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STRONGLY *E*-INVERSIVE SEMIRINGS

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Abstract

E-inversive semigroups have been the topic of research for many years. Properties of E-inversive semigroups were studied by Edward [1], Mitsch [9] and many others. In [2], Ghosh defined E-inversive semiring and studied its properties. According to him, an additively commutative semiring is called E-inversive semiring if and only if its additive reduct is an E-inversive semigroup. In this paper, we define strongly E-inversive semiring and study its properties.

Keywords: E-inversive semigroup, E-inversive semiring, strongly E-inversive semiring, skew-ring.

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1. Introduction

A semigroup S is said to be E-inversive if for every $a \in S$ there is an element $x \in S$ such that $ax \in E(S)$, where $E(S)$ is the set of all idempotents of the semigroup S. The notion of E-inversive semigroup was first introduced by Thierrin [13]. Later on, properties of E-inversive semigroups were studied by Mitsch [9], Lallement and Petrich [6], Hall and Munn [4], Margolis and Pin [8]. Subdirect products of E-inversive semigroups were first studied by Mitsch [9]. In 1999, the author defined E-inversive semiring [2]. According to him, an additively commutative semiring is said to be E-inversive if its additive reduct is an E-inversive semigroup. Some properties of E-inversive semirings were studied by Ghosh in that same paper

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[2]. In this paper, we define strongly E-inversive semiring and study some of its interesting properties.

The preliminaries and prerequisites, we need for this paper, are discussed in Section 2. In Section 3, we define strongly E-inversive semiring and study its basic properties. Finally, Section 4 is devoted to the study of E-unitary covers of strongly E-inversive semirings.

2. Preliminaries

A semiring $(S,+,\cdot)$ is a type $(2,2)$-algebra such that both the additive reduct $(S,+)$ and the multiplicative reduct $(S,\cdot)$ are semigroups and multiplication distributes over addition from either side, that is, $a(b+c)=ab+ac$ and $(b+c)a=ba+ca$ for all $a,b,c\in S$. We do not assume that the additive reduct $(S,+)$ is commutative. Following [3], we denote a semiring $(S,+,\cdot)$ as a skew-ring if its additive reduct $(S,+)$ is a group, not necessarily an abelian group. A semiring $(S,+,\cdot)$ is said to be a hemiring if the additive reduct $(S,+)$ is a monoid.

Let $(S,+,\cdot)$ be a semiring. An element $a\in S$ is called:

- *additively regular* if there exists an element $x\in S$ such that $a+x+a=a$;
- *additively completely regular* if there exists an element $z\in S$ such that $a+z+a=a$ and $z+a=a+z$;
- *completely regular* [12] if there exists $x\in S$ such that $a=a+x+a$, $a+x=x+a$ and $a(a+x)=a+x$.

If $a\in S$ is additively regular, we denote the set of all inverses of a in the semigroup $(S,+)$ by $V^+(a)$. A semiring S is said to be completely regular if every element a of S is completely regular. A semiring S is called idempotent semiring if both the reducts $(S,+)$ and $(S,\cdot)$ are bands. A subsemiring I of a semiring S is called an ideal of S if $IS\subseteq I$ and $SI\subseteq I$. An ideal I of a semiring S is called a k-ideal of S if $a\in I$ and either $a+x\in I$ or $x+a\in I$ for some $x\in S$ implies $x\in I$. Throughout this paper, we always let $E^+(S)$ be the set of all additive idempotents of the semiring S. Let S and T be two semirings. A mapping $\varphi: S\longrightarrow T$ is called a *semiring homomorphism* if $(a+b)\varphi=a\varphi+b\varphi$ and $(ab)\varphi=(a\varphi)(b\varphi)$ hold for all $a,b\in S$. If $\varphi: S\longrightarrow T$ is a semiring homomorphism, then the kernel of φ, denoted by $\ker\varphi$, is defined by $\ker\varphi=\{s\in S|s\varphi\in E^+(T)\}$. A subsemiring T of the direct product $S_1\times S_2$ of two semirings S_1 and S_2 is called subdirect product of S_1 and S_2 if the two projection mappings $\pi_1: T\longrightarrow S_1$ and $\pi_2: T\longrightarrow S_2$ respectively given by $\pi_1(s_1,s_2)=s_1$ and $\pi_2(s_1,s_2)=s_2$ (where $s_1\in S_1$, $s_2\in S_2$) are surjective.

