Discussiones Mathematicae
General Algebra and Applications 44 (2024) 43–55
https://doi.org/10.7151/dmgaa.1432

ON TERNARY RING CONGRUENCES OF TERNARY SEMIRINGS

Jatuporn Sanborisoot[1]

Algebra and Applications Research Unit
Department of Mathematics, Faculty of Science
Mahasarakham University, Mahasarakham 44150, Thailand
e-mail: jatuporn.san@msu.ac.th

and

Pakorn Palakawong na Ayutthaya

Department of Mathematics, Faculty of Science
Khon Kaen University, Khon Kaen, 40002, Thailand
e-mail: pakorn1702@gmail.com

Abstract

In this work, we study the notions of k-ideals and h-ideals of ternary semirings and investigate some of their algebraic properties. Furthermore, we construct a congruence relation with respect to a full k-ideal on a ternary semiring for the purpose of forming a ternary ring from the quotient ternary semiring.

Keywords: ternary ring, ternary semiring, ring congruence, k-ideal, h-ideal.

2020 Mathematics Subject Classification: 16Y60, 06F25.

1. Introduction

The concept of an algebraic structure together with a ternary operation was introduced first by Lehmer [12] in 1932. Later, Sioson [16] defined the notion of a ternary semigroup and studied algebraic properties of ideals on a ternary semigroup. Afterward, in 1990, the notion of a regularity on a ternary semigroup

[1]Corresponding author.
This research project was financially supported by Mahasarakham University.

was investigated by Santiago [14]. The concept of an algebraic structure which contains a binary operation and a ternary operation was defined by Lister [13] as a ternary ring. As a generalization of a ternary ring, Dutta and Kar [5, 6, 7] defined the notion of a ternary semiring and investigated some of their properties such as regularity and Jacobson radical.

A semiring which is a notable generalization of rings and distributive lattices was defined first by Vandiver [17]. This algebraic structure appears in a natural manner in some applications to the theory of automata, formal languages, optimization theory and other branches of applied mathematics (for example, see [3, 4, 8, 9, 11]). In abstract algebra, it is not difficult to prove that the kernel of a ring homomorphism is an ideal and each ideal of a ring can be considered as the kernel of a ring homomorphism. Similarly, the kernel of a semiring homomorphism is an ideal as well. However, there is an ideal of a semiring such that it cannot be considered as the kernel of a semiring homomorphism [1, 2]. This condition can be true on a semiring by using a more restrict type of ideals (see [2]) namely a k-ideal defined by Henriksen [10]. Later, in [15], Sen and Adhikari defined the notion of a full k-ideal and used a full k-ideal to construct a congruence relation on a semiring such that the quotient semiring forms a ring. Furthermore, the notion of an h-ideal, a more special kind of k-ideals, was also defined by Henriksen [10].

It is easy to construct a ternary semiring from a given semiring; however, there is a ternary semiring such that it cannot be considered as a semiring. Consequently, we are able to study a ternary semiring as a generalization of a semiring. In this work, we study the concept of a k-ideal of a ternary semiring as a similar way of Sen and Adhikari [15] on a semiring. In other words, we define the notion of a full k-ideal of a ternary semiring and use a full k-ideal to construct a congruence relation such that the quotient ternary semiring forms a ternary ring. Moreover, we also show that every h-ideal of a ternary semiring is immediately full and the concepts of k-ideals and h-ideals are coincidence in additively inverse ternary semirings.

2. Preliminaries

A nonempty set S together with a binary operation $+ : S \times S \to S$ is called a *semigroup* if $a + (b + c) = (a + b) + c$ for all $a, b, c \in S$. A *ternary groupoid* is an algebra $\langle S; f\rangle$ such that $f : S \times S \times S \to S$ is a ternary operation on the nonempty set S. A ternary groupoid $\langle S; f\rangle$ is called a *ternary semigroup* if f satisfies the associative property on S, i.e., $f(f(a, b, c), d, e) = f(a, f(b, c, d), e) = f(a, b, f(c, d, e))$ for all $a, b, c, d, e \in S$. A *ternary semiring* is an algebra $\langle S; +, f\rangle$ type $(2, 3)$ for which $\langle S; +\rangle$ is a semigroup, $\langle S; f\rangle$ is a ternary semigroup and for all $a, b, x, y \in S$, $f(a+b, x, y) = f(a, x, y) + f(b, x, y)$, $f(x, a+b, y) = f(x, a, y) + f(x, b, y)$ and $f(x, y, a+b) = f(x, y, a) + f(x, y, b)$. A ternary semiring $\langle S; +, f\rangle$ is said to be *additively commutative* if $a + b = b + a$ for all $a, b \in S$.

The set of all negative integers together with the usual addition and the usual multiplication is an example of a ternary semiring such that it cannot be considered as a semiring because every product of two negative integers is not a negative integer.

