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π-INVERSE ORDERED SEMIGROUPS

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Abstract

This article deals with the generalization of π-inverse semigroups without order to ordered semigroups. Here we characterize π-inverse ordered semigroups by their ordered idempotents and bi-ideals.

Keywords: bi-ideals, ordered idempotent, π-regular, π-inverse, inverse.

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1. Introduction

A semigroup $(S, \cdot)$ with an order relation $\leq$ is called an ordered semigroup [2, 7] if for all $a, b, x \in S$, $a \leq b$ implies $xa \leq xb$ and $ax \leq bx$. It is denoted by $(S, \cdot, \leq)$. Let $(S, \cdot, \leq)$ be an ordered semigroup. For a subset A of S, let $(A] = \{x \in S : x \leq a, \text{ for some } a \in A\}$.

An element a of S is said to be regular (completely regular) [9] if there exists $x \in S$ such that $a \leq axa$ $(a \leq a^2xa^2)$. S is called a regular (completely regular) ordered semigroup if every element of S is regular (completely regular). Note that S is regular (completely regular) if and only if $a \in (aSa]$ $(a \in (a^2Sa^2])$ for all $a \in S$.

An element $b \in S$ is called an inverse [5] of a if $a \leq aba$ and $b \leq bab$. The set of all inverses of an element $a \in S$ is denoted by $V_{\leq}(a)$. a', a'' are the inverse of a unless otherwise stated.

An element $e \in S$ is said to be an ordered idempotent if $e \leq e^2$. The set of all ordered idempotents of S is denoted by $E_{\leq}(S)$.

Bhuniya and Hansda [1] studied the ordered semigroups in which any two inverses of an element are $\mathcal{H}$-related. Class of these ordered semigroups are natural generalization of the class of all inverse semigroups. Hansda and Jamadar [5] named these ordered semigroups as inverse ordered semigroups and studied their different aspects. In this paper, we further extend inverse ordered semigroups to π-inverse ordered semigroups.

A nonempty subset A of S is called a left (right) ideal [8] of S, if $SA \subseteq A$ ($AS \subseteq A$) and $(A] = A$. A nonempty subset A is called a (two-sided)ideal of S if it is both a left and a right ideal of S. Following Kehayopulu [9], a nonempty subset B of an ordered semigroup S is called a bi-ideal of S if $BSB \subseteq B$ and $(B] = B$. Hansda [4] studied algebraic properties of bi-ideals in completely regular and Clifford ordered semigroups.

The principal [8] left ideal, right ideal, ideal and bi-ideal [9] generated by $a \in S$ are denoted by $L(a)$, $R(a)$, $I(a)$and $B(a)$ respectively. It is easy to show that

$$L(a) = (a \cup Sa],\ R(a) = (a \cup aS],\ I(a) = (a \cup Sa \cup aS \cup SaS]\ and\ B(a) = (a \cup aSa].$$

Kehayopulu [8] defined Green's relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{J}$ and $\mathcal{H}$ on an ordered semigroup S as follows

$$a\mathcal{L}b\ if\ L(a) = L(b),\ a\mathcal{R}b\ if\ R(a) = R(b), a\mathcal{J}b\ if\ I(a) = I(b) \text{ and } \mathcal{H} = \mathcal{L} \cap \mathcal{R}.$$

These four relations are equivalence relations on S.

An ordered semigroup S is called π-regular (resp. completely π-regular) [3] if for every $a \in S$ there is $m \in \mathbb{N}$ such that $a^m \in (a^m S a^m]$ (resp. $a^m \in (a^{2m} S a^{2m}]$). The set of all regular, completely regular, inverse and π-regular elements in an ordered semigroup S is denoted by $Reg_{\leq}(S)$, $Gr_{\leq}(S)$,
$V_{\leq}(S)$ and $\pi Reg_{\leq}(S)$ respectively.

Let S be an ordered semigroup and ρ be an equivalence relation on S. Following Hansda and Jamadar [5], an element $a \in S$ of type τ is said to be a ρ-unique element in S if for every other element $b \in S$ of type τ we have $a\rho b$.

