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SOME LCD CYCLIC CODES OF LENGTH $2p$ OVER FINITE FIELDS

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Abstract

In this paper, we explicitly determine the LCD minimal and maximal cyclic codes of length $2p$ over finite fields $\mathbb{F}_q$ with p and q are distinct odd primes and $\phi(p) = p-1$ is the multiplicative order of q modulo $2p$. We show that, every LCD maximal cyclic code is a direct sum of LCD minimal cyclic codes.

Keywords: linear and cyclic codes, LCD codes, reversible codes.

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1. Introduction

Let $\mathbb{F}_q$ be a finite field with q elements, where q is a prime power. An $[n, k]$ linear code C over $\mathbb{F}_q$ is a linear subspace of $\mathbb{F}_q^n$ with dimension k. Let C be an $[n, k]$ linear code over $\mathbb{F}_q$. Then the dual code of C is defined as

$$C^{\perp} = \left\{ b \in \mathbb{F}_q^n : bc^T = 0 \quad \forall c \in C \right\},$$

where bc^T denotes the standard inner product of the two vectors b and c (see [6]).

A linear code with a complementary dual (an LCD code) was defined to be a linear code C whose dual code $C^{\perp}$ satisfies (see [8])

$$C \cap C^{\perp} = \{0\}.$$

The linear code C of length n over the finite field $\mathbb{F}_q$ is said to be cyclic if $(c_0, c_1, c_2, \ldots, c_{n-1}) \in C$ implies $(c_{n-1}, c_0, c_2, \ldots, c_{n-2}) \in C$. We can also regard

C as an ideal in the principal quotient ring $R_n := \mathbb{F}_q[x]/(x^n - 1)$. By identifying any vector $(c_0, c_1, c_2, \ldots, c_{n-1}) \in \mathbb{F}_q$ with

$$c_0 + c_1 x + \cdots + c_{n-1} x^{n-1} \in R_n.$$

A code C in R_n is a cyclic code over $\mathbb{F}_q$ if and only if C is an ideal in the principal ring R_n. Let C be a cyclic code of length n over $\mathbb{F}_q$. Then there exists an unique polynomial $g(x) \in \mathbb{F}_q[x]$ such that $C = \langle g(x) \rangle$ and the dual code of C is $C^{\perp} = \langle h^*(x) \rangle$ where

$$x^n - 1 = g(x)h(x)$$

and

$$h^*(x) = h(0)^{-1} x^{deg(h)} h\left(\frac{1}{x}\right)$$

is called the reciprocal polynomial of $h(x)$.

For any polynomials $f(x)$, $g(x) \in \mathbb{F}_q[x]$ we have

$$(fg)^* = f^* g^*.$$

LCD cyclic codes over finite fields called also reversible cyclic codes were first introduced and studied by Massey [7] in 1964. Yang and Massey gave a necessary and sufficient condition for a cyclic code to have a complementary dual [11].

In this paper, we are intersted to construct two classes of LCD cyclic codes of length $2p$ over $\mathbb{F}_q$, with p and q are distinct odd primes and $\phi(p) = p - 1$ is the multiplicative order of q modulo $2p$. (ϕ denotes Euler's phi-function). In the same conditions as above, we show that every LCD maximal cyclic code can be represented as an unique direct sum of three LCD minimal cyclic codes.

The objective of this paper is to determine two classes of LCD cyclic codes of length $2p$ over $\mathbb{F}_q$ and the relationship between them with p and q are distinct odd primes and $\phi(p) = p - 1$ is the multiplicative order of q modulo $2p$.

2. Preliminaries

Let $\mathbb{F}_q$ be the finite field with q elements and let n be a positive integer co-prime to q. A cyclic code C of length n over $\mathbb{F}_q$ is a linear subspace of $\mathbb{F}_q^n$, it is known that any ideal C in $R_n = \mathbb{F}_q[x]/(x^n - 1)$ is generated by an unique monic polynomial $g(x)$ of the least degree in C. The polynomial $g(x)$ is a divisor of $(x^n - 1)$, and is called the generating polynomial of the code C. The integer $k = n - \deg g(x)$ is called the dimension of the subspace C and $|C| = q^k$. A minimal ideal in R_n, is called minimal (or an irreducible) cyclic code of length n over $\mathbb{F}_q$. Again, a maximal ideal in R_n, is called a maximal cyclic code of length n over $\mathbb{F}_q$.

