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ON B^*-PURE ORDERED SEMIGROUP

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Abstract

We introduce the concept of B^**-pure* ordered semigroups, and give some properties of B^**-pure* ordered semigroups.

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1. Introduction

A bi-ideal A of a semigroup S is said to be B*-pure* if $A \cap xS = xA$ and $A \cap Sx = Ax$ for all $x \in S$. A semigroup S is said to be B^**-pure* if every bi-ideal of S is B-pure. The concept B^**-pure* semigroups was studied by Kuroki [3]. In this paper, the concept of B^**-pure* ordered semigroups is introduced. We shall give some properties of B^**-pure* ordered semigroups, and characterize B^*-pure Archimedean ordered semigroups. We prove that any B^*-pure ordered semigroup is a semilattices of Archimedean semigroups. Let us recall some certain definitions and results used throughout the paper. A semigroup $(S, \cdot)$ together with a partial order $\leq$ that is *compatible* with the semigroup operation, meaning that, for any x, y, z in S,

$$x \leq y \text{ implies } zx \leq zy \text{ and } xz \leq yz$$

is called a *partially ordered semigroup* (or simply an *ordered semigroup*)(see [2]). Under the trivial relation, $x \leq y$ if and only if $x = y$, it is observed that every semigroup is an ordered semigroup. Let $(S, \cdot, \leq)$ be an ordered semigroup. For A, B nonempty subsets of S, we write AB for the set of all elements xy in S where $x \in A$ and $y \in B$, and write $(A]$ for the set of all elements x in S such that $x \leq a$ for some a in A, i.e.,

$$(A] = \{x \in S \mid x \leq a \text{ for some } a \in A\}.$$

In particular, we write Ax for $A\{x\}$, and xA for $\{x\}A$. It was shown in [10] that the followings hold:

(1) $A \subseteq (A]$ and $((A]] = (A]$;

(2) $A \subseteq B \Rightarrow (A] \subseteq (B]$;

(3) $((A](B]] = ((A]B] = (A(B]] = (AB]$;

(4) $(A](B] \subseteq (AB]$;

(5) $(A]B \subseteq (AB]$ and $A(B] \subseteq (AB]$;

(6) $(A \cup B] = (A] \cup (B]$.

The concepts of left, right and two-sided ideals of an ordered semigroup can be found in [2]. Let $(S, \cdot, \leq)$ be an ordered semigroup. A nonempty subset A of S is called a *left* (resp., *right*) *ideal* of S if it satisfies the following conditions:

(i) $SA \subseteq A$ (resp., $AS \subseteq A$);

(ii) $A = (A]$, that is, for any x in A and y in S, $y \leq x$ implies $y \in A$.

If A is both a left and a right ideal of S, then A is called a *two-sided ideal*, or simply an *ideal* of S. It is known that the union or intersection of two ideals of S is an ideal of S.

Let $(S, \cdot, \leq)$ be an ordered semigroup. A left ideal A of S is said to be *proper* if $A \subset S$. The symbol $\subset$ stands for proper subset of sets. A proper right and two-sided ideals are defined similarly. S is said to be *left (resp., right) simple* if S does not contain proper left (resp., right) ideals. If S does not contain proper *ideals* then we call S *simple*. A proper ideal A of S is said to be *maximal* if for any ideal B of S, if $A \subset B \subseteq S$, then $B = S$.

For any element a of an ordered semigroup $(S, \cdot, \leq)$, the *principal ideal generated* by a is of the form $I(a) = (a \cup Sa \cup aS \cup SaS]$.

A nonempty subset B is called a *bi-ideal* of S if

(i) $BSB \subseteq B$;

(ii) for any x in B and y in S, $y \leq x$ implies $y \in B$ (see [5]).

For any element a of an ordered semigroup $(S, \cdot, \leq)$ the *bi-ideal generated* by a is of the form $B(a) = (\{a\} \cup aSa]$.