Throughout this work, we simply write S instead of an additively commutative ternary semiring $\langle S; +, f\rangle$ and the juxtaposition abc instead of $f(a, b, c)$ for all $a, b, c \in S$.

For any nonempty subsets A, B, and C of a ternary semiring S, we denote that $A + B = \{a + b \in S \mid a \in A, b \in B\}$ and $ABC = \{abc \in S \mid a \in A,\ b \in B,\ c \in C\}$.

A nonempty subset T of a ternary semiring S is called a *subalgebra* of S if $T + T \subseteq T$ and $TTT \subseteq T$.

Definition 2.1. A nonempty subset A of a ternary semiring S is called a *left ideal* (respectively, *lateral ideal*, *right ideal*) of S if $A + A \subseteq A$ and $SSA \subseteq A$ (respectively, $SAS \subseteq A$, $ASS \subseteq A$). A is called an *ideal* of S if A is a left ideal, a lateral ideal, and a right ideal of S.

An element a of a ternary semiring S is called *additively regular* if $a = a+b+a$ for some $b \in S$. If the element b is unique and satisfies $b = b+a+b$, then b is called an *additively inverse* of a in S and will be denoted by the notation a'. Particularly, if every element of S is additively regular, then S is called an *additively regular ternary semiring*. Furthermore, if every additively regular element of S has the unique additively inverse, then S is called an *additively inverse ternary semiring*.

Let S be an additively inverse ternary semiring. It is obvious that $x = (x')'$ and $(x + y)' = x' + y'$ for all $x, y \in S$.

Lemma 2.2. *Let S be an additively inverse ternary semiring. Then for any $x, y, z \in S$, $(xyz)' = x'yz = xy'z = xyz'$.*

Proof. Let $x, y, z \in S$. Since $xyz + x'yz + xyz = (x + x' + x)yz = xyz$ and $x'yz + xyz + x'yz = (x' + x + x')yz = x'yz$, we obtain that $(xyz)' = x'yz$. The cases of $(xyz)' = xy'z$ and $(xyz)' = xyz'$ can be proved similarly. ∎

An element x of a ternary semiring S is called *additively idempotent* if $x + x = x$. The set of all additively idempotent elements of S is defined by

$$E^+ = \{x \in S \mid x + x = x\}.$$

It is not difficult to verify that E^+ is an ideal of S.

A partially ordered set $(L, \leq)$ is said to be a *lattice* if every pair of elements a and b of L has both greatest lower bound and least upper bound. If every subset A of a lattice L has both greatest lower bound and least upper bound, then L is called a *complete lattice*. It is not difficult to verify that a lattice L is a complete lattice if L has the greatest element and every nonempty subset of L has the greatest lower bound.

A lattice L is called *modular* if L satisfies the following law; for all $a, b \in L$, $a \leq b$ implies $a \vee (x \wedge b) = (a \vee x) \wedge b$ for every $x \in L$ where $x \vee y$ and $x \wedge y$ is the least upper bound and the greatest lower bound of $x, y \in L$, respectively.

Lemma 2.3. *A lattice L is modular if and only if for any $a, b, c \in L$, $a \wedge b = a \wedge c$, $a \vee b = a \vee c$ and $b \leq c$ implies $b = c$.*

3. Full k-ideals and h-ideals of ternary semirings

The notions and some properties of full k-ideals and h-ideals of ternary semirings have been defined and studied in this section.

Definition 3.1. An ideal A of a ternary semiring S is called a *k-ideal* of S if for any $x \in S$, $x + a = b$ for some $a, b \in A$ implies $x \in A$. If A is a k-ideal of S and $E^+ \subseteq A$, then A is said to be a *full k-ideal* of S.

The following example is an example of an ideal of a ternary semiring which is not a k-ideal.

Example 3.2. Define a ternary operation f on the set of all natural numbers $\mathbb{N}$ by $f(x, y, z) = x \cdot y \cdot z$ for any $x, y, z \in \mathbb{N}$ where $\cdot$ is the usual multiplication. Then $\langle \mathbb{N}; \max, f \rangle$ is a ternary semiring. We have that $2\mathbb{N} := \{2, 4, 6, 8, \ldots\}$, the set of all positive even numbers, is an ideal of $\langle \mathbb{N}; \max, f \rangle$ but not a k-ideal because $\max\{1, 2\} = 2$ but $1 \notin 2\mathbb{N}$.

The following example is an example of a k-ideal of a ternary semiring which is not a full k-ideal.

Example 3.3. Define a ternary operation f on the set of all natural numbers $\mathbb{N}$ by $f(x, y, z) = \min\{x, y, z\}$ for any $x, y, z \in \mathbb{N}$. Then $\langle \mathbb{N}; \max, f \rangle$ is a ternary semiring and $E^+ = \mathbb{N}$ is the set of all additively idempotent elements of $\langle \mathbb{N}; \max, f \rangle$. It is easy to obtain that the set $\mathbb{I}_m = \{1, 2, 3, \ldots, m\}$ for each $m \in \mathbb{N}$, is a k-ideal of $\langle \mathbb{N}; \max, f \rangle$ but not a full k-ideal because $E^+ \nsubseteq \mathbb{I}_m$.

