Discussiones Mathematicae
General Algebra and Applications 44 (2024) 111–126
https://doi.org/10.7151/dmgaa.1440

A BISIMPLE INVERSE MONOID OF QUADRUPLES OF NON-NEGATIVE INTEGERS. THE MÖBIUS FUNCTION

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Abstract

The additive monoid of non-negative integers $\mathbb{N}$ is isomorphic to the right unit submonoid of the (bisimple) bicyclic semigroup $B = \mathbb{N} \times \mathbb{N}$. The aim of this note is to construct a similar pair of monoids ($B^{\dagger} = \mathbb{N} \times \mathbb{N}, B^{\ddagger} = \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$). The monoid $B^{\dagger}$ gives rise to a bisimple inverse monoid $B^{\ddagger}$ of quadruples of non-negative integers like as Warne's 2-dimensional bicyclic semigroup. The links with the monoid of non-negative integers $\mathbb{N}$ and with the bicyclic semigroup may turn out to be expedient also for the computation of the corresponding Möbius functions.

Keywords: bicyclic semigroup, bisimple inverse monoid, Möbius function.

2020 Mathematics Subject Classification: Primary: 20M18; Secondary: 06A11, 18A05, 05A99.

1. Introduction

A well-known and thus much cited combinatorial bisimple inverse monoid is the bicyclic semigroup B which is a monoid of pairs of non-negative integers, $B = \mathbb{N} \times \mathbb{N}$, equipped with the multiplication defined by

$$(k,m) \cdot (r,s) = \begin{cases} (k, m-r+s) & \text{if } \ m \geq r \\ (k-m+r, s) & \text{if } \ m < r. \end{cases}$$

As a monoid of transformations, the bicyclic semigroup B is generated by the following transformations $\alpha, \beta, \iota : \mathbb{N} \to \mathbb{N}$ defined by

$$(n)\alpha = \begin{cases} 0 & \text{if } \ n = 0 \\ n-1 & \text{if } \ n > 0 \end{cases} \qquad (n)\beta = n+1 \quad \text{and} \quad (n)\iota = n.$$

The bicyclic semigroup B is a monoid admitting the following presentation:

$$B = \langle a, b \mid ba = 1\rangle.$$

The elements are words of the form $a^k b^m$ for $k, m \in \mathbb{N}$ (with the understanding $a^0 = b^0 = 1$). The multiplication is given by

$$a^k b^m a^r b^s = a^{k-m+\max(m,r)} b^{\max(m,r)-r+s}.$$

The element $1 = (0, 0) \in \mathbb{N}\times\mathbb{N}$ is the identity of B and the submonoid of right units in B (the $\mathcal{R}$-class of B containing the identity) is isomorphic to the monoid of the non-negative integers $\mathbb{N}$ with the usual addition. It is well-know that bisimple inverse monoids are described in terms of their right unit submonoid (Clifford [1] see also [6, Chapter 10, Section 1]). The right unit submonoid of a bisimple inverse monoid is right cancellative monoid satisfying the Clifford condition. A monoid S is said to satisfy the Clifford condition if for all $x, y \in S$ there exists $z \in S$ such that $Sx \cap Sy = Sz$. Having a right cancellative monoid satisfying the Clifford condition there is a well-known process of constructing a bisimple inverse monoid from a such monoid. The pair ($\mathbb{N}$, $B = \mathbb{N} \times \mathbb{N}$) describe the standard example to illustrate Clifford's theory of bisimple inverse monoids.

In this note we consider an example of "Clifford's pair" ($B^\dagger = \mathbb{N} \times \mathbb{N}, B^\ddagger = \mathbb{N}\times\mathbb{N}\times\mathbb{N}\times\mathbb{N}$) which is close to the "bicyclic pair" ($\mathbb{N}, B$). Already the construction of $B^\dagger$ ensures this goal and Proposition 3.1 expresses the internal connections between them. Although the monoid $B^\dagger$ is described as the bicyclic semigroup B, this monoid is not inverse as the bicyclic semigroup, but it is right cancellative satisfying the Clifford condition, it is atomic and half factorial, it is Möbius (in the sense of Leroux [5, 7]), locally right Garside (in the sense of Dehornoy [2]) and ℓ-RILL monoid (in the sense of Schwab [8]), properties which are also satisfied by the additive monoid of non-negative integers. This monoid has a special place in the class of monoids considered in [3, Section 4]. Now, the monoid $B^\ddagger$ is combinatorial, bisimple, and inverse as the bicyclic semigroup; the monoid $B^\dagger$ being isomorphic to the right units of $B^\ddagger$ (exactly as in the case of $\mathbb{N}$ and B). The links with the bicyclic semigroup may be found throughout the paper. The Möbius function of the locally finite partially ordered set $(B^\ddagger, \leq)$ (where $\leq$ is the natural partial order of the inverse monoid $B^\ddagger$) is given in Section 4.

The computations are simple, and this note assumes only elementary knowledge of semigroup theory. A monoid T is called an inverse monoid if for each $t \in T$ there exists a unique inverse (denoted by t^{-1}) such that $tt^{-1}t = t$ and $t^{-1}tt^{-1} = t^{-1}$. Note that an inverse monoid T is combinatorial if and only if its group of units is trivial, and it is bisimple if and only if for each pair of elements $s, t \in S$ there exists an element $x \in S$ such that $ss^{-1} = xx^{-1}$ and $x^{-1}x = t^{-1}t$. We refer the reader to the book of Petrich [6] for results and terminologies in inverse semigroup theory.

