Discussiones Mathematicae
General Algebra and Applications 44 (2024) 277–285
https://doi.org/10.7151/dmgaa.1443

ON THE NUMBER OF GROUP HOMOMORPHISMS BETWEEN CERTAIN GROUPS

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Abstract

Let H be a finite abelian group and $Dih(H) = \langle H, b \mid b^2 = 1 \ \&\ bhb^{-1} = h^{-1};\ \forall h \in H\rangle$ be the generalized dihedral group of H. The aim of this paper is to compute the number of group homomorphisms between two generalized dihedral groups and a generalized dihedral group and an abelian group. One of these results generalized an earlier work by J.W. Johnson published in 2013.

Keywords: group homomorphism, generalized dihedral group, abelian group.

2020 Mathematics Subject Classification: Primary: 20D99; Secondary: 20F99.

1. Introduction

Throughout this paper we will make this assumption that all groups are assumed to be finite. If G and H are such groups then we interest the number of homomorphisms from G into H, denoted by $\gamma(H, G) = |Hom(H, G)|$. In the case that $H = G$, we will use the notation $\gamma(G)$ as $\gamma(G, G)$. The problem of computing the number of homomorphisms between two groups is so difficult in general, and so some mathematicians presented methods to compute $\gamma(H, G)$ for certain groups.

With the best of our knowledge, the first published paper in which the number of homomorphisms between two finite groups is considered into account was the

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joint paper of Gallian and Van Buskirk [3]. In the mentioned paper, the authors obtained closed formulas for the number of group homomorphisms, and also ring homomorphisms, from $\mathbb{Z}_n$ into $\mathbb{Z}_m$. If (m, n) denotes the greatest common divisor of two positive integers m and n, then they proved that:

Theorem 1. $\gamma(\mathbb{Z}_m, \mathbb{Z}_n) = (m, n)$.

Johnson [4], found the number of group homomorphisms from the dihedral group D_{2m} into the dihedral group D_{2n}. He proved that

Theorem 2.

$$\gamma(D_{2m}, D_{2n}) = \begin{cases} n(m,n)+1 & 2 \nmid mn \\ n(m,n)+2 & 2 \nmid m \ \& \ 2 \mid n \\ n(m,n)+4n+4 & 2 \mid m \ \& \ 2 \mid n \\ n(m,n)+2n+1 & 2 \mid m \ \& \ 2 \nmid n. \end{cases}$$

The most important works on the problem of counting group homomorphisms were given by Takegahara and his co-authors. Chigira and Takegahara [2], studied the number of homomorphisms from a finite group to a general linear group over a finite field, and the authors of [5, 9, 10] investigated the number of homomorphisms from a finite abelian group to a symmetric or alternating groups. Liebeck and Shalev [6] have been estimated the number of homomorphisms from a finite group A to the general linear group $GL(n, q)$, where q is a prime power coprime to $|A|$.

Bate [1] provided upper and lower bounds for the number of completely reducible homomorphisms from a finite group to general linear and unitary groups over arbitrary finite fields, and to orthogonal and symplectic groups over finite fields of odd characteristic. Matei and Suciu [7] presented a method for computing the number of epimorphisms from a finitely presented group to a finite solvable group, which generalizes a formula of Gaschütz.

An elementary abelian group of order p^n, p is prime, is denoted by $E(p^n)$. Suppose G is an abelian group with decomposition $G \cong \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_d}$ in which $n_{i+1} \,|\, n_i$, $1 \le i \le d-1$, and for all j, $n_j \ge 2$. Then we define $\mathcal{S}(G) = \{n_1, \ldots, n_d\}$. Note that this decomposition is unique for each abelian group and so our definition for $S(G)$ is well-defined. The number of factors of even orders in this decomposition of H into cyclic groups is denoted by $\varepsilon(H)$. If n is a positive integer, then $\phi(n)$ denotes the Euler totient function evaluated at n.

Throughout this paper our notations are standard and we refer to the famous book of Robinson [8] for concepts and notations not presented here. Our results are checked by the computer algebra package Gap [11].