We give an example of a proper full k-ideal of a ternary semiring as follows.

Example 3.4. Let $\mathbb{N}_0$ be the set of all nonnegative integers. Then $\langle \mathbb{N}_0; +, f \rangle$ is a ternary semiring such that $+$ is the usual addition and $f(x, y, z) = x \cdot y \cdot z$ for all $x, y, z \in \mathbb{N}_0$ where $\cdot$ is the usual multiplication. We have that the set of all additively idempotent elements of $\langle \mathbb{N}_0; +, f\rangle$ is $\{0\}$ and $2\mathbb{N}_0 = \{0, 2, 4, 6, \ldots\}$ is a full k-ideal.

The proofs of the following two remarks are routine.

Remark 3.5. Let $\{A\}_{i\in I}$ be a family of full k-ideals of a ternary semiring S. Then $\bigcap_{i\in I} A_i$ is also a full k-ideal if it is not empty.

Remark 3.6. Every k-ideal of an additively inverse ternary semiring S is an additively inverse subalgebra of S.

The *k-closure* of a nonempty subset A of a ternary semiring S is defined by

$$[A]_k = \{x \in S \mid x + a = b \text{ for some } a, b \in A\}.$$

It is easy to prove that for any $\emptyset \neq A \subseteq S$, $A \subseteq [A]_k$ if $A + A \subseteq A$. Furthermore, if A is closed under the addition, then $[A]_k$ is also closed. Now, we give some necessary properties of k-closure of nonempty subsets of a ternary semiring as follows.

Lemma 3.7. *Let A, B, and C be nonempty subsets of an n-ary semiring S. Then the following statements hold.*

(i) *If $A + A \subseteq A$, then $[A]_k = [[A]_k]_k$.*

(ii) *If $A \subseteq B$, then $[A]_k \subseteq [B]_k$.*

(iii) *$[A]_k + [B]_k \subseteq [A + B]_k$.*

(iv) *If A, B, and C are closed under the addition, then $[A]_kBC \subseteq [ABC]_k$, $A[B]_kC \subseteq [ABC]_k$ and $AB[C]_k \subseteq [ABC]_k$.*

Proof. (i) Let $\emptyset \neq A \subseteq S$ be such that $A + A \subseteq A$. Obviously, $[A]_k \subseteq [[A]_k]_k$. If $x \in [[A]_k]_k$, then $x + y = z$ for some $y, z \in [A]_k$ such that $y + a_1 = b_1$ and $z + a_2 = b_2$ for some $a_1, a_2, b_1, b_2 \in A$. Then

$$x + y + a_1 + a_2 = z + a_1 + a_2 = z + a_2 + a_1 = b_2 + a_1. \tag{1}$$

We have $y + a_1 + a_2 = b_1 + a_2 \in A + A \subseteq A$ and $b_2 + a_1 \in A + A \subseteq A$. Using (1), we get $x \in [A]_k$ and so $[[A]_k]_k \subseteq [A]_k$. Therefore, $[A]_k = [[A]_k]_k$.

(ii)–(iv) are straightforward. ■

Lemma 3.8. *If A is an ideal of a ternary semiring S, then $[A]_k$ is a k-ideal of S.*

Proof. Let A be an ideal of S. It is clear that $[A]_k$ is closed under addition. Using A being an ideal of S and Lemma 3.7(ii) and (iv), we obtain that $SS[A]_k \subseteq [SSA]_k \subseteq [A]_k$, $S[A]_kS \subseteq [SAS]_k \subseteq [A]_k$ and $[A]_kSS \subseteq [ASS]_k \subseteq [A]_k$. If $x \in S$ such that $x + a = b$ for some $a, b \in [A]_k$, then by Lemma 3.7(i), we get $x \in [[A]_k]_k = [A]_k$. Therefore, $[A]_k$ is a k-ideal of S. ■

The following corollary is directly obtained by Lemma 3.8.

Corollary 3.9. *Let S be a ternary semiring. Them the following statements hold.*

(i) *An ideal A of S is a k-ideal if and only if $A = [A]_k$.*

(ii) *$[E^+]_k$ is a full k-ideal of S.*

Lemma 3.10. *Let A and B be two full k-ideals of an additively inverse ternary semiring S. Then $[A+B]_k$ is a full k-ideal of S such that $A \subseteq [A+B]_k$ and $B \subseteq [A+B]_k$.*