In a semiring S, we denote Green's relations on the semigroup $(S,+)$ by $\mathcal{L}^+$, $\mathcal{R}^+$, $\mathcal{J}^+$, $\mathcal{D}^+$ and $\mathcal{H}^+$. In fact, the relations $\mathcal{L}^+$, $\mathcal{R}^+$, $\mathcal{J}^+$, $\mathcal{D}^+$ and $\mathcal{H}^+$ are all congruences on the multiplicative reduct $(S,\cdot)$. Thus, if any one of these happens to be a congruence on the additive reduct $(S,+)$, it will be a semiring congruence on the semiring $(S,+,\cdot)$. A completely regular semiring S is said to be completely simple [12] if $\mathcal{J}^+ = S \times S$.

Before closing this section, we recall an important result which will be helpful in our further discussion.

Theorem 1 [12]**.** *The following conditions on a semiring are equivalent:*

(i) *S is completely regular;*

(ii) *every $\mathcal{H}^+$-class is a skew-ring;*

(iii) *S is union (disjoint) of skew-rings;*

(iv) *S is a b-lattice of completely simple semirings.*

3. Basic properties of strongly E-inversive semirings

In this section, we define strongly E-inversive semiring and study some of its properties.

Definition. A semiring S is said to be a strongly E-inversive semiring if for each $a \in S$, there exists an element $x \in S$ such that (i) $a + x \in E^+(S)$, (ii) $ax = xa$, (iii) $a(a + x) = a + x$ and (iv) $a(x + a) = x + a$.

There are plenty of examples of strongly E-inversive semirings; for instance, every ring is a strongly E-inversive semiring, every skew-ring is a strongly E-inversive semiring, every completely regular semiring is a strongly E-inversive semiring (as every completely regular semiring is union of skew-rings), every idempotent semiring is a strongly E-inversive semiring. However, the converse of the above statement is not true in general as the following example shows.

Example 2. Define on the set $\mathbb{N}$, the following operations $\oplus$ and $\odot$:

$$a \oplus b = \min\{a + b, M\}$$
$$a \odot b = \min\{ab, M\},$$

where $M \in \mathbb{N}$ is a fixed natural number. It is easy to verify that M is an additive idempotent element as well as a multiplicative idempotent. Now, for any $a \in \mathbb{N}$ choose an element $x \in \mathbb{N}$ such that $a + x > M$. Then $a \oplus x = x \oplus a = M$, $a \odot x = x \odot a$, $a \odot (a \oplus x) = a \odot M = M = a \oplus x$, $a \odot (x \oplus a) = a \odot M = M = x \oplus a$. Hence $(\mathbb{N}, \oplus, \odot)$ is a strongly E-inversive semiring. Since any element $b(> M) \in \mathbb{N}$ is not additively regular, it follows that this semiring is not a quasi completely regular semiring [7] and hence not a completely regular semiring. Moreover, it is interesting to point out that this semiring is not even an idempotent semiring.

Example 3. Any semiring S with $(S,\cdot)$ a band is strongly E-inversive. In fact for any $a \in S$, $a+a \in E^+(S), a\cdot a = a\cdot a, a(a+a) = a+a$.

