Discussiones Mathematicae
General Algebra and Applications 44 (2024) 93–99
https://doi.org/10.7151/dmgaa.1445

SOME RESULTS ON DEPENDENT ELEMENTS IN SEMIRINGS

S. Sara and R. Uzma

Department of Mathematics
LCWU, Lahore, Pakistan

e-mail: saro_c18@yahoo.com
ramzanu16@gmail.com

Abstract

In this paper, we introduce the notion of dependent elements of derivation in MA-Semirings. We also generalize some results of dependent elements of derivation of rings for MA-Semirings.

Keywords: MA-semiring, semiprime MA-semiring, commutators, centralizer, derivation, dependent element, free action.

2020 Mathematics Subject Classification: 16Y60, 16W25.

1. Introduction

MA-Semirings were introduced by Javed, Aslam, Hussain [6] in 2012. In the last few years, various concepts related to Lie type theory have been investigated in the structure of MA-Semirings (see [5, 11, 12]). A semiring X is said to be inverse semiring if for every $a \in X$ there exist a unique element $\acute{a} \in X$ such that $a+\acute{a}+a = a$ and $\acute{a}+a+\acute{a} = \acute{a}$, where $\acute{a}$ is called pseudo inverse of a. MA-Semirings form a subclass of inverse semirings which satisfy condition $(A-2)$ stated by Bandelt and Petrich [2], i.e., $a + \acute{a} \in Z(X)$, for all $a \in X$, where $Z(X)$ denotes the center of X. Throughout this paper, X will represent an MA-Semiring. The commutator $[x, y]$ in an MA-Semiring is defined as $[x, y] = xy + \acute{y}x = xy + y\acute{x}$ [6]. We will use the basic commutator identities $[xy, z] = [x, z]y + x[y, z]$ and $[x, yz] = y[x, z] + [x, y]z$. X is prime if $aXb = 0$ implies $a = 0$ or $b = 0$ and semiprime if $aXa = 0$ implies $a = 0$. An additive mapping $d : X \to X$ is called a derivation on X if $d(xy) = d(x)y + xd(y)$ for all $x, y \in X$. By [5], an additive mapping $d : X \to X$ is called commuting if $[d(x), x] = 0$ for all $x \in X$. It is called central if $d(x) \in Z(X)$ for all $x \in X$. Let $a \in X$ be a fixed element then the mapping $d : X \to X$ given by $d(x) = [a, x]$ is an inner derivation on X.

Laradji and Thaheem [15] initiated the study of dependent elements of endomorphisms of semiprime rings and generalized a number of results for semiprime rings. Dependent elements were covertly used by Kallman [7] to extend the concept of free action of automorphisms of abelian von Neumann algebras of Murray and von Neumann [9]. In [3] and [15], the notion of dependent elements and free action was studied for prime and semiprime rings. This concept was recently introduced for Semirings [11]. Our objective is to introduce the concept of dependent element of derivation in MA-semirings.

Motivated by the work of Laradji, Thaheem [15], Vukman and Kosi-Ulbl [16], we define dependent elements of derivation in MA-Semirings as follows. An element $a \in X$ is dependent element of a derivation $d : X \to X$ if $d(x)a + [a, x]\acute{a} = 0$ for all $x \in X$. The set of all dependent elements of derivation d is denoted by $D(d)$. If the only dependent element of a mapping d is zero then d acts freely on X. We also generalize some important results of [1] for semiprime MA-Semirings.