2. Preliminaries

Suppose H is an abelian group. The generalized dihedral group $Dih(H)$ can be presented by $Dih(H) = \langle H, b_H \mid b_H^2 = 1 \ \& \ b_H h b_H^{-1} = h^{-1};\ \forall h \in H\rangle$. It is well-known that this group is the semidirect product of H by the cyclic group of order 2. In an exact phrase, $Dih(H) = H \rtimes_\alpha Z_2$ in which $\alpha(0)$ is the identity element of $Aut(H)$ and $\alpha(1) = f$ in which $f(x) = x^{-1}$, for arbitrary element $x \in H$. Note that for each subgroup M of H, $\overline{M} = \{(m, 0) \mid m \in M\}$ is a subgroup of $Dih(H)$ isomorphic to M. On the other hand, the set of all elements in the form of $(h, 0), h \in H$, constitutes a subgroup of index 2 in $Dih(H)$ isomorphic to H.

The first main result of this paper is as follows.

Theorem 3. *Let H and G be two finite abelian groups. The number of homomorphisms from $Dih(H)$ into $Dih(G)$ can be computed by the following formula:*

$$\begin{aligned}\gamma(Dih(H), Dih(G)) \;&=\; |G|\gamma(H,G) + \frac{|G|}{2^{\varepsilon(G)}}\left(2^{(1+\varepsilon(H))(1+\varepsilon(G))} - 2^{(1+\varepsilon(H))\varepsilon(G)}\right)\\ &\quad - \;|G|2^{\varepsilon(H)\varepsilon(G)} + 2^{(1+\varepsilon(H))\varepsilon(G)}.\end{aligned}$$

In particular, $\gamma(Dih(H)) = |H||End(H)| + \frac{|H|}{2^{\varepsilon H}}\left(2^{(\varepsilon(H))^2} - 2^{(\varepsilon(H)+1)\varepsilon(H)}\right) - |H|2^{\varepsilon(H)^2} + 2^{(1+\varepsilon(H))\varepsilon(H)}$.

We now apply this theorem to present a simple proof for Theorem 2 which is the main result of [4].

***New Proof for Theorem* 2.** Since for each natural number r, $Dih(\mathbb{Z}_r)$ is a dihedral group of order $2r$, it is enough to apply Theorem 3. By Theorem 1, $\gamma(\mathbb{Z}_m, \mathbb{Z}_n) = (m, n)$ and it is easy to see that for each cyclic group A, $\varepsilon(A) \in \{0, 1\}$ and $\varepsilon(A) = 0$ if and only if A has odd order. Therefore,

$$\begin{aligned}\gamma(D_{2m}, D_{2n}) \;&=\; \begin{cases} n(m,n) + (2-1)n - n + 1 & 2 \nmid mn\\ n(m,n) + (2^2 - 2^1)\frac{n}{2} - n + 2^1 & 2 \nmid m \ \& \ 2 \mid n\\ n(m,n) + (2^4 - 2^2)\frac{n}{2} - 2n + 2^2 & 2 \mid m \ \& \ 2 \mid n\\ n(m,n) + (2^2 - 1)n - n + 1 & 2 \mid m \ \& \ 2 \nmid n\end{cases}\\ &=\; \begin{cases} n(m,n) + 1 & 2 \nmid mn\\ n(m,n) + 2 & 2 \nmid m \ \& \ 2 \mid n\\ n(m,n) + 4n + 4 & 2 \mid m \ \& \ 2 \mid n\\ n(m,n) + 2n + 1 & 2 \mid m \ \& \ 2 \nmid n.\end{cases}\end{aligned}$$

We are now ready to state our second main result which can be proved in a similar way as the proof of Theorem 3.

Theorem 4. *Let H and G be finite abelian groups. Then the following hold.*

1. $\gamma(Dih(H), G) = 2^{(\varepsilon(H)+1)\varepsilon(G)}$;
2. $\gamma(H, Dih(G)) = \gamma(H, G) + \frac{|G|}{2^{\varepsilon(G)}}(2^{\varepsilon(H)(1+\varepsilon(G))} + 2^{\varepsilon(H)\varepsilon(G)})$.

3. MAIN RESULTS

Suppose S is a minimal generating set for H, then $Dih(H) = \langle S, b_H\rangle$. If G is an abelian group of even order, then we use the notation $E(G)$ to denote the set of all involutions together with the identity element of G. It is easy to see that $E(G)$ is the largest elementary abelian 2-subgroup of G.

Suppose G, H and K are three finite groups. It is well-known that $\gamma(G, H \times K) = \gamma(G, H)\gamma(G, K)$, see [7, p. 168]. Also, if A and B are abelian groups then it is well-known that $\gamma(A, B) = \gamma(B, A)$. The following lemma is an immediate consequence of these known results.