Proof. Obviously, $A+B$ is closed under the addition. We get that $SS(A+B) \subseteq SSA + SSB \subseteq A+B$, $S(A+B)S \subseteq SAS + SBS \subseteq A+B$, and $(A+B)SS \subseteq ASS + BSS \subseteq A+B$. Now, $A+B$ is an ideal of S. Using Lemma 3.8, we immediately obtain that $[A+B]_k$ is a k-ideal. Since $E^+ \subseteq A$ and $E^+ \subseteq B$, $E^+ = E^+ + E^+ \subseteq A+B \subseteq [A+B]_k$. Hence, $[A+B]_k$ is a full k-ideal of S. Let $a \in A$. Then $a = a + a' + a = a + (a' + a) \in A + E^+ \subseteq A + B \subseteq [A+B]_k$. Hence, $A \subseteq [A+B]_k$. Similarly, we are able to get that $B \subseteq [A+B]_k$. ∎

Theorem 3.11. *Let $K(S)$ be the set of all full k-ideals of an additively inverse ternary semiring S. Then $K(S)$ is a complete lattice which is also modular.*

Proof. We have that $K(S)$ is a partially ordered set with respect to usual set inclusion. Let $A, B \in K(S)$. By Remark 3.5 and Lemma 3.10, we obtain that $A \cap B \in K(S)$ and $[A+B]_k \in K(S)$, respectively. Define $A \wedge B = A \cap B$ and $A \vee B = [A+B]_k$. Obviously, $A \cap B$ is the greatest lower bound of A and B. Let $C \in K(S)$ such that $A \subseteq C$ and $B \subseteq C$. Then $A + B \subseteq C + C \subseteq C$. By Remark 3.7(ii) and Corollary 3.9(i), we get $[A+B]_k \subseteq [C]_k = C$. Hence, $[A+B]_k$ is the least upper bound of A and B. Now, $K(S)$ is a lattice.

Clearly, S is the greatest element of $K(S)$. Let $\{A_i\}_{in \in I}$ be a family of nonempty subsets of $K(S)$. Using Remark 3.5, we obtain that $\bigcap_{in \in I} A_i \in K(S)$ and immediately get that it is the greatest lower bounded of $\{A_i\}_{i \in I}$. These imply that $K(S)$ is a complete lattice.

Finally, let $A, B, C \in K(S)$ such that $A \wedge B = A \wedge C$, $A \vee B = A \vee C$, and $B \subseteq C$. Let $x \in C$. Then $x \in C \subseteq A \vee C = A \vee B = [A+B]_k$. It follows that there exist $a_1, a_2 \in A$ and $b_1, b_2 \in B$ such that $x + a_1 + b_1 = a_2 + b_2$. Then

$$(2) \quad x + a_1 + a_1' + b_1 = x + a_1 + b_1 + a_1' = a_2 + b_2 + a_1' = a_2 + a_1' + b_2.$$

Now, $x \in C$, $a_1 + a_1' \in E^+ \subseteq C$ and $b_1, b_2 \in B \subseteq C$. Using (2), $a_2 + a_1' \in [C]_k = C$. At this point, $a_1 + a_1', a_2 + a_1' \in A \cap C = A \wedge C = A \wedge B = A \cap B \subseteq B$. It follows that $a_1 + a_1' + b_1 \in B$ and $a_2 + a_1' + b_2 \in B$. Using (2) again, we obtain that $x \in [B]_k = B$ and so $C \subseteq B$. Hence, $B = C$. By Lemma 2.3, $K(S)$ is a modular lattice. ∎

Now, we introduce a more restrict class of ideals of a ternary semiring as follows.

Definition 3.12. An ideal A of a ternary semiring S is called an h-*ideal* of S if for any $x \in S$, $x + a + s = b + s$ for some $a, b \in A$ and $s \in S$ implies $x \in A$.

Every h-ideal of a ternary semiring is immediately full and so the notion of a full h-ideal need not to be defined.

Remark 3.13. If A is an h-ideal of a ternary semiring S, then $E^+ \subseteq A$.

Proof. Let A be an h-ideal of S and let $e \in E^+$. If $a \in A$, then $e+a+e = a+e$. Since A is an h-ideal, $e \in A$. Hence, $E^+ \subseteq A$. ∎

It is clear that every h-ideal of a ternary semiring is a k-ideal. In general, the converse is not true as shown by the following example.

Example 3.14. Let $S = \{a, b, c\}$. Define a ternary operation f on the power set $P(S)$ of S by $f(A, B, C) = A \cap B \cap C$ for any $A, B, C \in P(S)$. Then $\langle P(S); \cup, f\rangle$ is a ternary semiring. We have that $T = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ is a k-ideal of $\langle P(S); \cup, f\rangle$. However, T is not an h-ideal because $\{c\} \cup \{a, b\} \cup \{a, c\} = S = \{b\} \cup \{a, c\}$ where $\{a, b\}, \{b\} \in T$ but $\{c\} \notin T$.

Remark 3.15. Let $\{A\}_{i\in I}$ be a family of h-ideals of a ternary semiring S. Then $\bigcap_{i\in I} A_i$ is also an h-ideal if it is not empty.

Remark 3.16. Every h-ideal of an additively inverse ternary semiring S is an additively inverse subalgebra of S.