Let $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$ denote the ring of integers modulo n. For $a, b \in \mathbb{Z}_n$, we say that $a \sim b$ if $a \equiv bq^i \pmod{n}$ for some integer $i \geq 0$. The relation $\sim$ is an equivalence relation on the set $\mathbb{Z}_n$ and partitions the set $\mathbb{Z}_n$ into disjoint equivalence classes called the q-cyclotomic cosets.

For $s \in \mathbb{Z}_n$, the class of s denoted by C_s is given by

$$C_s := \left\{s, sq, sq^2, \ldots, sq^{n_s-1}\right\} \pmod{n},$$

where n_s is the smallest positive integer such that $sq^{n_s} \equiv s \pmod{n}$ (see [2]).

The smallest nonnegative integer in C_s is called the coset leader of C_s.

Let $\Gamma_{(n,q)}$ be the set of all the coset leaders.Then we have

$$\bigcup_{s \in \Gamma_{(n,q)}} C_s = \mathbb{Z}_n.$$

Let α be a generator of $\mathbb{F}_{q^m}^*$, where $m = ord_n(q)$, then the element $\beta = \alpha^{\frac{q^m-1}{n}}$ is a primitive n-th root of unity in $\mathbb{F}_{q^m}$, then for each integer s, the polynomial (see for example Ling and Xing [5])

$$m_s(x) = \prod_{j \in C_s} \left(x - \beta^j\right)$$

is the minimal polynomial of β^s over $\mathbb{F}_q$, which is irreducible over $\mathbb{F}_q$.

It then follows that

$$x^n - 1 = \prod_{s \in \Gamma_{(n,q)}} m_s(x)$$

gives the decomposition of $x^n - 1$ into irreducible factors over $\mathbb{F}_q$.

The cyclic code $\widehat{m_s}$ in R_n generated by $\frac{(x^n-1)}{m_s(x)}$ is called a minimal cyclic code of length n over $\mathbb{F}_q$ or irreducible cyclic codes and the cyclic code M_s in R_n, generated by $m_s(x)$, is called a maximal cyclic code of length n over $\mathbb{F}_q$.

For more information see, for example, [3, 4] and [6].

We recall some definitions as below:

- A linear code C over $\mathbb{F}_q$ is said to be linear complimentary dual (LCD) if $C \cap C^{\perp} = \{0\}$.
- A polynomial $f(x)$ is said to be self-reciprocal if $f(x) = f^*(x)$, where $f^*(x)$ is the reciprocal polynomial of $f(x)$.
- A linear code C of length n is said to be reversible if $(c_{n-1}, c_{n-2}, \ldots, c_1, c_0) \in C$ whenever $(c_0, c_1, \ldots, c_{n-1}) \in C$.
- A cyclic code $C = \langle f(x) \rangle$ of length n over $\mathbb{F}_q$ is reversible if $f(x)$ is a self-reciprocal polynomial.

Remark 1. *LCD cyclic codes were referred to as reversible cyclic codes in the literature.*

3. Factorization of $x^{2p}-1$ over $\mathbb{F}_q$ and auxiliaries

In the paper [10], the authors determined the q-cyclotomic cosets modulo $2p^n$ with $n \geq 1$ is an integer, and p is an odd prime over the finite fields $\mathbb{F}_q$ where q is a power of an odd prime number, with $(p,q)=1$ and $\phi(p^n)$ is the multiplicative order of q modulo $2p^n$. In this paper, we are intersted in the special case $n=1$ (see [1, 2, 4]).

Proposition 2. *Let $\mathbb{Z}_{2p}=\{0,1,2,\dots,2p-1\}$ denote the ring of integers modulo $2p$ and $\phi(p)$ is the multiplicative order of q modulo $2p$. Then $\mathbb{Z}_{2p}$, can be partitioned into 4 q-cyclotomic cosets.*

Proof. For $s \in \mathbb{Z}_{2p}$, the class of s denoted by C_s is given by

$$C_s := \left\{s, sq, sq^2, \dots, sq^{n_s-1}\right\} \pmod{2p}.$$

Since q has order $\phi(p) \pmod{2p}$, so q also has order

$$\phi\left(p^{2-i}\right) \pmod{2p^{2-i}}, 1 \leq \ i \leq 2.$$

Then

$$q^{\phi(p^{2-i})} \equiv 1 \pmod{2p^{2-i}}$$

or

$$p^{i-1}q^{\phi(p^{2-i})} \equiv p^{i-1} \pmod{2p}$$

and

$$2p^{i-1}q^{\phi(p^{2-i})} \equiv 2p^{i-1} \pmod{2p}.$$

Hence

$$C_{p^{i-1}} = \left\{p^{i-1}, p^{i-1}q, \dots, p^{i-1}q^{\phi(p^{2-i})-1}\right\}$$

and

$$C_{2p^{i-1}} = \left\{2p^{i-1}, 2p^{i-1}q, \dots, 2p^{i-1}q^{\phi(p^{2-i})-1}\right\}.$$