2. Main results

Theorem 2.1. *Let d be commuting derivation on a semiprime MA-semiring X. An element $a \in D(d)$ if and only if $[a, x] = 0$ and $d(x)a = 0$ $\forall x \in X$.*

Proof. Consider $a \in D(d)$. Then

$$d(x)a + [a, x]\acute{a} = 0. \tag{1}$$

Replacing x by xy and using the fact that d is a derivation, we get $d(x)ya + [a, x]y\acute{a} + x(d(y)a + [a, y]\acute{a}) = 0$. By (1), we obtain

$$d(x)ya + [a, x]y\acute{a} = 0. \tag{2}$$

Multiply (2) by z on the right

$$d(x)yaz + [a, x]y\acute{a}z = 0. \tag{3}$$

Put $y = yz$ in (2), we have $d(x)yza + [a, x]yz\acute{a} = 0$. Adding a pseudo inverse of the last equation and equation (3), we have

$$d(x)y[a, z] + [a, x]y[z, a] = 0. \tag{4}$$

Multiplying (4) by x on left, we get

$$xd(x)y[a, z] + x[a, x]y[z, a] = 0 \tag{5}$$

Replacing y by xy in (4), we have $d(x)xy[a,z]+[a,x]xy[z,a]=0$. Adding the last equation and equation (5). Then, we have $[x,d(x)]y[a,z]+[x,[a,x]]y[z,a]=0$. Since d is commuting so $[x,d(x)]=0$ thus the last equation becomes $[x,[a,x]]$ $y[z,a]=0$. This gives

$$[[a,x],x]y[a,z]=0. \tag{6}$$

Multiply (6) with z on the right

$$[[a,x],x]y[a,z]z=0. \tag{7}$$

Replace y by yz in (6), we have $[[a,x],x]yz[a,z]=0$. Adding a pseudo inverse of the last equation in (7) and then by definition of commutator, we have $[[a,x],x]y[[a,z],z]=0$. Replacing z with x, we get $[[a,x],x]y[[a,x],x]=0$. Using semiprimeness of X, from the last equation we get $[[a,x],x]=0\ \forall x\in X$. As d is an inner derivation defined as $d(x)=[a,x]$. So last equation becomes

$$[d(x),x]=0. \tag{8}$$

So the inner derivation is commuting. Linearizing (8), we have

$$[d(x),y]+[d(y),x]=0. \tag{9}$$

Replacing x by xy in (9), we get $d(x)[y,y]+[d(x),y]y+x[d(y),y]+[x,y]d(y)+x[d(y),y]+[d(y),x]y=0$. By (8), $d(x)[y,y]+[d(x),y]y+[x,y]d(y)+[d(y),x]y=0$ or $d(x)(y+\acute{y}+y)y+\acute{y}d(x)y+[x,y]d(y)+[d(y),x]y=0$ or $d(x)yy+\acute{y}d(x)y+[x,y]d(y)+[d(y),x]y=0$ or $([d(x),y]+[d(y),x])y+[x,y]d(y)=0$. From (9), we have $[x,y]d(y)=0$. Replacing x by xz in the last relation and using it again, we obtain $[x,y]zd(y)=0\ \forall x,y,z\in X$. So we have, $[x,y]Xd(y)=0$. For $a\in D(d)$, $[a,y]Xd(y)=0$. As $d(y)=[a,y]$, so $[a,y]X[a,y]=0$, using semiprimeness of X in above equation, we have $[a,y]=0$ for all $y\in X$. Further from (1), we get $d(x)a=0$.

Conversely, consider $[a,x]=0$ and $d(x)a=0$. Post multiply $[a,x]=0$ with $\acute{a}$ and adding $d(x)a=0$, we get $d(x)a+[a,x]\acute{a}=0$. So $a\in D(d)$. Hence proved.

Theorem 2.2. *Let d be a commuting derivation of semiprime MA-Semiring X. If $a\in D(d)$ then $d(a)=0$.*

Proof. Since $a\in D(d)$, therefore

$$d(x)a=0\ \ \forall x\in X. \tag{10}$$

By replacing x by $d(x)$ in (10)

$$d^2(x)a=0\ \ \forall x\in X. \tag{11}$$

From (10), we get $0 = d(0) = d(d(x)a) = d^2(x)a + d(x)d(a)$. Using (11)

$$d(x)d(a) = 0. \tag{12}$$

Replacing x by ax in (12) $d(a)xd(a) + ad(x)d(a) = 0$. Using (12), we have $d(a)xd(a) = 0$ forall $x \in X$. Using semiprimeness of X, from the last equation we get $d(a) = 0$. This proves the result.