Proof. Let H be an h-ideal of S. Clearly, H is a subalgebra of S. Let $a \in H$. Then $(a + a') + a + s = a + s$ for all $s \in S$. So, $a + a' \in H$. This means that $a' + a = b$ for some $b \in H$ and thus $a' + a + t = b + t$ for any $t \in S$. This implies that $a' \in H$. Hence, H is additively inverse. ∎

The h-*closure* of a nonempty subset A of a ternary semiring S is defined by

$$[A]_h = \{x \in S \mid x + a + s = b + s \text{ for some } a, b \in A, s \in S\}.$$

It is obvious that $[A]_k \subseteq [A]_h$ for any $\emptyset \neq A \subseteq S$. Moreover, it is not difficult to verify that for any $\emptyset \neq A \subseteq S$, $A \subseteq [A]_h$ if $A + A \subseteq A$. Furthermore, if A is closed under the addition, then $[A]_h$ is also closed. Now, we give some necessary properties of h-closure of nonempty subsets on a ternary semiring as follows.

Lemma 3.17. *Let A, B, and C be nonempty subsets of a ternary semiring S. Then the following statements hold.*

(i) *If $A + A \subseteq A$, then $[A]_h = [[A]_h]_h$.*

(ii) *If* $A \subseteq B$, *then* $[A]_h \subseteq [B]_h$.

(iii) $[A]_h + [B]_h \subseteq [A+B]_h$.

(iv) *If* A, B, *and* C *are closed under the addition, then* $[A]_h BC \subseteq [ABC]_h$, $A[B]_h C \subseteq [ABC]_h$ *and* $AB[C]_h \subseteq [ABC]_h$.

Proof. (i) Let $\emptyset \neq A \subseteq S$ be such that $A + A \subseteq A$. Obviously, $[A]_h \subseteq [[A]_h]_h$. If $x \in [[A]_h]_h$, then $x + y + s = z + s$ for some $y, z \in [A]_h$ and $s \in S$ where $y + a_1 + u = b_1 + u$ and $z + a_2 + v = b_2 + v$ for some $a_1, a_2, b_1, b_2 \in A$ and $u, v \in S$. Then

$$\begin{aligned} x + y + s + a_1 + u + a_2 + v &= x + (y + a_1 + u) + a_2 + s + v \\ &= x + b_1 + u + a_2 + s + v \\ &= x + b_1 + a_2 + u + s + v \qquad (3) \\ x + y + s + a_1 + u + a_2 + v &= z + s + a_1 + u + a_2 + v \\ &= a_1 + (z + a_2 + v) + s + u \\ &= a_1 + b_2 + v + s + u. \qquad (4) \end{aligned}$$

Using (3) and (4), we get that $x + (b_1 + a_2) + u + s + v = (a_1 + b_2) + u + s + v$ where $b_1 + a_2, a_1 + b_2 \in A + A \subseteq A$ and $u + s + v \in S$ implies $x \in [A]_h$ and so $[[A]_h]_h \subseteq [A]_h$. Therefore, $[A]_h = [[A]_h]_h$.

(ii)–(iv) are straightforward. ■

Lemma 3.18. *If* A *is an ideal of a ternary semiring* S, *then* $[A]_h$ *is an* h*-ideal of* S.

Proof. Let A be an ideal of S. Clearly, $[A]_h$ is closed under the addition. Using A being an ideal of S and Lemma 3.17(ii) and (iv), we obtain that $SS[A]_k \subseteq [SSA]_k \subseteq [A]_k$, $S[A]_k S \subseteq [SAS]_k \subseteq [A]_k$ and $[A]_k SS \subseteq [ASS]_k \subseteq [A]_k$. If $x \in S$ such that $x + a + s = b + s$ for some $a, b \in [A]_h$ and $s \in S$, then by Lemma 3.17(i), we get $x \in [[A]_h]_h = [A]_h$. Therefore, $[A]_h$ is an h-ideal of S. ■

The following corollary is directly obtained by Lemma 3.18.

Corollary 3.19. *Let* S *be an* n*-ary semiring. Then the following statements hold.*

(i) *An ideal* A *of* S *is an* h*-ideal if and only if* $A = [A]_h$.

(ii) $[E^+]_h$ *is an* h*-ideal of* S.

Lemma 3.20. *Let* A *and* B *be two* h*-ideals of an additively inverse ternary semiring* S. *Then* $[A+B]_h$ *is an* h*-ideal of* S *such that* $A \subseteq [A+B]_h$ *and* $B \subseteq [A+B]_h$.