Theorem 1 [5]. *The following conditions are equivalent on an ordered semigroup S.*

1. *S is an inverse ordered semigroup;*
2. *S is regular and its idempotents are $\mathcal{H}$-commutative;*
3. *For every $e, f \in E_{\leq}(S)$, $e\mathcal{L}f(e\mathcal{R}f)$ implies $e\mathcal{H}f$.*

2. π-INVERSE ORDERED SEMIGROUP

This section deals with the characterization of the class of π-inverse ordered semigroups.

Let S be a π-regular ordered semigroup. Then for every $a \in S$ there is $m \in \mathbb{N}$ such that $a^m \leq a^m x a^m \leq a^m(xa^m x)a^m$ and $xa^m x \leq xa^m x(a^m)xa^m x$. Thus for every $a \in S$ there is $m \in \mathbb{N}$ such that $V_{\leq}(a^m) \neq \phi$.

Definition. A π-regular ordered semigroup S is called π-inverse if for every $a \in S$, there is $m \in \mathbb{N}$ such that any two inverses of a^m are $\mathcal{H}$-related.

For $a \in S$, there is $m \in \mathbb{N}$ such that every principal left ideal and every principal right ideal generated by a^m in a π-inverse ordered semigroup have $\mathcal{H}$-unique ordered idempotent generator. This has been shown in the following theorem.

Theorem 2. *A π-regular ordered semigroup S is π-inverse if and only if for every $a \in S$ there is $m \in \mathbb{N}$ such that $(Sa^m]$ and $(a^mS]$ are generated by an $\mathcal{H}$-unique ordered idempotent.*

Proof. Suppose that S is π-inverse. Let $a \in S$. Since S is π-regular, there is $m \in \mathbb{N}$ such that $a^m \leq a^m z a^m$ for some $z \in S$. Let $I = (Sa^m]$. Then clearly $I = (Sa^m z a^m] = (Se]$, where $e = za^m \in E_{\leq}(S)$. If possible let $I = (Sf]$ for some $f \in E_{\leq}(S)$. Then $e\mathcal{L}f$ and so $e \leq xf$ and $f \leq ye$ for some $x, y \in S$. Now $e \leq ee \leq eee \leq exfe$. Therefore $exf \leq exfexf$ so that $exf \in E_{\leq}(S)$. Also $exf \leq exfexf \leq exf(fe)exf$ and $fe \leq feee \leq fexfe \leq fe(exf)fe$. Therefore $fe \in V_{\leq}(exf)$. Also $exf \in V_{\leq}(exf)$. Since S is π-inverse for $fe, exf \in V_{\leq}(exf)$ we have $fe\mathcal{H}exf$. Then $e \leq ee \leq eee \leq exfe \leq exffe \leq fett_1exf$ for some $t, t_1 \in S$ and so $e \leq fz_1$, where $z_1 = ett_1exf$. Similarly $f \leq ez_2$ for some $z_2 \in S$. So $e\mathcal{R}f$. Hence $e\mathcal{H}f$. Likewise $(a^mS]$ is generated by an $\mathcal{H}$-unique ordered idempotent.

Conversely assume that given condition holds in S. Then S is π-regular. Let $a \in S$ and $a', a'' \in V_{\leq}(a^m)$ for some $m \in \mathbb{N}$. Clearly $(Sa^m] = (Sa'a^m] = (Sa''a^m]$. Since $a'a^m, a''a^m \in E_{\leq}(S)$ we have that $a'a^m\mathcal{H}a''a^m$, by given condition. Then there are $s, v \in S$ such that $a' \leq a'a^m a' \leq a''a^m sa'$ and $a'' \leq a'a^m va''$. Thus $a'\mathcal{R}a''$. Likewise $a'\mathcal{L}a''$, that is $a'\mathcal{H}a''$. Hence S is a π-inverse ordered semigroup. ■

The following theorem shows some equivalent conditions for an ordered semigroup S to be π-inverse.