2. The right cancellative monoid $B^\dagger$

Define the mappings $\alpha, \beta, \iota : \mathbb{N} \to \mathbb{N}$ as follows

$$(n)\alpha = \begin{cases} 0 & \text{if } n = 0 \\ n+1 & \text{if } n > 0 \end{cases} \qquad (n)\beta = n+1 \quad \text{and} \quad (n)\iota = n.$$

Let $B^\dagger$ be the monoid generated by these transformations. We observe that

$$(\forall n \in \mathbb{N}): \quad (n)\beta\alpha = (n+1)\alpha = n+2 = (n)\beta\beta$$

that is,

$$\beta\alpha = \beta^2.$$

If k and m are positive integers then

$$(n)\alpha^k = \begin{cases} 0 & \text{if } n = 0 \\ n+k & \text{if } n > 0 \end{cases} \qquad \text{and} \quad (n)\beta^m = n+m.$$

More generally,

$$(n)\alpha^k\beta^m = \begin{cases} m & \text{if } n = 0 \\ n+k+m & \text{if } n > 0. \end{cases}$$

Let γ be an element of $B^\dagger$, $\gamma = \gamma_1\gamma_2...\gamma_n$, where $\gamma_i = \alpha$ or $\gamma_i = \beta$. If j $(\le n)$ is the smallest index such that $\gamma_j = \beta$ then $\gamma = \alpha^{j-1}\beta^{n-j+1}$ since $\beta\alpha = \beta^2$ (if such a positive integer j does not exist then $\gamma = \alpha^n$, and if $j = 1$ then $\gamma = \beta^n$). Hence

$$B^\dagger = \left\{\alpha^k\beta^m \mid k, m \in \mathbb{N}\right\}$$

(where $\alpha^0 = \beta^0 = \iota$). Note that $\alpha^k\beta^m = \alpha^r\beta^s$ if and only if $k = r$ and $m = s$. This means that

$$B^\dagger = \langle a, b \mid ba = b^2\rangle$$

is a monoid presentation of $B^\dagger$. The elements are words of the form a^kb^m for $k, m \in \mathbb{N}$ (with the understanding $a^0 = b^0 = 1$). The multiplication is defined as follows

$$a^kb^ma^rb^s = \begin{cases} a^{k+r}b^s & \text{if } m = 0 \\ a^kb^{m+r+s} & \text{if } m > 0. \end{cases}$$

So $B^\dagger$ is the monoid of pairs of non-negative integers,

$$B^\dagger = \mathbb{N} \times \mathbb{N},$$

with multiplication determined by the rule

$$(k,m)(r,s) = \begin{cases} (k+r, s) & \text{if } m = 0 \\ (k, m+r+s) & \text{if } m > 0. \end{cases}$$

Proposition 1. *The monoid $B^\dagger$ is right cancellative and satisfies the Clifford condition.*

Proof. Let $x = a^k b^m$, $x' = a^{k'} b^{m'}$ and $y = a^r b^s$.

Case 1. $m = 0$ and $m' \neq 0$ (similarly if $m' = 0$ and $m \neq 0$). Then $xy = x'y$ implies

$$a^{k+r} b^s = a^{k'} b^{m'+r+s} \quad \text{that is } k + r = k' \text{ and } s = m' + r + s,$$

which is impossible since $m' > 0$.

Case 2. $m = m' = 0$. Then $xy = x'y$ implies

$$a^{k+r} b^s = a^{k'+r} b^s \quad \text{that is } k = k',$$

and therefore $x = x'$

Case 3. $m > 0$ and $m' > 0$. Then $xy = x'y$ implies

$$a^k b^{m+r+s} = a^{k'} b^{m'+r+s} \quad \text{that is } k = k' \text{ and } m = m',$$

and therefore $x = x'$. This proves the first part of the assertion.

Now, it is straightforward to check that

$$B^\dagger a^k b^m = \begin{cases} \{a^u b^{v+m} \mid u, v \in \mathbb{N}\} & \text{if } \ k = 0, \\ X_{k,m} \cup Y_{k,m} & \text{if } \ k > 0, \end{cases}$$

where

$$X_{k,m} = \left\{a^{u+k} b^m \mid u \in \mathbb{N}\right\} \quad \text{and} \quad Y_{k,m} = \left\{a^u b^{v+k+m} \mid u \in \mathbb{N}, v \in \mathbb{N}^*\right\}.$$

Let $x = a^k b^m$, $y = a^r b^s$ and assume that

$$k + m \geq r + s.$$

Case 1. $m = s$. Then $k \geq r$. The equality $k = r$ implies $x = y$ and therefore $B^\dagger x = B^\dagger y$. Assume that $k > r$, and therefore $k > 0$. Then

$$B^\dagger x \cap B^\dagger y = [X_{k,m} \cup Y_{k,m}] \cap B^\dagger y.$$

(i) If $r = 0$, then

$$\begin{aligned} B^\dagger x \cap B^\dagger y &= [\{a^{u+k} b^m \mid u \in \mathbb{N}\} \cup \{a^u b^{v+k+m} \mid u \in \mathbb{N}, v \in \mathbb{N}^*\}] \\ &\quad \cap \{a^u b^{v+s} \mid u, v \in \mathbb{N}\} = B^\dagger x. \end{aligned}$$

(ii) If $r > 0$ then

$$B^\dagger x \cap B^\dagger y = [X_{k,m} \cup Y_{k,m}] \cap [X_{r,s} \cup Y_{r,s}] = B^\dagger x$$

since $X_{k,m} \subseteq X_{r,s}$ and $Y_{k,m} \subseteq Y_{r,s}$.

Case 2. $m > r + s$.

