal{L}$ , either  $[\mathcal{L}, r_1] = (0)$  or  $[(\mathcal{L})^\Delta, r_1] = (0)$ . This implies that either  $[\mathcal{L}, \mathcal{L}] = (0)$  or  $[(\mathcal{L})^\Delta, \mathcal{L}] = (0)$ . In view of Lemma 5, the former case gives  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$  and by the latter case, we have  $[(l_1)^\Delta, l_2] = 0$ ,  $\forall l_1, l_2 \in \mathcal{L}$ . Further, putting  $2rs[l_2, r_1]$  in place of  $l_2$ , we conclude that

$$(4.11) \quad [(l_1)^\Delta, r]s[l_2, r_1] = 0$$

$\forall l_1, l_2, r_1 \in \mathcal{L}, r, s \in \mathcal{A}$ . Since  $\mathcal{A}$  is prime, so (4.11) implies that, either  $(l_1)^\Delta \in \mathcal{Z}(\mathcal{A})$ ,  $\forall l_1 \in \mathcal{L}$  or  $[\mathcal{L}, \mathcal{L}] = (0)$ . If  $(l_1)^\Delta \in \mathcal{Z}(\mathcal{A})$ ,  $\forall l_1 \in \mathcal{L}$ , then by Proposition 28,  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$ . On the other hand, if  $[\mathcal{L}, \mathcal{L}] = (0)$ , then by Lemma 5,  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$ . Therefore,  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$ .  $\blacksquare$

**Corollary 30.** *If  $\mathcal{D}$  is nonzero and  $[(l_1, l_2)]^\Delta + [(l_1)^\Delta, l_2] \in \mathcal{Z}(\mathcal{A})$ ,  $\forall l_1, l_2 \in \mathcal{A}$ , then  $\mathcal{A}$  is commutative.*

**Theorem 31.** *If one of the following conditions hold:*

$$(i) \quad (l_1l_2)^\Delta + (l_1)^\theta(l_2)^\Delta + (l_1l_2)^\theta \in \mathcal{Z}(\mathcal{A})$$

$$(ii) \quad (l_1l_2)^\Delta - (l_1)^\theta(l_2)^\Delta + (l_1l_2)^\theta \in \mathcal{Z}(\mathcal{A})$$

$\forall l_1, l_2 \in \mathcal{L}$ , then  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$  or  $\mathcal{D} = 0$ .



**Proof.** (i) In case  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$ , then we are done. Assume that  $\mathcal{L} \not\subseteq \mathcal{Z}(\mathcal{A})$  and by hypothesis, we have  $[(l_1 l_2)^\Delta + (l_1)^\theta (l_2)^\Delta + (l_1 l_2)^\theta, \mathcal{A}] = (0)$ ,  $\forall l_1, l_2 \in \mathcal{L}$ . Now, replacing  $l_1$  by  $l_1 + z_1$  and using the fact that  $\text{char}(\mathcal{A}) \neq 2$ , we obtain that

$$(4.12) \quad [(l_1 l_2, z_1 l_2)^\Psi, r] = 0$$

$\forall l_1, l_2, z_1 \in \mathcal{L}, r \in \mathcal{A}$ . Taking  $2j l_1$  in place of  $l_1$  in (4.12) and again using  $\text{char}(\mathcal{A}) \neq 2$ , we get

$$[(j)^\theta, r](l_1 l_2, z_1 l_2)^\Psi + [(l_1 l_2)^\eta(j, z_1 l_2)^\mathcal{S}, r] = 0$$

$\forall l_1, l_2, z, j \in \mathcal{L}, r \in \mathcal{A}$ . On putting  $r = (j)^\theta r$  in the last equation, we find  $[(l_1 l_2)^\eta(j, z_1 l_2)^\mathcal{S}, (j)^\theta]r = 0$ . This implies that

$$(4.13) \quad [(l_1 l_2)^\eta(j, z_1 l_2)^\mathcal{S}, (j)^\theta] \mathcal{A} [(l_1 l_2)^\eta(j, z_1 l_2)^\mathcal{S}, (j)^\theta] = (0)$$