An equivalence relation σ on S is called *congruence* if $(a, b) \in \sigma$ implies $(ac, bc) \in \sigma$ and $(ca, cb) \in \sigma$ for every $c \in S$. A congruence σ on S is called *semilattice congruence* if $(a^2, a) \in \sigma$ and $(ab, ba) \in \sigma$ for every $a, b \in S$. A semilattice congruence σ on S is called *complete* if $a \leq b$ implies $(a, ab) \in \sigma$. An ordered semigroup S is called a *semilattice of Archimedean semigroups (resp., complete semilattice of Archimedean semigroups)* if there exists a semilattice congruence (resp., complete semilattice congruence) σ on S such that the σ-class $(x)_\sigma$ of S containing x is a Archimedean subsemigroup of S for every $x \in S$.

A subsemigroup F is called a *filter* of S if

(i) $a, b \in S$, $ab \in F$ implies $a \in F$ and $b \in F$;

(ii) if $a \in F$ and b in S, $a \leq b$, then $b \in F$ (see [6]).

For an element x of S, we denote by $N(x)$ the filter of S generated by x and $\mathcal{N}$ the equivalence relation on S defined by $\mathcal{N} := \{(x, y) \mid N(x) = N(y)\}$. The relation $\mathcal{N}$ is the least complete semilattice congruence on S. An element e of an ordered semigroup $(S, \cdot, \leq)$ is called an *ordered idempotent* if $e \leq e^2$. We call an ordered semigroup S *idempotent ordered semigroup* if every element of S is an ordered idempotent (see [1]). The set of all ordered idempotent of an ordered semigroup S denoted by $E(S)$ and the set of all positive integers denoted by N.

An ordered semigroup $(S, \cdot, \leq)$ is called *Archimedean* if for any a, b in S there exists a positive integer n such that $a^n \in (SbS]$ (see [8]). An ordered semigroup S is called *regular* if for every $a \in S$, there exists $x \in S$ such that $a \leq axa$. Equivalent definitions are as follows: (1) $A \subseteq (ASA]$ for any $A \subseteq S$ or (2) $a \in (aSa]$ for any $a \in S$ (see [7]). An ordered semigroup S is said to be *normal* if $(xS] = (Sx]$ for all $x \in S$. An ordered semigroup S is said to be *weakly commutative* if for any $a, b \in S$, then there exists positive integer n such that $(ab)^n \in (bSa]$ (see [4]). An ideal A of an ordered semigroup S is called *globally idempotent* if $A = (A^2]$ (see [9]). An ideal A of an ordered semigroup S is called *complete* if $A = (AS] = (SA]$ (see [9]).

Definition. Let $(S, \cdot, \leq)$ be an ordered semigroup. A bi-ideal A of S is said to be *B-pure* if $A \cap (xS] = (xA]$ and $A \cap (Sx] = (Ax]$ for all $x \in S$. An ordered semigroup S is said to be *B^*-pure* if every bi-ideal of S is B-pure.

Example 1. Let $S = \{a, b\}$, $xy = b$ for all $x, y \in S$, $\leqslant = \{(a, a), (b, b), (a, b)\}$. It is clear that S is an ordered semigroup. We show that S is B^*-pure. We determine all bi-ideals in S. We have two candidates: $\{a\}$ and S. Of course, S is a bi-ideal, but $\{a\}$ is not a bi-ideal, because $\{a\}S\{a\} = \{b\}$. So there exists only one bi-ideal in S, namely S. Bi-deal S is B-pure, because $S \cap (Sx] = (Sx]$ and $S \cap (xS] = (xS]$ for all $x \in S$.

2. Main results

First, we have the following lemma.

Lemma 2. *Any normal ordered semigroups are weakly commutative.*

Proof. Let S be a normal ordered semigroup and $a, b \in S$. We have

$$(ab)^3 = ababab \in (SbSaS] \subseteq ((Sb]S(aS]] \subseteq ((bS]S(Sa]] \subseteq (bSa].$$

Hence S is weakly commutative. ■

Lemma 3. *Let S be a B^*-pure ordered semigroup. Then S has the following properties:*

(1) $(aS] = (a^2S]$ *and* $(Sa] = (Sa^2]$ *for all* $a \in S$*;*

(2) S *is normal;*

(3) S *is weakly commutative;*

(4) *for each* $x \in S$, $N(x) = \{y \in S \mid x^n \in (ySy]$ *for some* $n \in N\}$*;*

(5) a^2 *is regular for all* $a \in S$.