Theorem 3. *The following conditions are equivalent on an ordered semigroup S.*

1. *S is a π-inverse ordered semigroup;*
2. *S is π-regular and for every $e, f \in E_{\leq}(S)$, there is $m \in \mathbb{N}$ such that $(ef)^m \in (fSe]$;*
3. *S is π-regular and for every $e, f \in E_{\leq}(S)$, $e\mathcal{L}f(e\mathcal{R}f)$ implies $e\mathcal{H}f$.*

Proof. (1) $\Rightarrow$ (2) First suppose S is π-inverse. Then S is π-regular. Let $e, f \in E_{\leq}(S)$. Since S is π-regular, for $ef \in S$ there is $x \in S$ such that $x \in V_{\leq}(ef)^m$ for some $m \in \mathbb{N}$. We consider the following cases.

Case 1. If $m = 1$ then $ef \in (fSe]$ holds, by Theorem 1.

Case 2. If $m > 1$ then $x \leq x(ef)^m x$ implies that $fxe \leq fxe(ef)^m fxe$. Also $(ef)^m \leq (ef)^m x(ef)^m$ implies that $(ef)^m \leq (ef)^m (fxe)(ef)^m$. Thus $(ef)^m \in V_{\leq}(fxe)$. Now $x \leq x(ef)^m x = xe(fe)^{m-1} fx$ so that $fxe \leq fxe(fe)^{m-1} fxe \leq fxe(fe)^{m-1} fxe(fe)^{m-1} fxe$ and $(fe)^{m-1} fxe(fe)^{m-1} \leq (fe)^{m-1} fxe(fe)^{m-1} fxe$ $(fe)^{m-1} \leq (fe)^{m-1} fxe(fe)^{m-1} fxe(fe)^{m-1} fxe\ (fe)^{m-1}$. This gives $(fe)^{m-1}$ $fxe(fe)^{m-1} \in V_{\leq}(fxe)$. Thus $(ef)^m, (fe)^{m-1} fxe(fe)^{m-1} \in V_{\leq}(fxe)$. Since S is π-inverse, we have that $(fe)^{m-1} fxe(fe)^{m-1} \mathcal{H} (ef)^m$. Then there are $s_1, s_2 \in S$ such that $(ef)^m \leq (fe)^{m-1} fxe(fe)^{m-1} s_1$ and $(ef)^m \leq s_2 (fe)^{m-1} fxe(fe)^{m-1}$. Thus from the inequality $(ef)^m \leq (ef)^m x(ef)^m$ we have that $(ef)^m \leq (fe)^{m-1}$ $fxe(fe)^{m-1}\ s_1 x s_2 (fe)^{m-1} fxe(fe)^{m-1} \leq f(fe)^{m-1} fxe(fe)^{m-1} s_1 x s_2 (fe)^{m-1}$ $fxe(fe)^{m-1} e$. Therefore $(ef)^m \leq fye$, where $y = (fe)^{m-1} fxe(fe)^{m-1}\ s_1 x s_2$ $(fe)^{m-1} fxe(fe)^{m-1} \in S$. Hence $(ef)^m \in (fSe]$.

(2) $\Rightarrow$ (3) Let $e, f \in E_{\leq}(S)$ be such that $e\mathcal{L}f$. Then $e \leq xf$ and $f \leq ye$ for some $x, y \in S$. Now $e \leq xf$ implies $e \leq exf$ and so $e \leq ee \leq exfe$, which implies that $exf \leq exfexf$. So $exf \in E_{\leq}(S)$. Similarly$f \leq fye$ and $fye \in E_{\leq}(S)$. Now

$$e \leq exf \leq exff \leq (exf)(fye). \tag{1}$$

Since $exf, fye \in E_{\leq}(S)$, there exists $m \in \mathbb{N}$ such that $(exffye)^m \in ((fye)S(exf)]$, by condition (2). Then there exists $z \in S$ such that $(exffye)^m \leq (fye)z(exf)$. Thus $e \leq e^m$ together with (1) implies that $e \leq (exffye)^m$ and therefore $e \in ((fye)S(exf)] \subseteq (fS]$. Likewise $f \in (eS]$, that is, $e\mathcal{R}f$. Hence $e\mathcal{H}f$.

For $e\mathcal{R}f$, $e\mathcal{H}f$ follows dually.