Since $|C_0|=|C_p|=1$ and $|C_1|=|C_2|=p-1$, we have

$$C_0 \cup C_p \cup C_1 \cup C_2 = \mathbb{Z}_{2p}.$$

∎

In this section, we consider the complete factorization of $x^{2p}-1$ over $\mathbb{F}_q$, with p and q are distinct odd primes and $\phi(p)=p-1$ is the multiplicative order of q modulo $2p$.

Proposition 3. *Let $\mathbb{F}_q$ be a finite field with q elements and p be an odd prime coprime to q. Let $2p|q^m-1$, where $m = ord_{2p}(q)$, then*

$$x^{2p}-1=\prod_{s\in\Gamma_{(2p,q)}} m_s(x),$$

where

$$m_0(x) = x-1,$$

$$m_p(x) = x+1,$$

$$m_1(x) = x^{p-1} - x^{p-2} + \cdots - x + 1,$$

$$m_2(x) = x^{p-1} + x^{p-2} + \cdots + x + 1.$$

Since the classes C_0, C_p, C_1, C_2 are all the distinct q-cyclotomic cosets modulo $2p$, we have

$$M_0 = \langle m_0(x)\rangle,\ M_p = \langle m_p(x)\rangle,\ M_1 = \langle m_1(x)\rangle,\ M_2 = \langle m_2(x)\rangle,$$

are precisely all the distinct maximal cyclic codes of length $2p$ over $\mathbb{F}_q$.

And we have

$$\widehat{m_0} = \left\langle \tfrac{(x^{2p}-1)}{m_0(x)} \right\rangle,\ \widehat{m_p} = \left\langle \tfrac{(x^{2p}-1)}{m_p(x)} \right\rangle,\ \widehat{m_1} = \left\langle \tfrac{(x^{2p}-1)}{m_1(x)} \right\rangle,\ \widehat{m_2} = \left\langle \tfrac{(x^{2p}-1)}{m_2(x)} \right\rangle,$$

are precisely all the distinct minimal cyclic codes of length $2p$ over $\mathbb{F}_q$.

In this paragraph we are intersted to determine two classes of LCD cyclic codes of length $2p$ over $\mathbb{F}_q$, with p and q are distinct odd primes and $\phi(p) = p-1$ is the multiplicative order of q modulo $2p$.

The following tables, gives the generating polynomial and the corresponding reciprocal polynomial of the above maximal and minimal codes.

Table 1. The reciprocal polynomial of the generating polynomial of the maximal cyclic codes of length $2p$ over $\mathbb{F}_q$.

Codes	Generating polynomial $g(x)$	The reciprocal polynomial $g^*(x)$ of $g(x)$
M_0	$m_0(x)$	$m_0(x)$
M_p	$m_p(x)$	$m_p(x)$
M_1	$m_1(x)$	$m_1(x)$
M_2	$m_2(x)$	$m_2(x)$

Table 2. The reciprocal polynomial of the generating polynomial of the minimal cyclic codes of length $2p$ over $\mathbb{F}_q$.

Codes	Generating polynomial $g(x)$	The reciprocal polynomial $g^*(x)$ of $g(x)$
$\widehat{m_0} = \langle \frac{(x^{2p}-1)}{m_0(x)} \rangle$	$m_p(x) \times m_1(x) \times m_2(x)$	$m_p(x) \times m_1(x) \times m_2(x)$
$\widehat{m_p} = \langle \frac{(x^{2p}-1)}{m_p(x)} \rangle$	$m_0(x) \times m_1(x) \times m_2(x)$	$m_0(x) \times m_1(x) \times m_2(x)$
$\widehat{m_1} = \langle \frac{(x^{2p}-1)}{m_1(x)} \rangle$	$m_0(x) \times m_p(x) \times m_2(x)$	$m_0(x) \times m_p(x) \times m_2(x)$
$\widehat{m_2} = \langle \frac{(x^{2p}-1)}{m_2(x)} \rangle$	$m_0(x) \times m_p(x) \times m_1(x)$	$m_0(x) \times m_p(x) \times m_1(x)$

In [11], a necessary and sufficient condition for the existence of LCD cyclic codes of length n over $\mathbb{F}_q$ is given.