(i) If $k = r = 0$, then

$$B^\dagger x \cap B^\dagger y = \{a^u b^{v+m} \mid u, v \in \mathbb{N}\} \cap \{a^u b^{v+s} \mid u, v \in \mathbb{N}\} = B^\dagger x,$$

since $m > s$.

(ii) If $k = 0$ and $r > 0$, then

$$B^\dagger x \cap B^\dagger y = \{a^u b^{v+m} \mid u, v \in \mathbb{N}\} \cap [X_{r,s} \cup Y_{r,s}] = B^\dagger x,$$

since $B^\dagger x \subseteq Y_{r,s}$.

(iii) If $k > 0$ and $r = 0$, then

$$B^\dagger x \cap B^\dagger y = [X_{k,m} \cup Y_{k,m}] \cap \{a^u b^{v+s} \mid u, v \in \mathbb{N}\} = B^\dagger x,$$

since $X_{k,m}, Y_{k,m} \subseteq B^\dagger y$.

(iv) If $k > 0$ and $r > 0$, then

$$B^\dagger x \cap B^\dagger y = [X_{k,m} \cup Y_{k,m}] \cap [X_{r,s} \cup Y_{r,s}] = B^\dagger x,$$

since $X_{k,m}, Y_{k,m} \subseteq Y_{r,s}$.

Case 3. $m \leq r + s$ and $m \neq s$. Then $0 \leq r + s - m \leq k$.

(i) If $r = 0$ then $m < s$ and $k > 0$. It follows

$$B^\dagger x \cap B^\dagger y = [X_{k,m} \cup Y_{k,m}] \cap \{a^u b^{v+s} \mid u, v \in \mathbb{N}\} = Y_{k,m} = B^\dagger b^{k+m+1},$$

since $X_{k,m} \cap B^\dagger y = \emptyset$ and $Y_{k,m} \subseteq B^\dagger y$.

(ii) If $r > 0$ and $k = 0$, then $m = r + s$ and

$$B^\dagger x \cap B^\dagger y = \{a^u b^{v+m} \mid u, v \in \mathbb{N}\} \cap [X_{r,s} \cup Y_{r,s}] = Y_{r,s} = B^\dagger b^{r+s+1} = B^\dagger b^{k+m+1},$$

since $B^\dagger x \cap X_{r,s} = \emptyset$ and $Y_{r,s} \subset B^\dagger x$.

(iii) If $r > 0$ and $k > 0$, then

$$B^\dagger x \cap B^\dagger y = [X_{k,m} \cup Y_{k,m}] \cap [X_{r,s} \cup Y_{r,s}] = Y_{k,m} = B^\dagger b^{k+m+1},$$

since $X_{k,m} \cap B^\dagger y = \emptyset$, $B^\dagger x \cap X_{r,s} = \emptyset$ and $Y_{k,m} \subseteq Y_{r,s}$.

In conclusion,

$$B^\dagger a^k b^m \cap B^\dagger a^r b^s = \begin{cases} B^\dagger a^k b^m & \text{if } m > r+s \quad \text{or} \quad [m = s \text{ and } k \geq r] \\ B^\dagger a^r b^s & \text{if } s > k+m \quad \text{or} \quad [m = s \text{ and } k \leq r] \\ B^\dagger b^{k+m+1} & \text{if } 0 \leq r+s-m \leq k \quad \text{and} \quad m \neq s \\ B^\dagger b^{r+s+1} & \text{if } 0 \leq k+m-s \leq r \quad \text{and} \quad m \neq s. \end{cases}$$

Hence the right cancellative monoid $B^\dagger$ satisfies the Clifford condition. ■

Remarks. Since the identity $(0,0)$ is indecomposable in $B^\dagger$, the right cancellative law implies that the right divisibility is an ordering on $B^\dagger$. Clifford's condition involves that two elements with common left multiple admit a least common left multiple (LL-condition). Thus the monoid $B^\dagger$ is a RILL monoid (i.e., it is right cancellative (R), having the identity indecomposable (I), and satisfying the LL-condition). The monoid $B^\dagger$ is atomic with two atoms $a = (1,0)$, $b = (0,1)$; and it is half-factorial (i.e., two decomposition into atoms of a non-identity element (m,n) have the same length, namely $\ell(m,n) = m+n$). By [8, Proposition 2.1], $B^\dagger$ is a locally right Garside monoid in the sense of Dehornoy [2], and from [7, Proposition 3.1] it follows that $B^\dagger$ is a Möbius monoid (i.e., a Möbius category in the sense of Leroux [5] with a single object). A small category $\mathcal{C}$ is Möbius if 1) it is decomposition finite (i.e., for any morphism $\alpha \in Mor\mathcal{C}$ there is only a finite number of pairs $(\beta,\gamma) \in Mor\mathcal{C} \times Mor\mathcal{C}$ such that $\beta\gamma = \alpha$), 2) each identity morphism is indecomposable (i.e., $1 = \beta\gamma \Rightarrow \beta = \gamma = 1$), and 3) $\beta\gamma = \gamma$ always implies that β is an identity morphism (see [5]). Note that in [4] the Möbius monoid was defined as the $\mathcal{R}$-class containing the identity of a combinatorial bisimple inverse monoid.