Theorem 2.3. *Let X be a commutative semiprime MA-Semiring. Then $D(d)$ is a commutative semiprime subsemiring of X.*

Proof. Take $a, b \in D(d)$. Then by Theorem 2.1 $[a+b, x] = [a, x]+[b, x] = 0+0 = 0$ and $d(x)(a + b) = d(x)a + d(x)b = 0 + 0 = 0\ \forall x \in X$. So, $a + b \in D(d)$. Also $[ab, x] = [a, x]b + a[b, x] = (0)b + a(0) = 0$ and $d(x)ab = (d(x)a)b = (0)b = 0$ implies $ab \in D(d)$. Since $a, b \in D(d)$ then by Theorem 2.1 $[a, x] = [b, x] = 0$, which means that a, b are in center. Thus $D(d)$ is commutative. Also if $a \in D(d)$, then $d(x)a + [a, x]\acute{a} = 0$. Taking a pseudo inverse of above equation $d(x)\acute{a} + [a, x]\acute{\acute{a}} = 0$ that is $\acute{a} \in D(d)$. So $D(d)$ is a commutative subsemiring of X. To show semiprimeness of $D(d)$, consider $aD(d)a = 0$, $a \in D(d)$. Then $axa = 0$ for all $x \in D(d)$. In particular $a^3 = 0$, which implies $a = 0$ (because X has no central nilpotent). Thus $D(d)$ is a commutative semiprime subsemiring of semiring X.

Theorem 2.4. *If d is a commuting derivation of semiprime commutative MA-Semiring X. Then $D(d)$ is an ideal of X.*

Proof. Consider $a, b \in D(d)$ and by Theorem 2.3 $a + b$ and $\acute{a}, \acute{b}$ are also in $D(d)$. Let $a \in D(d)$ and using Theorem 2.2 $d(x)a = 0$ and $[a, x] = 0$ for all $x \in X$. For $d(x)a = 0$, post multiply with $r \in X$, we get $d(x)ar = 0$. Since $ar = ra$ as X is commutative. So $d(x)ar = d(x)ra = 0$ for all $x \in X$. Also $[ra, x] = [r, x]a + r[a, x] = [r, x]a = rxa + \acute{x}ra$. Since X is commutative $[ra, x] = rax + r\acute{x}a = r[a, x] = 0$. Hence $ar = ra \in D(d)$. Thus $D(d)$ is an ideal of X.

Observation 2.5. (i) *If X is a semiprime MA-Semiring then any ideal I of X is a semiprime subsemiring of X.*

If X is a semiprime MA-Semiring then an obvious calculation show that any ideal I of X is a subsemiring of X. To show semiprimeness of I, let $t \in I$. Consider $txt = 0$ for all $x \in I$. Replacement of x by xr, $r \in X$ implies $txrt = 0$. Post multiplying by x, we have $txrtx = 0$. By semiprimeness of X, we have $tx = 0$, for all $x \in I$. Replace x by rx, $r \in X$, we get $trx = 0$. Replace x by t, we have $trt = 0$. By semiprimenes of X, we arrived at desired result. That is I is a semiprime subsemiring of X.

(ii) *If d is a commuting derivation on X then $ad(x) = 0$ for all $x \in X$. To show this, consider $a \in D(d)$ and by Theorem 2.1, we have $d(x)a = 0$ for* *all $x \in X$. Replacing x with xy, we have $d(xy)a = 0$. As d is a derivation so $d(x)ya + xd(y)a = 0$ thus $d(x)ya = 0$. Pre multiplying by a and post multiplying by $d(x)$ in the last equation, we get $ad(x)yad(x) = 0$. Using semiprimeness of X, we have $ad(x) = 0$. Hence proved.*