Theorem 4 [6]**.** *Let C be a cyclic code of length n over $\mathbb{F}_q$ with generator polynomial $g(x)$ and $\gcd(n, q) = 1$. Then the following statements are equivalent.*

1. *C is an LCD cyclic code.*
2. *$g(x)$ is self-reciprocal, i.e., $g^*(x) = g(x)$.*

Proposition 5. *Every maximal cyclic code of length $2p$ over $\mathbb{F}_q$ is an LCD maximal cyclic code of length $2p$ over $\mathbb{F}_q$, where p and q are distinct odd primes and $\phi(p) = p - 1$ is the multiplicative order of q modulo $2p$.*

Proof. Let $C = \langle g(x) \rangle$ be a maximal cyclic code of length $2p$ over $\mathbb{F}_q$. Then, from Table 1, $g(x)$ is a self-reciprocal. By Theorem 4, the code C is an LCD cyclic code. ■

Proposition 6. *Every minimal cyclic code of length $2p$ over $\mathbb{F}_q$ is an LCD minimal cyclic code of length $2p$ over $\mathbb{F}_q$, p and q are distinct odd primes and $\phi(p) = p - 1$ is the multiplicative order of q modulo $2p$.*

Proof. Let $C = \langle g(x) \rangle$ be a minimal cyclic code of length $2p$ over $\mathbb{F}_q$. Then, from Table 2, $g(x)$ is a self-reciprocal. By Theorem 4, the code $C$is an LCD cyclic code. ■

4. RESULTS CONCERNING SOME LCD CYCLIC CODES OF LENGTH $2p$

In this section we determine the relationship between the LCD maximal cyclic codes and the LCD minimal cyclic codes of length $2p$ over $\mathbb{F}_q$, with p and q are distinct odd primes and $\phi(p) = p - 1$ is the multiplicative order of q modulo $2p$, we show that every LCD maximal cyclic code of length $2p$ can be represented as a direct sum of three LCD minimal cyclic codes.

Table 3. The generating polynomial of the dual of maximal cyclic codes of length $2p$ over $\mathbb{F}_q$.

Code C	Generating polynomial of C	Generating polynomial of $C^{\perp}$
M_0	$m_0(x)$	$m_p(x) \times m_1(x) \times m_2(x)$
M_p	$m_p(x)$	$m_0(x) \times m_1(x) \times m_2(x)$
M_1	$m_1(x)$	$m_0(x) \times m_p(x) \times m_2(x)$
M_2	$m_2(x)$	$m_0(x) \times m_p(x) \times m_1(x)$

Table 4. The generating polynomial of the dual of minimal cyclic codes of length $2p$ over $\mathbb{F}_q$.

Codes C	Generating polynomial C	Generating polynomial of $C^{\perp}$
$\widehat{m_0} = \langle \frac{(x^{2p}-1)}{m_0(x)} \rangle$	$m_p(x) \times m_1(x) \times m_2(x)$	$m_0(x)$
$\widehat{m_p} = \langle \frac{(x^{2p}-1)}{m_p(x)} \rangle$	$m_0(x) \times m_1(x) \times m_2(x)$	$m_p(x)$
$\widehat{m_1} = \langle \frac{(x^{2p}-1)}{m_1(x)} \rangle$	$m_0(x) \times m_p(x) \times m_2(x)$	$m_1(x)$
$\widehat{m_2} = \langle \frac{(x^{2p}-1)}{m_2(x)} \rangle$	$m_0(x) \times m_p(x) \times m_1(x)$	$m_2(x)$

Proposition 7. *Let C_i be a cyclic code of length n over $\mathbb{F}_q$ for $i = 1$ and 2. Then the sum $C_1 + C_2$ is direct, if and only if $C_1 \cap C_2 = \{0\}$.*

Proposition 8. *Let C_i be a cyclic code of length n over $\mathbb{F}_q$ for $i \in \{1, 2, 3\}$. Then $C_1 + C_2 + C_3$ is a direct sum if and only if*

$$\dim(C_1 + C_2 + C_3) = \dim(C_1) + \dim(C_2) + \dim(C_3).$$

Theorem 9 [9]**.** *Let C_i be a cyclic code of length n over $\mathbb{F}_q$ with generator polynomial $g_i(x)$ for $i = 1$ and 2. Then the cyclic code $C_1 + C_2$ has generator polynomial* $\gcd(g_1(x), g_2(x))$.