Proof. (1) Let $a \in S$. Since S is B^*-pure, the bi-ideal $(aS]$ is B-pure. Thus $(aS] = (aS] \cap (aS] = (a(aS]] \subseteq (a^2S]$. The converse is obvious. Hence $(aS] = (a^2S]$. Similarly, we have $(Sa] = (Sa^2]$.

(2) Let $a \in S$. By (1), we have

$$(aS] = (a^2S] \subseteq (SaS] \subseteq ((Sa]S] = (Sa] \cap (SS] \subseteq (Sa].$$

Similarly, we have $(Sa] \subseteq (aS]$. It follows that $(aS] = (Sa]$. Hence S is normal.

(3) This follows by (2) and Lemma 2.

(4) This follows by (3) and lemma in [4].

(5) Let a be any element of S. By (1) and (2) we have

$$a^2 \in (aS] = (a^2S] = (a^4S] \subseteq (a^2(a^2S]] = (a^2(Sa^2]] \subseteq (a^2Sa^2].$$

Thus a^2 is regular. ■

The following Corollary 4 can be obtained from Lemma 2 and theorem in [4].

Corollary 4. *Any normal ordered semigroups are semilattices of Archimedean semigroups.*

The following Theorem 5 can be obtained from Lemma 3 and theorem in [4].

Theorem 5. *Any B^*-pure ordered semigroups are semilattices of Archimedean semigroups.*

Theorem 6. *Let $(S, \cdot, \leq)$ be an ordered semigroup such that $(aS] = (a^2S]$ and $(Sa] = (Sa^2]$ for all a in S. The following statements are equivalent:*

(1) $(Se] = (eS]$ *for all* e *in* $E(S)$*;*

(2) S *is normal;*

(3) S *is weakly commutative;*

(4) *for each* $x \in S$, $N(x) = \{y \in S \mid x^n \in (ySy]$ *for some* $n \in N\}$.

Proof. By Lemma 2, (2) implies (3). We have that (3) and (4) are equivalent by lemma in [4].

(1)⇒(2). Let $a \in S$. We have $a^2 \in (aS] = (a^2S] = (a^4S]$ and $a^2 \in (Sa] = (Sa^2] = (Sa^4]$. Thus $a^2 \leq a^4x$ and $a^2 \leq ya^4$ for some x, y in S. This implies that $a^4 \leq a^4xya^4$. Hence $xya^4 \in E(S)$. Let $b \in (aS] = (a^2S] = (a^4S]$. Then $b \leq a^4z$ for some z in S. We have

$$\begin{aligned} b \leq a^4z \leq a^4xya^4z \in (a^4xya^4S] &\subseteq (a^4(xya^4S]] \\ &= (a^4(Sxya^4]] \\ &\subseteq (a^4Sxya^4] \\ &\subseteq (Sa^4] \\ &\subseteq (Sa]. \end{aligned}$$

Similarly, we have $(Sa] \subseteq (aS]$. Hence S is normal.

(3)⇒(1). Let $e \in E(S)$ and $x \in (eS]$. Then $x \leq ea$ for some $a \in S$. Since S is weakly commutative, then there exists a positive integer n such that $(ea)^n \in (aSe]$. It follows that

$$x \leq ea \leq eea \in (Sea] \subseteq (S(ea)^n] \subseteq (S(aSe]] \subseteq (SaSe] \subseteq (SSSe] \subseteq (Se].$$

Similarly, we have $(Se] \subseteq (eS]$. Hence $(Se] = (eS]$. This complete the proof. ■

Now we have shown that if an ordered semigroup S is B^*-pure, then the converse of Lemma 2 holds.

The following Theorem 7 can be obtained from Lemma 3 and Theorem 6.