(3) $\Rightarrow$ (1) Let $a \in S$ and $a', a'' \in V_{\leq}(a^m)$ for some $m \in \mathbb{N}$. Now $a^m a' \leq a^m a'' a^m a'$ and $a^m a'' \leq a^m a' a^m a''$ which gives $a^m a' \mathcal{R} a^m a''$ so that $a^m a' \mathcal{H} a^m a''$, by the condition (3). Likewise $a' a^m \mathcal{H} a'' a^m$. Then $a' \leq a' a^m a'$ gives that $a' \leq a'' a^m x a^m$ for some $x \in S$. Therefore $a' \leq a'' t$ where $t = a^m x a^m$. In a similar manner it is possible to get $u, v, w \in S$ such that $a' \leq u a''$, $a'' \leq a' v$ and $a'' \leq w a'$. So $a' \mathcal{H} a''$. Hence S is a π-inverse ordered semigroup. ■

Let S be a π-regular ordered semigroup. Then for every $a \in S$ there is $m \in \mathbb{N}$ such that $a^m \leq a^m x a^m$ for some $x \in S$ which gives that $a^m \leq a^m x (a^m) x a^m$. Here $a^m x, x a^m \in E_{\leq}(S)$ so that $a^m \in (eSf]$, for $e = a^m x$ and $f = x a^m$.

Following this idea we find a condition for a π-regular ordered semigroup to be π-inverse.

Theorem 4. *A π-regular ordered semigroup S is π-inverse if and only if for every $e, f \in E_{\leq}(S)$ and $x \in S$ whenever $x^m \in (eSf]$ for some $m \in \mathbb{N}$, then $x' \in (fSe]$ for every $x' \in V_{\leq}(x^m)$.*

Proof. First suppose that S is a π-inverse ordered semigroup. Then there is $m \in \mathbb{N}$ such that $V_{\leq}(x^m) \neq \phi$. Let $x' \in V_{\leq}(x^m)$. Suppose $x^m \in (eSf]$ for $e, f \in E_{\leq}(S)$. Then $x^m \leq e s_1 f$ for some $s_1 \in S$. Now $x' \leq x' x^m x' \leq x' e s_1 f x'$

and so $es_1fx' \leq es_1fx'es_1fx'$, that is $es_1fx' \in E_{\leq}(S)$. Similarly $x'es_1f \in E_{\leq}(S)$. Therefore $x' \leq x'(es_1fx')^r$ and $x' \leq (x'es_1f)^r x'$ for all $r \in \mathbb{N}$. Now since S is π-inverse, for $f, x'es_1f \in E_{\leq}(S)$ there are $s_2 \in S$ and $n \in \mathbb{N}$ such that $(x'es_1ff)^n \leq fs_2x'es_1f$, by Theorem 3(2). Similarly for $e, es_1fx' \in E_{\leq}(S)$ we have $(ees_1fx')^k \leq es_1fx's_3e$, for some $s_3 \in S$ and $k \in \mathbb{N}$. Then $x' \leq x'x^mx'$ implies that $x' \leq (x'es_1ff)^n x'(ees_1fx')^k \leq fs_2x'es_1x'fes_1fx's_3e$. Hence $x' \in (fSe]$.

Conversely, assume that the given condition holds in S. Let $e, f \in E_{\leq}(S)$ be such that $e\mathcal{L}f$, this yields that $e \leq ee \leq ezf$ for some $z \in S$. Therefore $e^m \in (eSf]$. Since $e \in V_{\leq}(e^m)$ we have $e \in (fSe]$, by given condition. Likewise $f \in (eSf]$. This implies that $e\mathcal{R}f$ and so $e\mathcal{H}f$. Thus by Theorem 3, S is a π-inverse ordered semigroup. ■

Corollary 5. *The following conditions are equivalent on a π-regular ordered semigroup S.*

1. *S is a π-inverse ordered semigroup;*
2. *Let $a \in S$. Then there are $m, n \in \mathbb{N}$ such that $(a^ma'a'a^m)^n \in (a'Sa']$, for every $a' \in V_{\leq}(a^m)$;*
3. *Any two inverses of an ordered idempotent in S are $\mathcal{H}$-related;*
4. *All inverses of e are $\mathcal{H}$-commutative, for every $e \in E_{\leq}(S)$;*
5. *For any $e \in E_{\leq}(S)$ and $e' \in V_{\leq}(e)$, $ee'e'e \in (e'Se']$.*

Proof. (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3), (3) $\Rightarrow$ (4) These are obvious.