3. The bisimple inverse monoid $B^\ddagger$

In this section we apply Clifford's [1] construction of bisimple inverse monoids from a right cancellative monoid satisfying the Clifford condition, namely from the monoid $B^\dagger$. Since 1 is indecomposable in $B^\dagger$, this bisimple inverse monoid $B^\ddagger$ is given by

$$B^\ddagger = B^\dagger \times B^\dagger$$

equipped with the operation $\diamond$ defined by

$$(x,y) \diamond (z,w) = (px, qw),$$

where $B^\dagger y \cap B^\dagger z = B^\dagger t$ and $py = qz = t$, for some $p,q,t \in B^\dagger$. More concretely, if $y = a^k b^m$ and $z = a^r b^s$ then the multiplication in $B^\ddagger$ is given by

$$(x,y) \diamond (z,w) = \begin{cases} (x, a^{k-r}w) & \text{if } k \geq r \text{ and } m = s \\ (x, a^k b^{m-r-s}w) & \text{if } m > r+s \\ (bx, b^{k+m-r-s+1}w) & \text{if } 0 \leq r+s-m \leq k \text{ and } m \neq s \\ (a^{r-k}x, w) & \text{if } k \leq r \text{ and } m = s \\ (a^r b^{s-k-m}x, w) & \text{if } s > k+m \\ (b^{r+s-k-m+1}x, bw) & \text{if } 0 \leq k+m-s \leq r \text{ and } m \neq s. \end{cases}$$

One of the realisation of this monoid is the Cartesian product $\mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ with respect to the multiplication

$$(p,q,k,m) \diamond (r,s,u,v) =$$

$$\begin{cases} (p,q,k-r+u,v) & \text{if } k \geq r \text{ and } m = s \\ (p,q,k,m-r-s+u+v) & \text{if } m > r+s \\ (0,p+q+1,0,k+m-r-s+u+v+1) & \text{if } 0 \leq r+s-m \leq k \text{ and } m \neq s \\ (r-k+p,q,u,v) & \text{if } k \leq r \text{ and } m = s \\ (r,s-k-m+p+q,u,v) & \text{if } s > k+m \\ (0,r+s-k-m+p+q+1,0,u+v+1) & \text{if } 0 \leq k+m-s \leq r \text{ and } m \neq s. \end{cases}$$

Proposition 2. *Let* $\varepsilon_1 : \mathbb{N} \longrightarrow B^\dagger$, $\varepsilon_2 : \mathbb{N} \longrightarrow B$, $\varepsilon_3 : B^\dagger \longrightarrow B^\ddagger$, $\varepsilon_4 : B \longrightarrow B^\ddagger$ *four maps defined by*

$$(n)\varepsilon_1 = (n,0),\ (n)\varepsilon_2 = (0,n),\ (k,m)\varepsilon_3 = (0,0,k,m),\ (k,m)\varepsilon_4 = (k,0,m,0).$$

Then ε_1, ε_2, ε_3 *and* ε_4 *are all injective monoid homomorphisms* (*embeddings;* $\mathbb{N}$ *being the additive monoid of non-negative integers*) *such that the following diagram commutes:*

$$\begin{tikzcd} \mathbb{N} \arrow[r, "\varepsilon_1"] \arrow[d, "\varepsilon_2"] & B^\dagger \arrow[d, "\varepsilon_3"] \\ B \arrow[r, "\varepsilon_4"'] & B^\ddagger \end{tikzcd}$$

Proof. Clearly, ε_1 and ε_2 are embeddings of the additive monoid $\mathbb{N}$ of non-negative integers in $B^\dagger$ and B, respectively.

If $(k,m),(r,s) \in B^\dagger$, then

$$(k,m)\varepsilon_3 \diamond (r,s)\varepsilon_3 = (0,0,k,m) \diamond (0,0,r,s) = \begin{cases} (0,0,k+r,s) & \text{if } m = 0 \\ (0,0,k,m+r+s) & \text{if } m > 0, \end{cases}$$

and

$$(k,m)(r,s) = \begin{cases} (k+r,s) & \text{if } m = 0 \\ (k,m+r+s) & \text{if } m > 0. \end{cases}$$

Hence $\varepsilon_3 : B^\dagger \longrightarrow B^\ddagger$ is an injective monoid homomorphism.

If $(k,m),(r,s) \in B$, then

$$(k,m)\varepsilon_4 \diamond (r,s)\varepsilon_4 = (k,0,m,0) \diamond (r,0,s,0) = \begin{cases} (k,0,m-r+s,0) & \text{if } m \geq r \\ (r-m+k,0,s,0) & \text{if } m < r, \end{cases}$$

and

$$(k,m)\cdot(r,s) = \begin{cases} (k,m-r+s) & \text{if } m \geq r \\ (k-m+r,s) & \text{if } m < r. \end{cases}$$

Hence $\varepsilon_4 : B \longrightarrow B^\ddagger$ is also an injective monoid homomorphism. Now, it is straightforward to check that the above diagram is commutative. ■

Proposition 3. *Let $\iota \times \iota \times \tau : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ be the transformation given by*

$$(k,m,r,s)(\iota \times \iota \times \tau) = (k,m,s,r),$$

and let $\varepsilon_5 : B \longrightarrow B^\ddagger$ be defined by

$$(k,m)\varepsilon_5 = (0,k,0,m).$$

Then ε_5 is (also) an injective monoid homomorphism (embedding) such that the following diagram commutes:

$$\begin{tikzcd} \mathbb{N} \arrow[r, "\varepsilon_1"] \arrow[d, equal, "\iota"] & B^\dagger = \mathbb{N} \times \mathbb{N} \arrow[r, "\varepsilon_3"] & B^\ddagger = \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} \arrow[d, leftrightarrow, "\iota\times\iota\times\tau"] \\ \mathbb{N} \arrow[r, "\varepsilon_2"'] & B = \mathbb{N} \times \mathbb{N} \arrow[r, "\varepsilon_5"'] & B^\ddagger = \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}. \end{tikzcd}$$

Proof. We prove that ε_5 is a monoid homomorphism. If $(k,m),(r,s) \in B$, then

$$(k,m)\varepsilon_5 \diamond (r,s)\varepsilon_5 = (0,k,0,m) \diamond (0,r,0,s) = \begin{cases} (0,k,0,s) & \text{if } m = r \\ (0,k,0,m-r+s) & \text{if } m > r \\ (0,r-m+k,0,s) & \text{if } m < r, \end{cases}$$

and

$$(k,m)\cdot(r,s) = \begin{cases} (k,m-r+s) & \text{if } m \geq r \\ (k-m+r,s) & \text{if } m < r. \end{cases}$$

Hence $\varepsilon_5 : B \longrightarrow B^\ddagger$ is an injective monoid homomorphism.