Proof. Since $SS(A+B) \subseteq SSA+SSB \subseteq A+B$, $S(A+B)S \subseteq SAS+SBS \subseteq A+B$, $(A+B)SS \subseteq ASS+BSS \subseteq A+B$, and $A+B$ is closed under the addition, we get that $A+B$ is an ideal of S. Using Lemma 3.18, we obtain that $[A+B]_h$ is an h-ideal. Let $a \in A$. Then $a = a+a'+a = a+(a'+a) \in A+E^+ \subseteq A+B \subseteq [A+B]_h$. Hence, $A \subseteq [A+B]_h$. Similarly, we are able to get that $B \subseteq [A+B]_h$. ■

Theorem 3.21. *Let $H(S)$ be the set of all h-ideals of an additively inverse ternary semiring S. Then $H(S)$ is a complete lattice which is also modular.*

Proof. We have that $H(S)$ is a partially ordered set with respect to the usual set inclusion. Let $A, B \in H(S)$. By Remark 3.15 and Lemma 3.20, we obtain that $A \cap B \in H(S)$ and $[A+B]_h \in H(S)$, respectively. Define $A \wedge B = A \cap B$ and $A \vee B = [A+B]_h$. Obviously, $A \cap B$ is the greatest lower bound of A and B. Let $C \in H(S)$ such that $A \subseteq C$ and $B \subseteq C$. Then $A+B \subseteq C+C \subseteq C$. By Remark 3.17(ii) and Corollary 3.19(i), we get $[A+B]_h \subseteq [C]_h = C$. Hence, $[A+B]_h$ is the least upper bound of A and B. Now, $H(S)$ is a lattice.

Clearly, S is the greatest element of $H(S)$. Let $\{A_i\}_{in\in I}$ be a family of nonempty subsets of $H(S)$. Using Remark 3.15, we obtain that $\bigcap_{in\in I} A_i \in H(S)$ and immediately get that it is the greatest lower bounded of $\{A_i\}_{i\in I}$. These imply that $H(S)$ is a complete lattice.

Finally, let $A, B, C \in H(S)$ such that $A \wedge B = A \wedge C$, $A \vee B = A \vee C$, and $B \subseteq C$. Let $x \in C$. Then $x \in C \subseteq A \vee C = A \vee B = [A+B]_h$. It follows that there exist $a_1, a_2 \in A$, $b_1, b_2 \in B$ and $s \in S$ such that $x + a_1 + b_1 + s = a_2 + b_2 + s$. Then

$$\begin{aligned} x + a_1 + a_1' + b_1 + s &= x + a_1 + b_1 + s + a_1' \\ &= a_2 + b_2 + s + a_1' \\ &= a_2 + a_1' + b_2 + s. \end{aligned} \tag{5}$$

Since, $x \in C$, $a_1 + a_1' \in E^+ \subseteq C$ and $b_1 \in B \subseteq C$, we have $x + a_1 + a_1' + b_1 \in C$. Using (5) and $b_2 \in B \subseteq C$, we get $a_2 + a_1' \in [C]_h = C$. At this point, $a_1 + a_1', a_2 + a_1' \in A \cap C = A \wedge C = A \wedge B = A \cap B \subseteq B$. It follows that $a_1+a_1'+b_1 \in B$ and $a_2+a_1'+b_2 \in B$. Using (5) again, we obtain that $x \in [B]_h = B$ and so $C \subseteq B$. Hence, $B = C$. By Lemma 2.3, $H(S)$ is a modular lattice. ■

4. Ternary ring congruences

In this section, we characterize a ternary ring congruence with respect to a full k-ideal of an additively inverse ternary semirings.

Definition 4.1. A binary relation ρ on a ternary semiring $\langle S; +, f\rangle$ is said to be a *congruence* if ρ is an equivalence relation on S and satisfies the following properties; for any $a, b, x, y \in S$, $(a, b) \in \rho$ implies $(a + x, b + x)$, (axy, bxy), (xay, xby), $(xya, xyb) \in \rho$.

Definition 4.2. A ternary semiring $\langle S; +, f\rangle$ is called a *ternary ring* if $\langle S; +\rangle$ is a group. In other words, the following conditions are satisfied.

(i) There exists $0 \in S$ such that $x + 0 = x = 0 + x$ for all $x \in S$.

(ii) For each $x \in S$, there is $y \in S$ such that $x + y = 0 = y + x$.

If $\langle S; +, f\rangle$ is a ternary ring, then the element y in (2) is usually denoted by $-x$.

Definition 4.3. A congruence ρ on a ternary semiring S is called a *ternary ring congruence* if the quotient ternary semiring $S/\rho := \{a\rho \mid a \in S\}$ is a ternary ring.

Theorem 4.4. *Let A be a full k-ideal of an additively inverse ternary semiring S. Then the relation*

$$\rho_A = \{(a, b) \in S \times S \mid a + b' \in A\}$$

is a ternary ring congruence such that $-(a\rho_A) = a'\rho_A$.