Lemma 5. *Let $G_1, \ldots, G_n$ and $H_1, \ldots, H_m$ be abelian groups. Then*

$$\gamma(G_1 \times \cdots \times G_n, H_1 \times \cdots \times H_m) = \prod_{i=1}^{n}\prod_{j=1}^{m}\gamma(G_i, H_j).$$

Corollary 6. *Let G and H be finite abelian groups. Then*

$$\gamma(H, G) = \prod_{i\in S(G)}\prod_{j\in S(H)}(i, j).$$

Suppose U_n denotes the unit group of the ring $\mathbb{Z}_n$ of integers modulo n. If $n = 2^{\alpha}{p_1}^{n_1}\cdots{p_r}^{n_r}$, where p_i's are different odd primes, then by the Chinese remainder theorem $U_n \cong U_{2^{\alpha}} \times U_{p_1^{n_1}} \times \cdots \times U_{p_r^{n_r}}$. Moreover, U_2 is trivial group, $U_4 \cong \mathbb{Z}_2$, $U_{2^n} \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}}$, $n > 2$ is an integer, and for each odd prime p and positive integer m, $U_{p^m} \cong \mathbb{Z}_{p^m - p^{m-1}}$.

Corollary 7. *Let $n > 2$ and $m > 2$ be two positive integers with prime factorizations $n = 2^{\alpha}{p_1}^{n_1}\cdots{p_r}^{n_r}$ and $m = 2^{\beta}{q_1}^{l_1}\cdots{q_s}^{l_s}$, where p_i, $1 \le i \le r$, as well as q_j, $1 \le j \le s$ are different odd primes. Moreover, α, β, r, s, n_i, $1 \le i \le r$, and m_j, $1 \le j \le s$ are non-negative integers. Without loss of generality we can assume that $\beta \le \alpha$. Then the following hold:*

1. *if $\alpha, \beta \in \{0, 1\}$, then $\gamma(U_n, U_m) = \prod_{i=1}^{r}\prod_{j=1}^{s}(\phi(p_i), \phi(q_j))$;*
2. *if $\alpha = 2$ and $\beta \in \{0, 1\}$, then $\gamma(U_n, U_m) = 2^s\prod_{i=1}^{r}\prod_{j=1}^{s}(\phi(p_i), \phi(q_j))$;*
3. *if $\alpha = \beta = 2$ then $\gamma(U_n, U_m) = 2^{r+s+1}\prod_{i=1}^{r}\prod_{j=1}^{s}(\phi(p_i), \phi(q_j))$;*

4. *if* $\alpha > 2$ *and* $\beta \in \{0, 1\}$*, then*

$$\gamma(U_n, U_m) = 2^s \prod_{i=1}^{r} \prod_{j=1}^{s} (\phi(p_i), \phi(q_j)) \prod_{j=1}^{s} (2^{\alpha-2}, \phi(q_j)).$$

In particular, if $\alpha = 3$*, then* $\gamma(U_n, U_m) = 2^{2s} \prod_{i=1}^{r} \prod_{j=1}^{s} (\phi(p_i), \phi(q_j))$*;*

5. *if* $\alpha > 2$ *and* $\beta = 2$*, then*

$$\gamma(U_n, U_m) = 2^{r+s+2} \prod_{i=1}^{r} \prod_{j=1}^{s} (\phi(p_i), \phi(q_j)) \prod_{j=1}^{s} (2^{\alpha-2}, \phi(q_j)).$$

In particular, if $\alpha = 3$*, then* $\gamma(U_n, U_m) = 2^{r+2s+2} \prod_{i=1}^{r} \prod_{j=1}^{s} (\phi(p_i), \phi(q_j))$*;*

6. *if* $\alpha > 2$ *and* $\beta > 2$*, then* $\gamma(U_n, U_m) = 2^{r+s+\beta+1} \prod_{i=1}^{r} \prod_{j=1}^{s} (\phi(p_i), \phi(q_j)) \prod_{j=1}^{s} (2^{\alpha-2}, \phi(q_j)) \times \prod_{i=1}^{r} (2^{\beta-2}, \phi(p_i))$*. In the special case that* $\alpha = \beta = 3$ *we will have* $\gamma(U_n, U_m) = 2^{2r+2s+4} \prod_{i=1}^{r} \prod_{j=1}^{s} (\phi(p_i), \phi(q_j))$ *and if* $\alpha > 3$ *and* $\beta = 3$ *then* $\gamma(U_n, U_m) = 2^{2r+s+4} \prod_{i=1}^{r} \prod_{j=1}^{s} (\phi(p_i), \phi(q_j)) \prod_{j=1}^{s} (2^{\alpha-2}, \phi(q_i))$*.*