Now, it is straightforward to check (again) that this new diagram is also commutative. ■

Both monoids B and $B^\ddagger$ are bisimple inverse monoids. Both are combinatorial since 0 and (0,0) are indecomposable in $\mathbb{N}$ and $B^\dagger$, respectively. The bicyclic semigroup B is E-unitary since $\mathbb{N}$ is cancellative, but $B^\ddagger$ is not E-unitary since the right cancellative monoid $B^\dagger$ is not left cancellative.

4. The Möbius function

A locally finite poset is a partially ordered set $(P, \leq)$ for which every closed interval $[x, y] = \{z \mid x \leq z \leq y\}$ is finite. The incidence algebra of a locally finite poset $(P, \leq)$ is the set $\mathcal{A}(P) = \{f : I(P) \to \mathbb{C}\}$ of all complex valued maps defined on the set $I(P)$ of all nonempty intervals $[x, y]$, equipped with the pointwise addition and scalar multiplication and a convolution (multiplication) given by

$$(f * g)[x, y] = \sum_{z \in [x,y]} f[x, z]g[z, y].$$

(Here, in this section, we shall write function symbols on the left.) The Kronecker delta function δ_P defined by $\delta_P[x, y] = 1$ if $x = y$ and 0 otherwise, is the identity with respect to the convolution, and the convolution inverse $\mu_{P_\leq}$ of the zeta function ζ_P given by $\zeta_P[x, y] = 1$ for any nonempty interval $[x, y]$, is the Möbius function of the locally finite poset $(P, \leq)$.

Now, if M is a right cancellative Möbius monoid (that is $1 \in M$ is indecomposable and for any $a \in M$ there are a finite number of pairs $(b, c) \in M \times M$ such that $a = bc$) then the convolution $f * g$ of two elements $f, g \in \mathcal{A}(M) = \{f : M \to \mathbb{C}\}$ is defined by

$$(f * g)(a) = \sum_{a=bc} f(b)g(c).$$

The identity with respect to the convolution is the delta function δ_M defined by $\delta_M(a) = 1$ if a is the identity of M and 0 otherwise, and the convolution inverse μ_M of the zeta function ζ_M given by $\zeta_M(a) = 1$ for every $a \in M$, is the Möbius function of the Möbius monoid M. In the case of a right cancellative Möbius category $\mathcal{C}$, all these considerations can be formulated without difficulty (M being substituted by the set of all morphisms of $\mathcal{C}$).

The notations used in the following $\mu_{P_\leq}$, μ_M and $\mu_{\mathcal{C}}$ will indicate the order or algebraic structures to which the Möbius function is related.

It is straightforward to see that the additive monoid of non-negative integers $\mathbb{N}$ and the monoid $B^\dagger$ are right cancellative Möbius monoids. In the case of the additive monoid $\mathbb{N}$, the incidence algebra $\mathcal{A}(\mathbb{N})$ is the algebra of arithmetic functions with Cauchy convolution and the Möbius function $\mu_{\mathbb{N}}$ is given by

$$\mu_{\mathbb{N}}(m) = \begin{cases} 1 & \text{if } m = 0 \\ -1 & \text{if } m = 1 \\ 0 & \text{if } m > 1. \end{cases}$$

In the case of the right cancellative Möbius monoid $B^\dagger$ the algebra $\mathcal{A}(B^\dagger)$ is an algebra of arithmetic functions of two variables. By [10, Proposition 4.2] the

Möbius function $\mu_{B^\dagger}$ is given by

$$\mu_{B^\dagger}(m,n) = \begin{cases} \ \ 1 & \text{if } [m=0, n=0] \text{ or } [m=0, n=2]; \\ -1 & \text{if } [m=1, n=0] \text{ or } [m=0, n=1] \\ \ \ 0 & \text{otherwise.} \end{cases}$$

A Möbius function, as an arithmetic function of two variables, more related to $\mu_{\mathbb{N}}$ via the Möbius monoid $B^\dagger$ can be obtained by a breaking process described in [9]. It is straightforward to see that $B_b^\dagger = \{a^k b^m \in B^\dagger \mid k = 0\}$ is a (Möbius) submonoid of $B^\dagger$ such that the binary operation on $B^\dagger$ induces a right action of $B_b^\dagger$ on $B^\dagger - B_b^\dagger$. Thus a two objects (denoted 1 and 2) Möbius category $\mathcal{C}(B^\dagger/B_b^\dagger)$ is formed for which the set of morphisms $B^\dagger$ is broken up into two parts, one of which $Hom_{\mathcal{C}(B^\dagger/B_b^\dagger)}(1,1) = B_b^\dagger$ and the second one is $Hom_{\mathcal{C}(B^\dagger/B_b^\dagger)}(1,2) = B^\dagger - B_b^\dagger$ ($Hom_{\mathcal{C}(B^\dagger/B_b^\dagger)}(2,2)$ is a singleton and $Hom_{\mathcal{C}(B^\dagger/B_b^\dagger)}(2,1) = \emptyset$). The composition of morphisms in $\mathcal{C}(B^\dagger/B_b^\dagger)$ is completely determined by the binary operation on $B^\dagger$ (via the induced right action of $B_b^\dagger$ on $B^\dagger - B_b^\dagger$).