Theorem 7. *For a B^*-pure ordered semigroup S. The following statements are equivalent:*

(1) $(Se] = (eS]$ *for all* e *in* $E(S)$;

(2) S *is normal;*

(3) S *is weakly commutative;*

(4) *for each* $x \in S$, $N(x) = \{y \in S \mid x^n \in (ySy]$ *for some* $n \in N\}$.

Theorem 8. *For a B^*-pure ordered semigroup S. The following statements are equivalent:*

(1) *every ideal of* S *is globally idempotent;*

(2) *every ideal of* S *is complete.*

Proof. By Theorem 2.3 in [9], (1) implies (2).

(2)⇒(1). Let A be any ideal of S and $b \in A$. Since A is complete, $A = (AS]$. We have $b \in (aS]$ for some $a \in A$. Since S is B^*-pure and every ideal is a bi-ideal, $A \cap (aS] = (aA]$. We have

$$b \in A \cap (aS] = (aA] \subseteq (A^2].$$

Thus $A \subseteq (A^2]$. As is easily seen, $(A^2] \subseteq A$. Hence $A = (A^2]$. ■

Theorem 9. *For an idempotent ordered semigroup S. The following statements are equivalent:*

(1) *S is B^*-pure;*

(2) *S is normal and $(Sa] = (Sa^2]$ for all $a \in S$.*

Proof. By Lemma 3, (1) implies (2).

(2) $\Rightarrow$ (1). Let A be any bi-ideal of S, $x \in S$. Let $a \in A \cap (Sx] = A \cap (Sx^2]$. Then $a \leq yx^2$ for some $y \in S$. Since $ay \in (aS] = (Sa] = (Sa^2]$, $ay \leq za^2$ for some $z \in S$. We have

$$\begin{aligned} a \leq a^2 \leq ayx^2 \leq za^2x^2 &\in (SaaSx] \\ &\subseteq ((Sa](aS]x] \\ &= ((aS](Sa]x] \\ &\subseteq (aSSax] \\ &\subseteq (ASSAx] \\ &\subseteq (Ax]. \end{aligned}$$

Thus $A \cap (Sx] \subseteq (Ax]$. Let $b \in (Ax]$. Then $b \leq ax$ for some a in A. We have

$$b \leq ax \in (aS] = (Sa] = (Sa^2] \subseteq (aSa] \subseteq (ASA] \subseteq A,$$

and so $(Ax] \subseteq A$. Since $(Ax] \subseteq (Sx]$, then $(Ax] \subseteq A \cap (Sx]$. Thus $A \cap (Sx] = (Ax]$. Similarly, we have $A \cap (xS] = (xA]$. Hence A is B-pure. ■

Theorem 10. *Any normal regular ordered semigroups are B^*-pure.*

Proof. Let S be a normal regular ordered semigroup, A be a bi-deal of S and $x \in S$. Let $b \in (xA]$. Then $b \leq xa$ for some a in A. Since S is regular, then $a \leq aya$ for some y in S. We have

$$\begin{aligned} b \leq xa \leq xaya \in (SaSa] \subseteq ((Sa]Sa] &= ((aS]Sa] \\ &\subseteq (aSSa] \\ &\subseteq (aSa] \\ &\subseteq (ASA] \subseteq A. \end{aligned}$$

Thus $(xA] \subseteq A$. Since $(xA] \subseteq (xS]$, then $(xA] \subseteq A \cap (xS]$. Let $a \in A \cap (xS]$. Then $a \leq xb$ for some b in S. Since S is regular, then $a \leq aya$ for some y in S. We have

$$\begin{aligned} a \le aya \le ayaya \le xbyaya = x(by)aya &\in (xSaya] \\ &\subseteq (x(Sa]ya] \\ &\subseteq (x(aS]SA] \\ &\subseteq (xaSSA] \\ &\subseteq (xASSA] \\ &\subseteq (xA]. \end{aligned}$$

Thus $A \cap (xS] = (xA]$. Similarly, we have $A \cap (Sx] = (Ax]$. Hence A is a B-pure. ■

The following Corollary 11 can be obtained from Lemma 3 and Theorem 10.