(4) $\Rightarrow$ (5) Let $e \in E_{\leq}(S)$ and $e' \in V_{\leq}(e)$. Then $ee'e'e \leq e's_1ees_2e'$ for some $s_1, s_2 \in S$. Hence $ee'e'e \in (e'Se']$.

(5) $\Rightarrow$ (1) Let $a \in S$ and $a', a'' \in V_{\leq}(a^m)$ for some $m \in \mathbb{N}$. Then $a' \leq a'a^ma' \leq a'a^ma''a^ma' \leq a''a^ms_4a'a^ma'$, for some $s_4 \in S$. Therefore $a' \leq a''t_1$ where $t_1 = a^ms_4a'a^ma'$. Similarly there exists $t_2 \in S$ such that $a' \leq t_2a''$. Also there are $t_3, t_4 \in S$ such that $a'' \leq t_3a'$ and $a'' \leq a't_4$. Thus $a'\mathcal{H}a''$. Hence S is a π-inverse ordered semigroup. ■

Corollary 6. *Let S be a π-inverse ordered semigroup and $a, b \in S$. If $m, n \in \mathbb{N}$ are such that $V_{\leq}(a^m)$, $V_{\leq}(b^n) \neq \phi$, then the following statements hold in S.*

1. *$a^m\mathcal{L}b^n$ if and only if $a'a^m\mathcal{H}b'b^n$ for every $a' \in V_{\leq}(a^m)$ and $b' \in V_{\leq}(b^n)$;*
2. *$a^m\mathcal{R}b^n$ if and only if $a^ma'\mathcal{H}b^nb'$ for every $a' \in V_{\leq}(a^m)$ and $b' \in V_{\leq}(b^n)$;*
3. *$a^m\mathcal{H}b^n$ if and only if $a'a^m\mathcal{H}b'b^n$ and $a^ma'\mathcal{H}b^nb'$ for every $a' \in V_{\leq}(a^m)$ and $b' \in V_{\leq}(b^n)$.*

Proof. (1) Let $a, b \in S$. Since S is π-inverse, there are $m, n \in \mathbb{N}$ such that $V_{\leq}(a^m)$, $V_{\leq}(b^n) \neq \phi$. Let $a' \in V_{\leq}(a^m)$, $b' \in V_{\leq}(b^n)$. Let $a^m\mathcal{L}b^n$. Since $a^m \leq$

$a^m a' a^m$ and $a' a^m \leq a' a^m a' a^m$, we have $a^m \mathcal{L} a' a^m$, which implies that $b^n \mathcal{L} a' a^m$. Also $b^n \mathcal{L} b' b^n$. Hence $a' a^m \mathcal{L} b' b^n$. Since $a' a^m, b' b^n \in E_{\leq}(S)$ and S is π-inverse we have $a' a^m \mathcal{H} b' b^n$, by Theorem 3(3).

Conversely suppose that given condition holds in S. Let $a, b \in S$ with $a' \in V_{\leq}(a^m)$ and $b' \in V_{\leq}(b^n)$ for some $m, n \in \mathbb{N}$. Then by given condition $a' a^m \mathcal{H} b' b^n$. Also we have $a^m \mathcal{L} a' a^m$ and $b^n \mathcal{L} b' b^n$ so that $a^m \mathcal{L} b^n$.

(2) and (3) These follow dually. ■

3. Bi-ideals in π-inverse ordered semigroups

In this section we characterize a π-inverse ordered semigroup S by the principal bi-ideals of S.