Corollary 8. *Let* G *and* H *be abelian groups. Then*

$$\gamma(G, H) = \prod_{i \in S(G)} \prod_{i \in S(H)} (i, j).$$

In particular, $\gamma(H, \mathbb{Z}_2) = 2^{\varepsilon(H)}$*.*

Proof. By our definition $S(\mathbb{Z}_2) = \{2\}$ and so $\gamma(H, \mathbb{Z}_2) = \prod_{i \in S(H)} (i, 2) = 2^{\varepsilon(H)}$, proving the result. ∎

Suppose G is a finite group. It is clear that there is a one to one correspondence between the set of all subgroups of index 2 in G and non-zero homomorphisms from G into the cyclic group $\mathbb{Z}_2$. This proves that there is exactly $\gamma(G, \mathbb{Z}_2) - 1$ subgroups of index 2 in G. We now apply this simple result to prove the following lemma.

Lemma 9. *Let* H *be an abelian group. Then* $\gamma(Dih(H), \mathbb{Z}_2) = 2^{\varepsilon(H)+1}$*.*

Proof. To prove the lemma, it is enough to count the number of subgroup of index 2 in $Dih(H)$. By definition of the generalized dihedral group, H is a subgroup of index 2 in $Dih(H)$. Choose a subgroup H' of index 2 in H, $x \in Dih(H) \setminus H$ and $y \in Dih(H) \setminus (Dih(H') \cup H)$. It can be easily seen that H, $\langle H', x \rangle$ and $\langle H', y \rangle$ are the only proper subgroups of $Dih(H)$ containing H' and the last two subgroups are isomorphic to $Dih(H')$. Therefore, $\gamma(Dih(H), \mathbb{Z}_2) = 2|\{K \leq H \mid |H : K| = 2\}| + 2 = 2(\gamma(H, \mathbb{Z}_2) - 1) + 2 = 2\gamma(H, \mathbb{Z}_2)$. By Corollary 8, $\gamma(H, \mathbb{Z}_2) = 2^{\varepsilon(H)}$ and so $\gamma(Dih(H), \mathbb{Z}_2) = 2^{\varepsilon(H)+1}$, proving the lemma. ∎

Corollary 10. $\gamma(Dih(H), \mathbb{Z}_2^n) = 2^{n(\varepsilon(H)+1)}$.

Proof. By Lemma 9, $\gamma(Dih(H), \mathbb{Z}_2) = 2^{\varepsilon(H)+1}$ and by Lemma 5, $\gamma(Dih(H), \mathbb{Z}_2^n) = 2^{n(\varepsilon(H)+1)}$. ■

We are now ready to present the proof of our first main result.

***Proof of Theorem* 3.** To calculate the number of homomorphisms $h : Dih(H) \longrightarrow Dih(G)$, we consider four cases that in which the order of G or H are odd or even.

(i) *Both of G and H have odd orders.* Note that if we have the image of h under b_H and each element of S, then the homomorphism h will be completely determined. It is clear that all elements $b_G g \in Dih(G)$ are involutions. Since G has odd order, these are all elements of even order in $Dih(G)$ which shows that $h(b_H) = e_G$ or $h(b_H) = b_G g$, for some $g \in G$. If $h(b_H) = e_G$, then h is the zero homomorphism, and so we can assume that there exists $g \in G$ such that $h(b_H) = b_G g$. Furthermore, $h(H) \subseteq G$ and so h induces a homomorphism from H into G. On the other hand, we assume that $h_1 : H \longrightarrow G$ is a group homomorphism. We extend h_1 to the homomorphism $\overline{h_1} : Dih(H) \longrightarrow Dih(G)$ by $\overline{h_1}(b_H x) = b_G y h_1(x)$, where $y \in H$ is arbitrary. Therefore, we will have $|G|$ different choices for defining $h(b_H)$ and $\gamma(H, G)$ different choices for the group homomorphism h_1. This proves that there are $|G|\gamma(H, G) + 1$ homomorphisms from $Dih(H)$ into $Dih(G)$.