Corollary 11. *For a regular ordered semigroup S. The following statements are equivalent:*

(1) *S is B^*-pure;*

(2) *S is normal.*

Theorem 12. *For a B^*-pure ordered semigroup S. The following statements are equivalent:*

(1) *S is Archimedean;*

(2) *$(SaS] = (SbS]$ for all $a, b \in S$;*

(3) *$(aS] = (bS]$ for all $a, b \in S$;*

(4) *$(aSa] = (bSb]$ for all $a, b \in S$;*

(5) *for any $e, f \in E(S)$, $(e, f) \in \mathcal{N}$;*

(6) *every bi-ideal of S is Archimedean.*

Proof. It is clear that (6) implies (1).

(1)⇒(2). Let $a, b \in S$. Since S is Archimedean, then there exists positive integer n such that $a^n \in (SbS]$. By Lemma 3, we have

$$(SaS] \subseteq (Sa^nS] \subseteq (S(SbS]S] \subseteq (SSbSS] \subseteq (SbS].$$

Similarly, we have $(SbS] \subseteq (SaS]$. Hence $(SaS] = (SbS]$. It follows from Lemma 3 (1) and (3) that (2) implies (3) and (3) implies (4).

(4)⇒(5). Let $e, f \in E(S)$. Then $(eSe] = (fSf]$. This implies that $N(e) = N(f)$. Hence $(e, f) \in \mathcal{N}$.

(5)⇒(6). Let A be a bi-deal of S and $a, b \in A$. Since S is B^*-pure, a^2 and b^2 are regular by Lemma 3. Then $a^2 \le a^2xa^2$ and $b^2 \le b^2yb^2$ for some $x, y \in S$. This implies that $a^2x, b^2y \in E(S)$. We have $b^2y \in N(a^2x)$. Then $(a^2x)^n \in (b^2ySb^2y]$ for some positive integer n. Thus $(a^2x)^n \le b^2yzb^2y$ for some $z \in S$. We have

$$\begin{aligned} a^3 \le aa^2xa^2 \le aa^2xa^2xa^2 &= a(a^2x)a^2xa^2 \\ &\le a(a^2x)^n a^2 \\ &\le a(b^2yzb^2y)a^2 \\ &= ab(b(yzb^2ya)a) \\ &\in (Ab(ASA)] \\ &\subseteq (AbA]. \end{aligned}$$

Hence A is Archimedean. This completes the proof of the theorem. ■

Theorem 13. *Any B^*-pure Archimedean regular ordered semigroup S does not contain proper bi-ideals.*

Proof. Let A be any bi-ideal of S. Let $a \in A$ and $b \in S$. Since S is Archimedean, then there exists positive integer n such that $b^n \in (SaS]$. Since S is B^*-pure, $(aSa]$ is B-pure. Then by the regularity of S and Lemma 3, we have

$$\begin{aligned} b \in (bSb] \subseteq (b^nSb^n] &\subseteq ((SaS]S(SaS]] \\ &\subseteq (SaSSSaS] \\ &\subseteq (SaSSS(aS]] \\ &\subseteq (SaSSS(Sa]] \\ &\subseteq (SaSSSSa] \\ &\subseteq (S(aSa]] \\ &= (SS] \cap (aSa] \\ &\subseteq (ASA] \\ &\subseteq A. \end{aligned}$$

Thus $S \subseteq A$. Hence $S = A$. ■

The following Theorem 14 can be obtained from Theorem 13 .

Theorem 14. *Any B^*-pure Archimedean regular ordered semigroups are left and right simple.*

Theorem 15. *For a B^*-pure Archimedean ordered semigroup S. The following statements are equivalent:*

(1) *S is regular;*

(2) *S does not contain proper bi-ideals;*

(3) *S are left and right simple.*

Proof. By Theorem 13, (1) implies (2). It is clear that (2) implies (3).

(3)⇒(1). Let $a \in S$. As is easily seen, $(Sa]$ is a left ideal and $(aS]$ is a right ideal. Since S are left and right simple, then $S = (Sa]$ and $S = (aS]$. We have $a \in (aS] = (a(Sa]] \subseteq (aSa]$. This completes the proof of the theorem. ■

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