Theorem 7. *Let S be a π-regular ordered semigroup. Then the following conditions are equivalent.*

1. *S is a π-inverse ordered semigroup;*
2. *For any $a \in S$, there is $m \in \mathbb{N}$ such that $B(a') = B(a'')$ for every $a', a'' \in V_{\leq}(a^m)$;*
3. *For any $e, f \in E_{\leq}(S)$, $B((ef)^m) \subseteq B(e) \cap B(f)$ for some $m \in \mathbb{N}$;*
4. *For any $e \in E_{\leq}(S)$ and $x \in V_{\leq}(e)$, $B(ex) = B(xe)$.*

Proof. (1) $\Rightarrow$ (2) First suppose that S is a π-inverse ordered semigroup. Let $a \in S$. Then there is $m \in \mathbb{N}$ such that $a', a'' \in V_{\leq}(a^m)$. Suppose $x \in B(a')$. Therefore $x \leq a'$ or $x \leq a' y a'$ for some $y \in S$. Since S is π-inverse, $a' \mathcal{H} a''$. If $x \leq a'$ then $x \leq a' a^m a' \leq a'' s_1 a^m s_2 a''$ for some $s_1, s_2 \in S$. Therefore $x \leq a'' s a''$ where $s = s_1 a^m s_2$. If $x \leq a' y a'$ then there is $s_3 \in S$ such that $x \leq a'' s_3 a''$. Thus in either case $x \in B(a'')$. Also $a' \in B(a'')$ implies that $B(a') \subseteq B(a'')$. Similarly $B(a'') \subseteq B(a')$. Hence $B(a') = B(a'')$.

(2) $\Rightarrow$ (3) Let $e, f \in E_{\leq}(S)$ and $x \in V_{\leq}(ef)^m$ for some $m \in \mathbb{N}$. Clearly $(ef)^m, (fe)^{m-1} fxe (fe)^{m-1} \in V_{\leq}(fxe)$ and so by the condition (2) it follows that $B((ef)^m) = B((fe)^{m-1} fxe (fe)^{m-1})$. Now $(ef)^m \in B((fe)^{m-1} fxe (fe)^{m-1})$ implies $(ef)^m \leq (fe)^{m-1} fxe (fe)^{m-1}$ or $(ef)^m \leq (fe)^{m-1} fxe (fe)^{m-1} h (fe)^{m-1} fxe (fe)^{m-1}$ for some $h \in S$. So in either case $(ef)^m \leq h_1 (fe)^{m-1} fxe (fe)^{m-1}$ and $(ef)^m \leq (fe)^{m-1} fxe (fe)^{m-1} h_2$ for some $h_1, h_2 \in S$. Likewise there are $h_3, h_4 \in S$ such that $(fe)^{m-1} fxe (fe)^{m-1} \leq h_3 (ef)^m$ and $(fe)^{m-1} fxe (fe)^{m-1} \leq (ef)^m h_4$. Hence $(ef)^m \mathcal{H} (fe)^{m-1} fxe (fe)^{m-1}$.

Let $w \in B(ef)^m$. Then either $w \leq (ef)^m$ or $w \leq (ef)^m s_1 (ef)^m$ for some $s_1 \in S$. If $w \leq (ef)^m$ then $w \leq (ef)^m \leq (ef)^m x (ef)^m \leq (ef)^m x s_2 (fe)^{m-1} fxe (fe)^{m-1}$ for some $s_2 \in S$.

Also $w \leq (ef)^m s_1 (ef)^m$ gives $w \leq efs_1s_3(fe)^{m-1}fxe(fe)^{m-1}$ for some $s_3 \in S$. So in either case $w \in B(e)$. Likewise $w \in B(f)$. Therefore $w \in B(e) \cap B(f)$ and hence $B(ef)^m \subseteq B(e) \cap B(f)$.

(3) $\Rightarrow$ (4) Let $e \in E_{\leq}(S)$ and $x \in V_{\leq}(e)$. Then $e, xe, ex \in E_{\leq}(S)$. Now by condition (3) $B((exe)^m) \subseteq B(e) \cap B(xe)$ for some $m \in \mathbb{N}$. Let $y \in B(e)$. Then either $y \leq e$ or $y \leq es_3e$ for some $s_3 \in S$. If $y \leq e$ then $y \leq exe \leq eexe \leq exeexe \leq \cdots \leq (exe)^m$. So $y \in B((exe)^m)$. Likewise $y \in B((exe)^m)$ for the case $y \leq es_3e$. Therefore $B(e) = B((exe)^m)$ and so $B(e) \subseteq B(xe)$. Also $B((xee)^n) \subseteq B(e) \cap B(xe)$ for some $n \in \mathbb{N}$, then by a similar argument $B(xe) \subseteq B(e)$. Therefore $B(e) = B(xe)$. Likewise $B(e) = B(ex)$. Therefore $B(xe) = B(ex)$.