Proof. Let A be a full k-ideal of S. Firstly, we show that ρ is an equivalence relation on S. Let $a, b, c \in S$. Since $a + a' \in E^+ \subseteq A$, $(a, a) \in \rho_A$. Thus, ρ_A is reflexive. If $(a, b) \in \rho_A$, then $a + b' \in A$. By Remark 3.6, we get $b + a' = (b')' + a' = (b' + a)' = (a + b')' \in A$ and so $(b, a) \in \rho_A$. Thus, ρ_A is symmetric. Assume that $(a, b), (b, c) \in \rho_A$. It follows that $a + b' \in A$ and $b + c' \in A$. Then $a + c' + b + b' \in A$. Since $b + b' \in E^+ \subseteq A$, $a + c' \in [A]_k = A$. So, $(a, c) \in \rho_A$ and thus ρ_A is transitive. Now, ρ_A is an equivalence relation.

Secondly, let $a, b, x, y \in S$. Assume that $(a, b) \in \rho_A$ and so $a + b' \in A$. Then

$$(a + x) + (b + x)' = a + x + b' + x' = (a + b') + (x + x') \in A + E^+ \subseteq A + A \subseteq A.$$

Hence, $(a + x, b + x) \in \rho_A$. Using Lemma 2.2, we obtain that

$$axy + (bxy)' = axy + b'xy = (a + b')xy \in ASS \subseteq A.$$

Hence, $(axy, bxy) \in \rho_A$. Analogously, we are able to obtain that (xay, xby), $(xya, xyb) \in \rho_A$. Now, ρ_A is a congruence on S.

Finally, we show that S/ρ_A is a ternary ring together with the operations $\oplus$ and F on S/ρ_A defined by $a\rho_A \oplus b\rho_A = (a+b)\rho_A$ and $F(a\rho_A, b\rho_A, c\rho_A) = (abc)\rho_A$ for any $a, b, c \in S$. It is immediately to obtain that $\langle S/\rho_A; \oplus, F\rangle$ is a quotient ternary semiring of $\langle S; +, f\rangle$. Let $e \in E^+$ and $x \in S$. Then $(e + x) + x' = e + (x + x') \in E^+ + E^+ = E^+ \subseteq A$ and so $(e + x, x) \in \rho_A$. It follows that

$$e\rho_A \oplus x\rho_A = (e + x)\rho_A = x\rho_A.$$

Since $e + (x + x')' = e + x' + x \in A$, $(e, x + x') \in \rho_A$. It turns out that

$$x\rho_A \oplus x'\rho_A = (x + x')\rho_A = e\rho_A.$$

Therefore, S/ρ_A is a ternary ring. ∎

Theorem 4.5. *Let ρ be a congruence on an additively inverse ternary semiring S such that S/ρ is a ternary ring. Then there exists a full k-ideal A of S such that $\rho_A = \rho$.*

Proof. Let $A = \{a \in S \mid (a, e) \in \rho$ for some $e \in E^+\}$. Since ρ is reflexive, $E^+ \subseteq A \neq \emptyset$. Let $a, b \in A$. Then there exist $e, f \in E^+$ such that $(a, e) \in \rho$ and $(b, f) \in \rho$. Then $(a + b, e + f) \in \rho$ and $e + f \in E^+$. Hence $a + b \in A$ and thus $A + A \subseteq A$. If $x \in SSA$, then $x = stc$ for some $s, t \in S$ and $c \in A$ such that $(c, g) \in \rho$ for some $g \in E^+$. It follows that $(x, stg) = (stc, stg) \in \rho$. Since E^+ is an ideal of S, $stg \in SSE^+ \subseteq E^+$. So, $x \in A$ leads to $SSA \subseteq A$. Similarly, we are able to obtain that $SAS \subseteq A$ and $ASS \subseteq A$. Now, A is an ideal of S.

Let $x \in [A]_k$. Then $x + a = b$ for some $a, b \in A$ where $(a, e) \in \rho$ and $(b, f) \in \rho$ for some $e, f \in E^+$. However, $f\rho$ and $e\rho$ are additively idempotent in the ternary ring S/ρ. This implies that $e\rho = f\rho$ is the zero element of S/ρ. It follows that $f\rho = b\rho = (x + a)\rho = x\rho \oplus a\rho = x\rho \oplus e\rho = x\rho$. Thus, $(x, f) \in \rho$ where $f \in E^+$. Thus, $x \in A$ and so $[A]_k = A$. By Corollary 3.9(i), A is a full k-ideal of S.

Finally, we show that $\rho = \rho_A$. Let $(a, b) \in \rho$. Then $(a + b', b + b') \in \rho$. Since $b + b' \in E^+$, $a + b' \in A$ and thus $(a, b) \in \rho_A$. Hence, $\rho \subseteq \rho_A$. If $(a, b) \in \rho_A$, then $a + b' \in A$. Thus, $(a + b', e) \in \rho$ for some $e \in E^+$. We have that $b\rho = e\rho \oplus b\rho = (a + b')\rho \oplus b\rho = a\rho \oplus b'\rho \oplus b\rho = a\rho \oplus (b + b')\rho = a\rho$, since $b + b' \in E^+$. This shows that $(a, b) \in \rho$ and so $\rho_A \subseteq \rho$. Therefore, $\rho = \rho_A$. ∎

We note that the concepts of full k-ideals and h-ideals of an additively inverse ternary semiring are coincidence as the following remark.