Now, we prove our main results.

Proposition 10. *If C is an LCD maximal cyclic code of length $2p$ over $\mathbb{F}_q$, with p and q are distinct odd primes and $\phi(p) = p - 1$ is the multiplicative order of q modulo $2p$, then C can be represented as a direct sum of three LCD minimal cyclic codes of length $2p$ over $\mathbb{F}_q$, $C = C_1 \oplus C_2 \oplus C_3$, Moreover, $|C| = |C_1|\,|C_2|\,|C_3|$.*

Proof. Using the proposition [8], theorem [9] and properties of gcd of polynomials, we find: if

$$C_1 = \widehat{m_0}, \quad C_2 = \widehat{m_p}, \quad C_3 = \widehat{m_1},$$

then

$$\begin{aligned}\dim(C) &= \dim\left(C_1 + C_2 + C_3\right) = \dim\left(\left(C_1 + C_2\right) + C_3\right)\\ &= \dim\left(\langle \gcd\left(m_1(x) \times m_2(x), \quad m_0(x) \times m_p(x) \times m_2(x)\right)\rangle\right),\\ &\quad \text{since} \quad \widehat{m_0} + \widehat{m_p} = \langle m_1(x) \times m_2(x)\rangle\\ &= \dim\left(\langle m_2(x) \times \gcd\left(m_1(x), \quad m_0(x) \times m_p(x)\right)\right)\\ &= \dim\left(\langle m_2(x)\rangle\right) = \dim\left(M_2\right).\end{aligned}$$

Hence

$$\dim(C) = \dim(M_2) = p + 1 = \dim(C_1) + \dim(C_2) + \dim(C_3).$$

We find

$$C = C_1 \oplus C_2 \oplus C_3.$$

On the other hand

$$|C_1|\,|C_2|\,|C_3| = q \cdot q \cdot q^{p-1} = q^{p+1} = |C|\,.$$

Hence

$$|C| = |C_1|\,|C_2|\,|C_3|\,.$$

In a similar way, we find: if

$$C_1 = \widehat{m_0}, \quad C_2 = \widehat{m_p}, \quad C_3 = \widehat{m_2},$$

then

$$\begin{aligned}\dim(C) &= \dim\left(C_1 + C_2 + C_3\right) = \dim\left(\left(C_1 + C_2\right) + C_3\right)\\ &= \dim(\langle \gcd\left(m_1(x) \times m_2(x), \quad m_0(x) \times m_p(x) \times m_1(x)\right)\rangle),\\ &\quad \text{since} \quad \widehat{m_0} + \widehat{m_p} = \langle m_1(x) \times m_2(x)\rangle\\ &= \dim\left(\langle m_1(x) \times \gcd\left(m_2(x), \quad m_0(x) \times m_p(x)\right)\rangle\right)\\ &= \dim(\langle m_1(x)\rangle) = \dim(M_1) = p + 1 = \dim(C_1) + \dim(C_2) + \dim(C_3).\end{aligned}$$

Hence

$$C = C_1 \oplus C_2 \oplus C_3.$$

On the other hand

$$|C_1|\,|C_2|\,|C_3| = q \cdot q \cdot q^{p-1} = q^{p+1} = |C|\,.$$

Hence

$$|C| = |C_1|\,|C_2|\,|C_3|\,.$$

In a similar way, we find: if

$$C_1 = \widehat{m_0}, \quad C_2 = \widehat{m_1}, \quad C_3 = \widehat{m_2},$$

then

$$\begin{aligned}
\dim(C) &= \dim\left(C_1 + C_2 + C_3\right) = \dim\left(\left(C_1 + C_2\right) + C_3\right)\\
&= \dim(\langle \gcd\left(m_p(x) \times m_2(x), \quad m_0(x) \times m_p(x) \times m_1(x)\right)\rangle),\\
&\quad \text{since} \quad \widehat{m_0} + \widehat{m_1} = \langle m_p(x) \times m_2(x)\rangle\\
&= \dim\left(\langle \gcd\left(m_p(x) \times m_2(x), m_0(x) \times m_p(x) \times m_1(x)\right)\rangle\right)\\
&= \dim\left(\langle m_p(x) \times \ \gcd\left(m_2(x), m_0(x) \times m_1(x)\right)\rangle\right) = \dim\left(\langle m_p(x)\rangle\right)\\
&= \dim(M_p) = 2p - 1 = \dim(C_1) + \dim(C_2) + \dim(C_3).
\end{aligned}$$