$$\begin{tikzcd} & & 2 \arrow[r, dashed, "id"] & 2 \\ 1 \arrow[r, "B_b^\dagger"'] & 1 \arrow[ur, "B^\dagger - B_b^\dagger"] & & \end{tikzcd}$$

Obviously, for any pair of non-negative integers (k,m), we have (see also [9, Corollary 4.1]):

$$\mu_{\mathcal{C}(B^\dagger/B_b^\dagger)}(a^k b^m) = \begin{cases} \mu_{\mathbb{N}}(m) & \text{if } k = 0 \\ \mu_{\mathbb{N}}(m+1) & \text{if } k > 0. \end{cases}$$

The same equality holds, namely

$$\mu_{\mathcal{C}(B^\dagger/B_a^\dagger)}(a^k b^m) = \begin{cases} \mu_{\mathbb{N}}(k) & \text{if } m = 0 \\ \mu_{\mathbb{N}}(k+1) & \text{if } m > 0 \end{cases}$$

for any pair of non-negative integers (k,m), if the two objects broken Möbius category $\mathcal{C}(B^\dagger/B_a^\dagger)$ is the category in which the set of morphisms $B^\dagger$ is broken up into two parts, one of which $Hom_{\mathcal{C}(B^\dagger/B_a^\dagger)}(2,2) = B_a^\dagger = \{a^k b^m \in B^\dagger \mid m = 0\}$ and the second one is $Hom_{\mathcal{C}(B^\dagger/B_a^\dagger)}(1,2) = B^\dagger - B_a^\dagger$ ($Hom_{\mathcal{C}(B^\dagger/B_a^\dagger)}(1,1)$ being a singleton and $Hom_{\mathcal{C}(B^\dagger/B_a^\dagger)}(2,1) = \emptyset$).

$$\begin{tikzcd} & & 2 \arrow[r, "B_a^\dagger"] & 2 \\ 1 \arrow[r, dashed, "id"] & 1 \arrow[ur, "B^\dagger - B_a^\dagger"] & & \end{tikzcd}$$

Another is when we use the submonoid

$$B^{\dagger}_{ev.} = \left\{a^k b^m \in B^{\dagger} |\ k \text{ and } m \text{ are both even}\right\}$$

in the breaking process.

$$\mathcal{C}(B^{\dagger}/B^{\dagger}_{ev.}):\qquad \begin{tikzcd} & & 2 \arrow[r, dashed, "id"] & 2 \\ 1 \arrow[r, "B^{\dagger}_{ev.}"'] & 1 \arrow[ur, "B^{\dagger}-B^{\dagger}_{ev.}"] & & \end{tikzcd}$$

Proposition 4. *The Möbius function* $\mu_{\mathcal{C}(B^{\dagger}/B^{\dagger}_{ev.})}$ *of the broken Möbius category* $\mathcal{C}(B^{\dagger}/B^{\dagger}_{ev.})$ *is given by*

$$\mu_{\mathcal{C}(B^{\dagger}/B^{\dagger}_{ev.})}(a^k b^m) = \begin{cases} \mu_{B^{\dagger}}\left(\frac{k}{2}, \frac{m}{2}\right) & \text{if } a^k b^m \in B^{\dagger}_{ev.}; \\ -1 & \text{if } m = 1 \text{ or } [m = 0 \text{ and } k = 1]; \\ 1 & \text{if } m = 3 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since $B^{\dagger}$ and $B^{\dagger}_{ev.}$ are isomorphic it follows that $\mu_{\mathcal{C}(B^{\dagger}/B^{\dagger}_{ev.})}(a^k b^m) = \mu_{B^{\dagger}}\left(\frac{k}{2}, \frac{m}{2}\right)$ if $a^k b^m \in B^{\dagger}_{ev.}$.

Now, let $a^k b^m \in B^{\dagger} - B^{\dagger}_{ev.}$. Then,

$$\begin{aligned} 0 &= \left(\delta * \mu_{\mathcal{C}\left(B^{\dagger}/B^{\dagger}_{ev.}\right)}\right)\left(a^k b^m\right) \\ &= \sum_{a^k b^m = a^p b^q a^{2r} b^{2s}} \delta\left(a^p b^q\right) \mu_{\mathcal{C}(B^{\dagger}/B^{\dagger}_{ev.})}\left(a^{2r} b^{2s}\right) + \delta(id_2) \mu_{\mathcal{C}\left(B^{\dagger}/B^{\dagger}_{ev.}\right)}\left(a^k b^m\right), \end{aligned}$$

that is

$$\mu_{\mathcal{C}(B^{\dagger}/B^{\dagger}_{ev.})}\left(a^k b^m\right) = -\sum_{a^k b^m = a^p b^q a^{2r} b^{2s}} \mu_{B^{\dagger}}(r, s).$$

Case 1. $m = 0$. Then $k = 2\ell + 1$ and

$$\mu_{\mathcal{C}(B^{\dagger}/B^{\dagger}_{ev.})}(a^k) = -\sum_{r=0}^{\ell} \mu_{B^{\dagger}}(r, 0) = \begin{cases} -1 & \text{if } \ell = 0 \\ 0 & \text{if } \ell > 0. \end{cases}$$

Case 2. $m = 1$. Then

$$\mu_{\mathcal{C}(B^{\dagger}/B^{\dagger}_{ev.})}(a^k b) = -\mu_{B^{\dagger}}(0, 0) = -1$$

for any non-negative integer k.