Remark 4.6. The concepts of full k-ideals and h-ideals of an additively inverse ternary semiring are coincidence.

Proof. We immediately obtain that every h-ideal is a full k-ideal. Let A be a full k-ideal. By Theorem 4.4, we obtain that S/ρ is a ternary ring and A is its zero element. Let $x \in S$ and $x + a + s = b + s$ for some $a, b \in A$, $s \in S$. Then $x\rho + a\rho + s\rho = b\rho + s\rho$ and so $x\rho + 0 + s\rho = 0 + s\rho$. Hence, $x\rho = 0$ implies $x \in A$. Therefore, A is an h-ideal. ∎

5. Conclusion and discussion

The notions of a k-ideal and a full k-ideal of a ternary semiring were defined in Section 3. There is a k-ideal which is not full as it is shown by Example 3.3.

However, every h-ideal of a ternary semiring is immediately full. Moreover, h-ideals and full k-ideals are coincidence in an additively inverse ternary semiring and the set of all of them forms a complete lattice and also a modular lattice.

A group (ring) congruence is such a congruence relation on a semigroup (semiring) that the quotient semigroup (semiring) is a group (ring). Similarly, a ternary ring congruence is such a congruence relation on a ternary semiring that the quotient ternary semiring is a ternary ring. Constructing a relation with respect to a full k-ideal of an additively inverse ternary semiring is a way to obtain a ternary ring congruence.

We claim that all results of this work are also true for an n-ary semiring for any $n \geq 3$. However, some basic properties of an additively inverse n-ary semiring have to be defined and investigated.

References

[1] M.R. Adhikari, Basic Algebraic Topology and its Applications (New Delhi, Springer Publication, 2006).
https://doi.org/10.1007/978-81-322-2843-1

[2] M.R. Adhikari and A. Adhikari, Basic Modern Algebra with Applications (New Delhi, Springer Publication, 2014).
https://doi.org/10.1007/978-81-322-1599-8

[3] D.B. Benson, Bialgebras: some foundations for distributed and concurrent computation (Washington DC, Computer Science Department, Washington State University, 1987).
https://doi.org/10.3233/FI-1989-12402

[4] J.H. Conway, Regular Algebra and finite Machines (London, Chapman and Hall, 1971).

[5] T.K. Dutta and S. Kar, *On regular ternary semirings*, in: Advances in Algebra, Proceeding of ICM Satellite Conference in Algebra and Related Topics, (World Scientific, 2003) 343–355.
https://doi.org/10.1142/9789812705808_0027

[6] T.K. Dutta and S. Kar, *On the Jacobson radical of a ternary semiring*, Southeast Asian Bull. Math. **28(1)** (2004) 1–13.

[7] T.K. Dutta and S. Kar, *A note on the Jacobson radical of a ternary semiring*, Southeast Asian Bull. Math. **29(4)** (2004) 677–687.

[8] K. Glazek, A Guide to Literature on Semirings and their Applications in Mathematics and Information Sciences with Complete Bibliography (Dodrecht, Kluwer Academic Publishers, 2002).
https://doi.org/http://dx.doi.org/10.1007/978-94-015-9964-1

[9] J.S. Golan, Semirings and their Applications (Dodrecht, Kluwer Academic Publishers, 1999).
https://doi.org/http://dx.doi.org/10.1007/978-94-015-9333-5

[10] M. Henriksen, *Ideals in semirings with commutative addition*, Amer. Math. Soc. Notices. **6** (1958) 321.

[11] W. Kuich and W. Salomma, Semirings, Automata, Languages (Berlin, Springer Verlag, 1986).
https://doi.org/10.1007/978-3-642-69959-7

[12] D.H. Lehmer, *A ternary analogue of abelian groups*, American J. Math. **59** (1932) 329–338.
https://doi.org/10.2307/2370997

[13] W.G. Lister, *Ternary rings*, Trans. Amer. Math. Soc. **154** (1971) 37–55.
https://doi.org/10.2307/1995425

[14] M.L. Santiago, *Regular ternary semigroups*, Bull. Calcutta Math. Soc. **82** (1990) 67–71.

[15] M.K. Sen and M.R. Adhikari, *On k-ideals of semirings*, Internat. J. Math. & Math. Sci. **15(2)** (1992) 347–350.
https://doi.org/10.1155/S0161171292000437

[16] F.M. Sioson, *Ideal theory in ternary semigroups*, Math. Jpn. **10** (1965) 63–84.

[17] H.S. Vandiver, *Note on a simple type of algebra in which cancellation law of addition does not hold*, Bull. Amer. Math. Soc. **40** (1934) 914–920.
https://doi.org/10.1090/S0002-9904-1934-06003-8

Received 1 May 2022
Revised 19 July 2022
Accepted 9 September 2022