(ii) *$|G|$ is even and $|H|$ is odd.* Since $|H|$ is odd, all elements $h(s)$, $s \in S$, has odd orders. It is clear that $h(b_H) \in E(G) \cup (Dih(G) \setminus G)$. If $h(b_H) \in Dih(G) \setminus G$, then a similar argument as (i) shows that we have exactly $|G|\gamma(H, G)$ homomorphisms. We now assume that $h(b_H) \in E(G)$. Suppose there exists $s \in S$ such that $h(s) \notin E(G)$. Since $h(b_H s) = h(b_H)h(s) \in G$ and $O(b_H s) = 2$, $b_H s \in E(G)$ which leads to a contradiction. Therefore, elements of S map to elements of $E(G)$. Note that $O(h(s))|O(s)$ and $O(s)$ is odd which shows that $O(s) \neq 2$. This proves that $h(S) = \{e_G\}$. This creates $|E(G)|$ new homomorphisms and so $\gamma(Dih(H), Dih(G)) = |G|\gamma(H, G) + |2^{\varepsilon(G)}|$.

(iii) *$|G|$ is odd and $|H|$ is even.* In this case, G does not have an element of even order. Since $O(h(b_H)) = 1, 2$, b_H can be mapped to e_G or an element in the form of $b_G g$, $g \in G$. Suppose there are two elements $s_1, s_2 \in S$ such that $h(s_1) = b_G g_1$ and $h(s_2) = g_2$, where $g_1, g_2 \in G$ and $g_2 \neq e_G$. Since H is abelian, $b_G g_1 g_2 = h(s_1)h(s_2) = h(s_2)h(s_1) = g_2 b_G g_1 = b_G g_2^{-1} g_1$ which implies that $g_2^2 = e_G$. But G has odd order and so $g_2 = e_G$. This proves that if for an element $s_1 \in S$, $h(s_1) = b_G g_1$ then the image of all elements of S under the homomorphism h will be identity element of G or an element in the form of $b_G g$, where $g \in G$. Hence, we have one of the following cases.

1. $h(S) \subseteq G$. In this subcase, if $h(b_H) = e_G$, then h will be the zero homomorphism. Moreover, all mappings for which every element of S mapped to an element of G and b_H mapped to an element in the form of $b_G g$, $g \in G$, can be extended to a unique homomorphism from $Dih(H)$ into $Dih(G)$ and similar to (i) there are $|G|\gamma(H, G)$ of such homomorphisms.

2. *There exists $s_1 \in S$ such that $h(s_1) = b_G g_1$, for some $g_1 \in G$.* For each $s \in S$, $O(h(s)) = 1, 2$ and also $O(h(b_H)) = 1, 2$. This shows that $h(H)$ is an elementary abelian 2-subgroup of $Dih(G)$ and since $4 \nmid |Dih(G)|$, $h(H)$ is a subgroup of order 2 in $Dih(G)$. There are $\gamma(Dih(H), \mathbb{Z}_2) - 1$ non-trivial homomorphisms from $Dih(H)$ into $\mathbb{Z}_2$ and since we have $|G|$ involutions in $Dih(G)$, we will have $|G|[\gamma(Dih(H), \mathbb{Z}_2) - 1]$ homomorphisms. But there are $|G|$ homomorphisms for which $h(S) = \{e_G\}$ and b_H mapped to an element in the form of $b_G g$, $g \in G$. Therefore, the total number of homomorphisms from $Dih(H)$ into $Dih(G)$ is $|G|\gamma(H, G) + |G|[\gamma(Dih(H), \mathbb{Z}_2) - 1] - |G| + 1$.

We now apply Lemma 9 to complete the proof of (iii).

(iv) *Both of G and H have even orders.* Since $|H|$ and $|G|$ are both even and $O(h(b_H))|2$, there exists $g \in G$ such that $b_H = b_G g$ or $h(b_H) \in E(G)$. Our proof will consider two cases that $h(S) \subseteq G$ or there exists $s_1 \in S$ such that $h(s_1) = b_G g_1$.