$\forall l_1, l_2, z_1, j \in \mathcal{L}, r \in \mathcal{A}$ . Further the primeness of  $\mathcal{A}$  implies that  $0 = [(l_1 l_2)^\eta(j, z_1 l_2)^\mathcal{S}, (j)^\theta] = [(l_1 l_2)^\eta, (j)^\theta](j, z_1 l_2)^\mathcal{S} + (l_1 l_2)^\eta[(j, z_1 l_2)^\mathcal{S}, (j)^\theta]$ . Replacing  $l_1$  by  $2l_1 k$  and using  $\text{char}(\mathcal{A}) \neq 2$ , in the resulting equation, we have

$$[(l_1)^\eta, (j)^\theta](k)^\eta (l_2)^\eta(j, z_1 l_2)^\mathcal{S} = 0$$

$\forall l_1, l_2, z_1, j, k \in \mathcal{L}$ . Therefore, by Lemma 7, the previous equation infers that for each  $j \in \mathcal{L}$ , either  $[(\mathcal{L})^\eta, (j)^\theta] = (0)$  or  $(l_2)^\eta(j, z_1 l_2)^\mathcal{S} = 0$ ,  $\forall l_2, z_1 \in \mathcal{L}$ . This implies that  $[(\mathcal{L})^\eta, (\mathcal{L})^\theta] = (0)$  or  $(l_2)^\eta(j, z_1 l_2)^\mathcal{S} = 0$ ,  $\forall j, l_2, z_1 \in \mathcal{L}$ . By Lemma 12, the former case infers that  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$ , which is contradiction to our assumption. Thus, we have  $(l_2)^\eta(j, z_1 l_2)^\mathcal{S} = 0$ ,  $\forall j, l_2, z_1 \in \mathcal{L}$ . Taking  $l_2 + l_1$  in place of  $l_2$ , we get

$$(4.14) \quad (l_2)^\eta(j, z_1 l_1)^\mathcal{S} + (l_1)^\eta(j, z_1 l_2)^\mathcal{S} = 0$$

$\forall j, l_1, l_2, z_1 \in \mathcal{L}$ . Replacing  $l_2$  by  $2l_2 k$  we have

$$(l_2)^\eta(k)^\eta(j, z_1 l_1)^\mathcal{S} + (l_1)^\eta(z_1 l_2)^\theta(j, k)^\mathcal{S} + (l_1)^\eta(k)^\eta(j, z_1 l_2)^\mathcal{S} = 0$$

$\forall j, k, l_1, l_2, z_1 \in \mathcal{L}$ . By (4.14),

$$(k)^\eta(j, z_1 l_2)^\mathcal{S} = -(l_2)^\eta(j, z_1 k)^\mathcal{S}, (k)^\eta(j, z_1 l_1)^\mathcal{S} = -(l_1)^\eta(j, z_1 k)^\mathcal{S}$$

and using these in last relation, we have

$$-((l_1)^\eta \circ (l_2)^\eta)(j, z_1 k)^\mathcal{S} + (l_1)^\eta(z_1)^\theta (l_2)^\theta(j, k)^\mathcal{S} = 0.$$

By putting  $2j l_1$  in place of  $l_1$ , we have

$$[(j)^\eta, (l_2)^\eta](l_1)^\eta(j, z_1 k)^\mathcal{S} = 0$$

$\forall j, k, l_1, l_2, z_1 \in \mathcal{L}$ . Further, by Lemma 7, we obtain that for each  $j \in \mathcal{L}$ , either  $[(j)^\eta, (\mathcal{L})^\eta] = (0)$  or  $(j, z_1 k)^\mathcal{D} = (0)$ ,  $\forall k, z_1 \in \mathcal{L}$ . This concludes that either  $[(\mathcal{L})^\eta, (\mathcal{L})^\eta] = (0)$  or  $(j, z_1 k)^\mathcal{D} = 0$ ,  $\forall j, k, z_1 \in \mathcal{L}$ . By Corollary 9, the former case implies  $\mathcal{L} \subseteq \mathcal{Z}(\mathcal{A})$ , a contradiction. Therefore, we have  $(j, z_1 k)^\mathcal{D} = 0$ ,  $\forall j, k, z_1 \in \mathcal{L}$  and on replacing  $z_1$  by  $2z_1 l_2$ , this infers that  $(l_2)^\eta (k)^\eta (j, z_1)^\mathcal{D} = 0$ ,  $\forall j, k, l_2, z_1 \in \mathcal{L}$ . Since  $\eta$  is an automorphism of  $\mathcal{A}$ , so by Lemma 7 the running equation gives  $(\mathcal{L}, \mathcal{L})^\mathcal{D} = (0)$ . Moreover, by Proposition 24,  $\mathcal{D} = 0$ .

By using the same technique with necessary variations, we can obtain the same conclusion for the case (ii). ■

**Corollary 32.** *If  $(l_1 l_2)^\Delta \pm (l_1)^\theta (l_2)^\Delta + (l_1 l_2)^\theta \in \mathcal{Z}(\mathcal{A})$ ,  $\forall l_1, l_2 \in \mathcal{A}$ . Then  $\mathcal{A}$  is commutative or  $\mathcal{D} = 0$ .*

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