Case 3. $m = 2n$ $(n > 0)$. Then

$$\mu_{\mathcal{C}(B^\dagger/B^\dagger_{ev.})}\big(a^k b^m\big)$$

$$= -\mu_{B^\dagger}(0,0) - \sum_{r=1}^{n-1} \mu_{B^\dagger}(r,0) - \sum_{s=1}^{n} \mu_{B^\dagger}(0,s) - \sum_{r,s\geq 1, r+s<n} \mu_{B^\dagger}(r,s) = 0,$$

for any non-negative odd integer k.

Case 4. $m = 2n+1$ $(n > 0)$. Then

$$\mu_{\mathcal{C}(B^\dagger/B^\dagger_{ev.})}\big(a^k b^m\big)$$

$$= -\mu_{B^\dagger}(0,0) - \sum_{r=1}^{n} \mu_{B^\dagger}(r,0) - \sum_{s=1}^{n} \mu_{B^\dagger}(0,s) - \sum_{r,s\geq 1, r+s\leq n} \mu_{B^\dagger}(r,s)$$

$$= \begin{cases} -1+1+1 = 1 & \text{if } \ n = 1 \\ -1+1-(-1+1)-0 = 0 & \text{if } \ n > 1 \end{cases}$$

for any non-negative integer k.

The proof of Proposition 4 is now complete. ∎

Now, let (S,T) be a Clifford pair (i.e., S is a right cancellative monoid which satisfies the Clifford condition and T is a bisimple inverse monoid such that the subsemigroup of right units is isomorphic to the monoid S). If the right cancellative monoid S is a Möbius monoid (as above the monoids $\mathbb{N}$ and $B^\dagger$) then T is combinatorial (since $1 \in S$ is indecomposable). More like this, the pair $(T, \leq)$ is a locally finite poset, where $\leq$ is the well-known natural partial order for inverse semigroups ($t \leq u$ if and only if $tt^{-1}u = t$).

It is straightforward to check that for the bicyclic semigroup B the natural partial order $\leq$ is given by

$$(m,n) \leq (m',n') \ \text{ if and } \textit{only} \text{ if } \ m - m' = n - n' \in \mathbb{N}.$$

Proposition 5. *The Möbius function $\mu_{B_\leq}$ of the bicyclic semigroup with the natural partial order relation $\leq$ is given by*

$$\mu_{B_\leq}[(m,n),(m',n')] = \begin{cases} 1 & \textit{if } \ m = m' \\ -1 & \textit{if } \ m = m'+1 \\ 0 & \textit{if } \ m - m' > 1. \end{cases}$$

Proof. Taking into account [7, Proposition 2.6 (2)] we have $\mu_{B_\leq}[(m,n),(m',n')] = \mu_{\mathbb{N}}(m-m')$, and the statement is proved. ∎

The natural partial order in the case of the bisimple inverse monoid $B^{\ddagger}$ is somewhat more complicated. We have

$$(m,n,p,q) \leq (m',n',p',q') \text{ if and } only \text{ if } (m,n,m,n) \diamond (m',n',p',q') = (m,n,p,q)$$

that is

$$(m,n,p,q) \leq (m',n',p',q') \text{ if and } only \text{ if}$$
$$\begin{cases} n = n' \; q = q' \text{ and } m - m' = p - p' \in \mathbb{N}, \\ \text{or} \\ p = m, \;\; q = n - m' - n' + p' + q' \text{ and } n > m' + n'. \end{cases}$$

However the Möbius function $\mu_{B^{\ddagger}_{\leq}}$ of the bisimple inverse monoid $B^{\ddagger}$ with the natural partial order relation $\leq$ is more simple, and clearly expresses a close relationship with the Möbius function of the bicyclic semigroup but also with the Möbius function of the Möbius monoid $B^{\dagger}$

Proposition 6. *The Möbius function $\mu_{B^{\ddagger}_{\leq}}$ of the bisimple inverse monoid $B^{\ddagger}$ with the natural partial order relation $\leq$ is given by*

$$\mu_{B^{\ddagger}_{\leq}}[(m,n,p,q),(m',n',p',q')]$$
$$= \begin{cases} \mu_{B_{\leq}}[(m,p),(m',p')] & \text{if } n = n', \;\; q = q' \text{ and } m - m' = p - p' \in \mathbb{N} \\ \mu_{B^{\dagger}}(m, n - m' - n') & \text{if } p = m, \;\; q = n - m' - n' + p' + q' \text{ and } n > m' + n'. \end{cases}$$

Proof. By [7, Proposition 2.6 (2)] it follows that $\mu_{B^{\ddagger}_{\leq}}[(m,n,p,q),(m',n',p',q')] = \mu_{B^{\dagger}}(x,y)$, where $(x,y) \in B^{\dagger}$ such that $(x,y)(m',n') = (m,n)$. In the case $n = n'$, $q = q'$, $m - m' = p - p' \in \mathbb{N}$, we have $y = 0$ and $x = m - m'$. Thus $\mu_{B^{\ddagger}_{\leq}}[(m,n,p,q),(m',n',p',q')] = \mu_{B^{\dagger}}(m - m', 0) = \mu_{B_{\leq}}[(m,p),(m',p')]$. In the case $p = m$, $q = n - m' - n' + p' + q'$, $n > m' + n'$, we have $x = m$ and $y = n - m' - n'$. So, the assertion of Proposition 6 is proved completely. ■