Theorem 2.6. *Let X be a semiprime MA-Semiring. Then $C = \{a \in X : d(x)a = 0 \ \forall x \in X\}$ is an ideal of X.*

Proof. To show C is an ideal, let $a, b \in C$ then $d(x)a = 0$ and $d(x)b = 0$, for all $x \in X$. Thus $d(x)(a + b) = d(x)a + d(x)b = 0$, which implies $a + b \in C$. Now for $a \in C$, $d(x)a = 0$. Post multiply with $r \in X$, we get $d(x)ar = 0$ which implies $ar \in C$. Again for $d(x)a = 0$, replacing x by xr, we have $d(xr)a = 0$ or $d(x)ra + xd(r)a = 0$ or $d(x)ra = 0$. This implies $ra \in C$. Hence C is an ideal.

Lemma 2.7. *Let J be an ideal of an MA-Semiring X then $L = Ann(J)$ is also an ideal and $J \cap L = \{0\}$.*

Theorem 2.8. *Let d be a commutative derivation on a semiprime MA-Semiring X. Then there exist ideals U and V of X which satisfies following.*

(a) $U \cap V = \{0\}$,

(b) $d = 0$ *on* U *and* $d(V) \subseteq V$,

(c) $D(d|_V) = 0$, *where* $d|_V$ *is a restriction of* d *on* V. *In other words, the restriction of* d *on* V *is a free action.*

Proof. (a) Consider $D(d)$, the set of all dependent elements of commuting derivation d. Let U be the ideal of X generated by $D(d)$. Let $V = Ann(U)$. Then by Lemma 2.7, V is an ideal and $U \cap V = \{0\}$.

(b) Let $a \in D(d)$ then by Theorem 2.2 and Theorem 2.1, $d(a) = 0$ and $d(x)a = 0$ for all $x \in X$. By Observation 2.5(ii) $ad(x) = 0$. Thus $d(ax) = d(a)x + ad(x) = 0$, $d(xa) = d(x)a + xd(a) = 0$ and $d(xay) = d(x)ay + xd(a)y + xad(y) = 0$, for all $a \in D(d)$ and $x, y \in X$. Hence $d = 0$ on $U = < D(d) >$.

Also, let $v \in V$ which is $Ann(U)$. Thus $va = 0$ for all $a \in U = < D(d) >$. So, $d(va) = d(0) = 0$ or $d(v)a + vd(a) = 0$. $d(v)a = 0$ because $d = 0$ on U. So, $d(v) \in Ann(U) = V$. Hence $d(V) \subseteq V$.

(c) Since V is an ideal of X then from (b), $d(V) \subseteq V$. So $d_1 = d|_V$(is a restriction of d on V) is a derivation on V. Let $c \in V$ be a dependent element of d_1, so by Theorem 2.1, $[c, v] = 0$. Replacing v by vr in $[c, v] = 0$, we have $[c, vr] = 0$ or $[c, v]r + v[c, r] = 0$, which implies $v[c, r] = 0$. Replace v by vx, $x \in X$, we get $vx[c, r] = 0$. Put $v = [c, r]$, and then using semiprimeness of X, we have $[c, r] = 0$ for $r \in X$. Also from Theorem 2.1 $d_1(v)c = 0$ for all $v \in V$. Replacing v by xv in the last equation, we have $d_1(xv)c = 0$. Here d_1 is a restriction of d over V and $xv \in V$. So, we have $d(xv)c = 0$ or $d(x)vc + xd(v)c = 0$, which implies

$d(x)vc = 0$. Since V is an ideal of X, so $cvd(x) \in V$ for all $x \in X$. Replacing v by $cvd(x)$ in $d(x)vc = 0$, we get $d(x)cvd(x)c = 0$. By Observation 2.5(i) V is semiprime and by using semiprimeness of V, we get $d(x)c = 0$. Since $[c, x]$ and $d(x)c = 0$, for all $x \in X$, therefore by Theorem (2.1) $c \in D(d) \subseteq U$. So, $c \in U$ and $c \in V = Ann(U)$. Thus by (a) $c = 0$. Hence $D(d|_V) = 0$. That is, d acts freely on V.