1. $h(S) \subseteq G$. We first assume that $h(b_H) = b_G g$. By an argument similar to Part (i) of the proof of Theorem 3, we will have $|G|\gamma(H, G)$ homomorphisms. Suppose that $h(b_H) \in E(G)$. Similar to Part (ii) of the proof of Theorem 3, we assume that there exists $s \in S$ such that $x = h(s) \in G \setminus E(G)$ and so $O(h(s)) = O(x) \neq 1, 2$. Since $h(b_H) \in E(G)$ and $x = h(s) \in G$, $h(b_H s) = h(b_H)h(s) \in G$, and since $O(h(b_H s))|2$, $h(b_H s) \in E(G)$. On the other hand, $h(b_H s) = h(b_H)h(s) = h(b_H)x$ and $x \notin E(G)$ which is impossible. This contradiction shows that $h(S) \subseteq E(G)$ which show that $h(Dih(H)) \subseteq E(G)$. Therefore, we have to counted the number of homomorphisms from $Dih(H)$ into $E(G)$.

2. *There exists $s_1 \in S$ such that $h(s_1) = b_G g_1$.* Similar to what we have done in (iii), we assume that for another element $s_2 \in S$, $h(s_2) = g_2$ in which $g_2 \in G$. Since H is abelian, $g_2^2 = e_G$. This proves that the image of each element of S has the form of $b_G g$ or is an element of $E(G)$. In each case, it can be easily seen that $O(h(s))|2$, $s \in S$. Also, $O(h(b_H))|2$ and hence $h(Dih(H))$ is a trivial subgroup or an elementary abelian 2-group. Thus, $h(Dih(H))$ is isomorphic to a subgroup of $Dih(E(G)) \cong \mathbb{Z}_2 \times E(G)$ and we have $\frac{|G|}{|E(G)|}$ subgroups isomorphic to $Dih(E(G))$. In the last case, we have to reduce this case by the number of homomorphisms with this condition that $h(S) \subseteq G$. Therefore,

$$\gamma(Dih(H), Dih(G)) = |G|\gamma(H, G) + \frac{|G|}{|E(G)|}(\gamma(Dih(H), E(G) \times Z_2)$$

$$- \gamma(Dih(H), E(G))) - |G|\gamma(H, E(G)) + \gamma(Dih(H), E(G)).$$

We now apply Lemmas 5, 9 and Corollary 10 to get the result.

This completes the proof of (iv).

***Proof of Theorem* 4.** Suppose G and H are abelian groups. Our proof will consider two separate cases as follows.

1. A similar argument as Part (iv)(b) of the proof of Theorem 3 shows that $h(S) \subseteq E(G)$ and so $\gamma(Dih(H), G) = \gamma(Dih(H), E(G))$. Now by Corollary 10, $\gamma(Dih(H), G) = 2^{(\varepsilon(H)+1)\varepsilon(G)}$, as desired.

2. If $h(S) \subseteq E(G)$, then there are $\gamma(H, G)$ homomorphisms. Thus, we can assume that there exists $s_1 \in S$ such that $h(s_1) = b_G g_1$, for some $g_1 \in G$. Now a similar argument as Part (ii) of the proof of Theorem 3 shows that $h(s) \in (Dih(G) \setminus G) \cup E(G)$. Hence the image of $Dih(H)$ is isomorphic to a subgroup of $Dih(E(G)) \cong E(G) \times \mathbb{Z}_2$ and we have $\frac{|G|}{|E(G)|}$ such subgroups. Since $h(s_1) = b_G g_1$, $Dih(H) \nsubseteq G$. Therefore, $\gamma(H, Dih(G)) = \gamma(H, G) + \frac{|G|}{|E(G)|}(2^{(\varepsilon(H)+1)\varepsilon(G)} - 2^{\varepsilon(H)\varepsilon(G)})$.

4. Concluding remarks

In this paper, the number of homomorphisms between two generalized dihedral groups were calculated. This gives a generalization of a result by Johnson [4]. We also compute the number of homomorphisms between an abelian group and a generalized dihedral groups, and the number of homomorphisms between the unite rings of integers modulo n and m, respectively. The next step in this program is to calculate the number of homomorphisms between two generalized dicyclic groups, an abelian group and a generalized dicyclic group, and a generalized dihedral group and a generalized dicyclic group. We checked all results of this paper by gap programs. These programs are accessible from the authors upon request.

Acknowledgement

The authors are grateful to the referee for careful reading of the paper and valuable suggestions and comments.

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Received 17 November 2022
Revised 20 February 2023
Accepted 21 February 2023