Hence

$$C = C_1 \oplus C_2 \oplus C_3.$$

On the other hand

$$|C_1|\,|C_2|\,|C_3| = q \cdot q^{p-1} \cdot q^{p-1} = q^{2p-1} = |C|\,.$$

Hence

$$|C| = |C_1|\,|C_2|\,|C_3|\,.$$

In a similar way, we find: if

$$C_1 = \widehat{m_P}, \quad C_2 = \widehat{m_1}, \quad C_3 = \widehat{m_2},$$

then

$$\begin{aligned}
\dim(C) &= \dim\left(C_1 + C_2 + C_3\right) = \dim\left(\left(C_1 + C_2\right) + C_3\right)\\
&= \dim(\langle \gcd\left(m_0(x) \times m_2(x), \quad m_0(x) \times m_p(x) \times m_1(x)\right)\rangle),\\
&\quad \text{since} \quad \widehat{m_p} + \widehat{m_1} = \langle m_0(x) \times m_2(x)\rangle\\
&= \dim\left(\langle m_0(x) \times \gcd\left(m_1(x), m_0(x) \times m_p(x)\right)\rangle\right) = \dim\left(\langle m_0(x)\rangle\right)\\
&= \dim(M_0) = 2p - 1 = \dim(C_1) + \dim(C_2) + \dim(C_3).
\end{aligned}$$

Hence

$$C = C_1 \oplus C_2 \oplus C_3.$$

On the other hand

$$|C_1|\,|C_2|\,|C_3| = q \cdot q^{p-1} \cdot q^{p-1} = q^{2p-1} = |C|\,.$$

Hence

$$|C| = |C_1|\,|C_2|\,|C_3|\,.$$

■

Example 11. Take $q = 7$, $p = 11$. Then $\Gamma_{(2p,q)} = \{0, 1, 2, 11\}$, hence the LCD maximal cyclic codes M_0, M_{11}, M_1, M_2 of length 22 over $\mathbb{F}_7$ and the LCD minimal cyclic codes $\widehat{m_0}$, $\widehat{m_{11}}$, $\widehat{m_1}$, $\widehat{m_2}$ of length 22 over $\mathbb{F}_7$ are given below:

(a) There are the following minimal polynomials
$m_0(x) = x - 1$, $m_{11}(x) = x + 1$,
$m_1(x) = x^{11} - x^{10} + x^9 - x^8 + x^7 - x^6 + x^5 - x^4 + x^3 - x^2 + x - 1$,
$m_2(x) = x^{11} + x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$.

(b) If $g_s(x)$ is the generating polynomial of $\widehat{m_s}$ then we have
$g_0(x) = \frac{(x^{22}-1)}{m_0(x)} = x^{21} + x^{20} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$,
$g_{11}(x) = \frac{(x^{34}-1)}{m_{17}(x)} = x^{21} - x^{20} + x^{19} - x^{18} + x^{17} - x^{16} + x^{15} - x^{14} + x^{13} - x^{12} + x^{11} - x^{10} + x^9 - x^8 + x^7 - x^6 + x^5 - x^4 + x^3 - x^2 + x - 1$,
$g_1(x) = \frac{(x^{22}-1)}{m_1(x)} = x^{12} + x^{11} - x - 1$,
$g_2(x) = \frac{(x^{34}-1)}{m_2(x)} = x^{12} - x^{11} + x - 1$.

(c) Table 5. The generating polynomial and dimension of the LCD maximal cyclic codes of length 22 are given by

LCD Maximal cyclic code of length 22 over $\mathbb{F}_7$	M_0	M_{11}	M_1	M_2
Generating polynomial	$m_0(x)$	$m_{11}(x)$	$m_1(x)$	$m_2(x)$
Dimension	21	21	12	12

(d) Table 6. The generating polynomial and dimension of the LCD minimal cyclic codes of length 22 are given by

LCD Minimal cyclic code of length 22 over $\mathbb{F}_7$	$\widehat{m_0}$	$\widehat{m_{11}}$	$\widehat{m_1}$	$\widehat{m_2}$
Generating polynomial	$g_0(x)$	$g_{11}(x)$	$g_1(x)$	$g_2(x)$
Dimension	1	1	10	10

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