(4) $\Rightarrow$ (1) By condition (4) we have $ex\mathcal{H}xe$. Also $ex \in B(e)$ and $ex \in B(x)$. Then $ex \leq e$ or $ex \leq eb_1e$ and $ex \leq x$ or $ex \leq xb_2x$ for some $b_1, b_2 \in S$. Here following cases arise.

Case 1. If $ex \leq e$ and $ex \leq x$, then $ex \leq exex \leq xe \leq xexe = xae$ where $a = ex$.

Case 2. If $ex \leq e$ and $ex \leq xb_2x$, then $ex \leq exex \leq xb_2xe = xbe$ where $b = b_2x$.

Case 3. If $ex \leq eb_1e$ and $ex \leq x$, then $ex \leq exex \leq xeb_1e = xce$ where $c = eb_1$.

Case 4. If $ex \leq eb_1e$ and $ex \leq xb_2x$, then $ex \leq exex \leq xb_2xeb_1e = xde$ where $d = b_2xeb_1$. Therefore in either case $ex \leq xse$ for some $s \in S$. Similarly $xe \leq etx$ for some $t \in S$. Thus e, x are $\mathcal{H}$-commutative. Hence by Corollary 5, S is a π-inverse ordered semigroup. ■

Corollary 8. *A π-regular ordered semigroup S is π-inverse if and only if for any $e \in E_{\leq}(S)$ and $x \in V_{\leq}(e)$, $B(ex) = B(e) \cap B(x) = B(xe) = B(e) = B(x)$.*

Proof. This follows from Theorem 7. ■

Corollary 9. *A π-regular ordered semigroup S is π-inverse if and only if for any $e, f \in E_{\leq}(S)$, $e\mathcal{L}f(e\mathcal{R}f)$ implies $B(e) = B(f)$.*

Proof. Let S be a π-inverse ordered semigroup. Since S is π-inverse $e\mathcal{L}f(e\mathcal{R}f)$ implies $e\mathcal{H}f$ by Theorem 3. So it is easy to check that $B(e) = B(f)$.

Conversely suppose that the condition holds in S. Now $B(e) = B(f)$ gives that $e \in B(f)$ and $f \in B(e)$. Therefore $e \leq f$ or $e \leq fxf$ and $f \leq e$ or $f \leq eye$ for some $x, y \in S$. In either case $e\mathcal{R}f$. So $e\mathcal{L}f$ implies $e\mathcal{H}f$. Hence S is a π-inverse ordered semigroup, by Theorem 3. ■

Corollary 10. *Let S be a π-inverse ordered semigroup and $a, b \in S$. If $a' \in V_{\leq}(a^m)$, $b' \in V_{\leq}(b^n)$, for some $m, n \in \mathbb{N}$, then the following conditions hold on S.*

1. $a^m \mathcal{L} b^n$ *if and only if* $B(a'a^m) = B(b'b^n)$.
2. $a^m \mathcal{R} b^n$ *if and only if* $B(a^m a') = B(b^n b')$.

Proof. (1) Let S be a π-inverse ordered semigroup and $a, b \in S$. Also let $a' \in V_{\leq}(a^m)$, $b' \in V_{\leq}(b^n)$ for some $m, n \in \mathbb{N}$, such that $a^m \mathcal{L} b^n$. So by Corollary 6 $a'a^m \mathcal{H} b'b^n$. Let $x \in B(a'a^m)$. Therefore $x \leq a'a^m$ or $x \leq a'a^m s_1 a'a^m$ for some $s_1 \in S$. So it is easy to verify that $x \in B(b'b^n)$. Also $a'a^m \in B(b'b^n)$. So $B(a'a^m) \subseteq B(b'b^n)$. Similarly $B(b'b^n) \subseteq B(a'a^m)$. So $B(a'a^m) = B(b'b^n)$.

Converse follows easily.

(2) This is similar to (1). ∎

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