5. Final remarks

1) The computation of all examples of Möbius functions $\mu_{\mathcal{C}(\mathfrak{M}/\mathfrak{M}')}$ of broken Möbius categories $\mathcal{C}(\mathfrak{M}/\mathfrak{M}')$ is considered in [9], but also $\mu_{\mathcal{C}(B^{\dagger}/B^{\dagger}_a)}$ and $\mu_{\mathcal{C}(B^{\dagger}/B^{\dagger}_b)}$ from the previous section find the solution in [9, Corollary 4.1]. They are all Möbius functions for which for any $x \in \mathfrak{M} - \mathfrak{M}'$, $\mu_{\mathcal{C}(\mathfrak{M}/\mathfrak{M}')}(x) = -1$ if x is an atom over $\mathfrak{M}'$ (i.e., $x = yz$ with $z \in \mathfrak{M}'$ implies $z = 1_{\mathfrak{M}'}$) and $\mu_{\mathcal{C}(\mathfrak{M}/\mathfrak{M}')}(x)$ vanishes otherwise. By Corollary 4.1 of [9] this happens if for any $x \in \mathfrak{M} - \mathfrak{M}'$ the set of all right divisors $z \in \mathfrak{M}'$ of x contain a greatest element. It was necessary,

besides the examples that satisfy the stated condition, to present an example for which the result of Corollary 4.1 of [9] does not take place. Proposition 4.1 of the previous section responds to this requirement.

2) One of the purpose of Warne's paper [12] is to study a certain generalization of the bicyclic semigroup. Warne's 2-dimensional bicyclic semigroup (which is isomorphic to the quadrucyclic semigroup considered in [11]) is the Bruck product $B \circ B$, namely it is the Cartesian product $B \times B = \mathbb{N}^4$ with the operation $\triangleright$ defined by

$$(p,q,k,m) \triangleright (r,s,u,v) = \begin{cases} (p, q-r+s, k, m) & \text{if } q > r \\ (p, s, k-m+u, v) & \text{if } q = r \text{ and } m < u \\ (p, s, k, v) & \text{if } q = r \text{ and } m = u \\ (p, s, k, m-u+v) & \text{if } q = r \text{ and } m > u \\ (p-q+r, s, u, v) & \text{if } q < r. \end{cases}$$

Now, for any semigroup S it is defined a partial order $\leq$ on the set of idempotents $E(S)$ (if $E(S)$ is not empty) by the rule that $e \leq f$ if and only if $ef = e = fe$. In the case of an inverse semigroup S this is just the restriction to $E(S)$ of the natural partial order on S. It can be easily seen that if $S = B \circ B$ then the set of idempotents

$$E(B \circ B) = \big\{(p,q,k,m) \in \mathbb{N}^4 \mid q = p \text{ and } m = k\big\} = \{(p,p,k,k)\}_{p,k \in \mathbb{N}}$$

is lexicographically ordered, that is $E(B \circ B)$ is order isomorphic to $\mathbb{N} \times \mathbb{N}$ under the order defined by:

$$(p,k) \leq (r,u) \text{ if and only if } p > r \text{ or } p = r \text{ and } k \geq u.$$

It is well known (see [12, Corollary 2.1]) that an inverse monoid S is combinatorial bisimple with $E(S)$ lexicographically ordered if and only if $S \cong B \circ B$.

Now, $B^{\ddagger}$ is a combinatorial bisimple inverse monoid of quadruples of non-negative integers which is not isomorphic to the 2-dimensional bicyclic semigroup $B \circ B$. The monoid operation $\diamond$ on $B^{\ddagger}$ restricted to the set of idempotents

$$E(B^{\ddagger}) = \{(p,q,k,m) \in \mathbb{N}^4 \mid k = p \text{ and } m = q\} = \{(p,q,p,q)\}_{p,q \in \mathbb{N}}$$

is given by

$$(p,q,p,q) \diamond (r,s,r,s) = \begin{cases} (p,q,p,q) & \text{if } p \geq r \text{ and } q = s \\ (p,q,p,q) & \text{if } q > r+s \\ (0, p+q+1, 0, p+q+1) & \text{if } 0 \leq r+s-q \leq p, \ q \neq s \\ (r,s,r,s) & \text{if } p \leq r \text{ and } q = s \\ (r,s,r,s) & \text{if } s > p+q \\ (0, r+s+1, 0, r+s+1) & \text{if } 0 \leq p+q-s \leq r, \ q \neq s, \end{cases}$$

As can be read in the above table, $E(B^{\ddagger})$ is order isomorphic to $\mathbb{N}\times\mathbb{N}$ under a (reverse) lexicographic-type order relation given by

$$(p,q) \leq (r,s) \ \text{ if and } only \text{ if } \ q > s + r \text{ or } q = s \text{ and } p \geq r.$$

If we replace $B^{\dagger}$ by $B_1^{\dagger} =< a,b \mid ba = b >$ we get a combinatorial bisimple inverse monoid $B_1^{\ddagger}$ (in the same way as $B^{\ddagger}$) which is just Warne's 2-dimensional bicyclic semigroup. Note that $B_1^{\dagger}$ is not a Möbius monoid though $B_n^{\dagger} =< a,b \mid ba = b^n >$ is a Möbius monoid for all $n > 1$. The monoid $B^{\dagger}(= B_2^{\dagger})$ is the only half-factorial monoid in the class $\{B_n^{\dagger} =< a,b \mid ba = b^n >\}_{n>1}$. An explicit description of the system of sets of lengths of the non-commutative atomic monoids $B_n^{\dagger}$, $n > 1$, is given in [3, Section 4].

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Received 10 October 2022
Revised 12 November 2022
Accepted 14 November 2022