By Theorem 2.4 and Theorem 2.8, we have the following corollary.

Corollary 2.9. *Let d be a commutative derivation on a semiprime MA-Semiring X. Then there exist ideals $U = D(d)$ and $V = Ann(D(d))$ of X such that*

(a) $U \cap V = 0$,

(b) $d = 0$ *on* U, $d(V) \subseteq V$,

(c) $D(d|_V) = 0$, *where* $d|_V$ *is a restriction of* d *on* V. *That is,* d *acts freely on* V.

Acknowledgement

The authors are cordially indebted to the referee for valuable suggestions.

References

[1] F. Ali and M.A. Chaudhry, *Dependent elements of derivations on semiprime rings*, Int. J. Math. and Math. Sci. (2009).
https://doi.org/10.1155/2009/696737

[2] H.J. Bandelt and M. Petrich, *Subdirect products of rings and distributive lattices*, Proceedings of the Edinburgh Mathematical Society **25(2)** (1982) 155–171.
https://doi.org/10.1017/s0013091500016643

[3] M.A. Chaudhry and M.S. Samman, *Free actions on semiprime rings*, Math. Bohemica **133(2)** (2008) 197–208.
https://doi.org/10.21136/mb.2008.134055

[4] J. Golan, The Theory of Semirings with Application in Math. and Theoretical Computer Science **54** (Monographs and Surveys in Pure and Applied Mathematics, 1992).

[5] M.A. Javed and M. Aslam, *Some commutativity conditions in prime MA-semirings*, Ars. Combin. **114** (2014) 373–384.

[6] M.A. Javed, M. Aslam and M. Hussain, *On condition (A2) of Bandlet and Petrich for inverse semirings*, In Int. Math. Forum **7(59)** (2012) 2903–2914.

[7] R.R. Kallman, *A generalization of free action*, Duke Math. J. **36(4)** (1969) 781–789.
http://gdmltest.u-ga.fr/item/1077378641/

[8] P.H. Karvellas, *Inversive semirings*, J. Australian Math. Soc. **18(3)** (1974) 277–288.
https://doi.org/10.1215/s0012-7094-69-03693-x

[9] F.J. Murray and J. von Neumann, *On rings of operators* II, Trans. Amer. Math. Soc. **41(2)** (1937) 208–248.

[10] R. Murugesan, K.K. Sindhu and P. Namasivayam, *Free actions of semi derivations on semiprime semirings*, Int. J. Sci. and Eng. Inv. **2(04)** (2016).

[11] S. Sara, M. Aslam and M.A. Javed, *On dependent elements and free actions in inverse semirings*, In Int. Math. Forum **12** (2016) 557–564.
https://doi.org/10.12988/imf.2016.6441

[12] S. Sara, M. Aslam and M. Javed, *On centralizer of semiprime inverse semiring*, Discuss. Math. Gen. Alg. and Appl. **36(1)** (2016) 71–84.
https://doi.org/10.7151/dmgaa.1252

[13] S. Sara and M. Aslam, *On Posners Second theorem in additively inverse semiring*, Hacettepe J. Math. and Stat. **48(4)** (2019) 996–1000.
https://doi.org/10.15672/hjms.2018.576

[14] S. Sara and M. Aslam, *On Lie ideals of inverse semirings*, Italian J. Pure Appl. Math. **44** (2020) 22–29.
https://doi.org/10.1142/s1793557121501813

[15] Laradji and A. Thaheem, *On dependent elements in semiprime rings*, Math. Japonica **47(1)** (1998) 29–31.

[16] J. Vukman and I. Kosi-Ulbl, *On dependent elements in rings*, Int. J. Math. and Math. Sci. **54** (2004) 2895–2906.
https://doi.org/10.1155/s0161171204311221

Received 24 June 2022
Revised 25 October 2022
Accepted 25 October 2022