Theorem 4. *A semiring S is a strongly E-inversive semiring if and only if for all $a \in S$ there exists $y \in S$ such that $a+y, y+a \in E^+(S)$; $ay = ya$; $(a+y)a = a+y$; $(y+a)a = y+a$.*

Proof. First suppose that the semiring S is a strongly E-inversive semiring. Then for any $a \in S$, there exists an element $x \in S$ such that $a+x \in E^+(S)$, $ax = xa$, $a(a+x) = a+x$ and $a(x+a) = x+a$. Now, if we choose $y = x+a+x$, then it can be easily verified that $y+a+y = y$ and hence $a+y, y+a \in E^+(S)$. Since $ax = xa$, we must have $ay = ya$. Again, $(a+y)a = a^2 + ya = a^2 + ay = a(a+y) = a(a+x+a+x) = a(a+x)+a(a+x) = a+x+a+x = a+y$. Hence $(a+y)a = a+y$. Similarly, we can show that $(y+a)a = y+a$.

Converse part follows easily. ■

Theorem 5. *A semiring S is a strongly E-inversive semiring if and only if for all $a \in S$ there exists $y \in S$ such that $y+a+y = y$, $ay = ya$, $a(a+y) = a+y$, $a(y+a) = y+a$.*

Proof. First suppose that the semiring S is a strongly E-inversive semiring. Then for any $a \in S$, there exists an element $x \in S$ such that $a+x \in E^+(S)$, $ax = xa$, $a(a+x) = a+x$ and $a(x+a) = x+a$. Now, if we choose $y = x+a+x$, then it can be easily verified that $y+a+y = y$, $ay = ya$, $a(a+y) = a+y$ and $a(y+a) = y+a$.

Converse part is obvious. ■

Remark 6. The definition of strongly E-inversive semiring is not one-sided. Moreover, a semiring S is strongly E-inversive if and only if $W(a) = \big\{x \in S : x+a+x = x, ax = xa, a(a+x) = a+x, a(x+a) = x+a\big\} \neq \emptyset$ for every $a \in S$. Now following [9], we get that if $(S,+,\cdot)$ is a strongly E-inversive semiring then $(S,+)$ is an E-inversive semigroup as $I(a) = \big\{x \in S : a+x, x+a \in E^+(S)\big\} \neq \emptyset$ for every $a \in S$ and it is obvious that $W(a) \subseteq I(a)$ for all $a \in S$.

Though the additive reduct of a strongly E-inversive semiring is always an E-inversive semigroup, but the converse may not be true as the following example shows.

Example 7. Consider the semiring $(\mathbb{N},+,\cdot)$, where addition '+' of two elements is maximum of two elements in $\mathbb{N}$ and multiplication '·' is the usual multiplication of natural numbers. Then $(\mathbb{N},+)$ is an E-inversive semigroup and $(\mathbb{N},+,\cdot)$ is an E-inversive semiring but not a strongly E-inversive semiring.

Proposition 8. *Every left (right, both sided) k-ideal of a strongly E-inversive semiring is strongly E-inversive.*

Proof. Let I be a left k-ideal of a strongly E-inversive semiring S and $a \in I$. Since S is strongly E-inversive, it follows that $W(a) \neq \emptyset$. Let $x \in W(a)$. As I is a left ideal of S and $a \in I$, we must have $a^2, xa \in I$. This implies $a^2 + xa \in I$, i.e., $a + x \in I$. Again, I is a left k-ideal of S and $a, a + x \in I$ imply that $x \in I$. Consequently, I is a strongly E-inversive semiring. ■

The following well known result can be found, for instance, in [10].

Lemma 9. *The following conditions on a semigroup $(S, \cdot)$ are equivalent.*

(1) *S is a rectangular band;*

(2) *S is regular and satisfies the identity $ab = axb$;*

(3) *S is a completely simple band.*

Proposition 10. *Let S be a strongly E-inversive semiring such that the multiplicative reduct $(S, \cdot)$ is a completely simple semigroup and $e \in W(e)$, for all $e \in E^+(S)$. Then $(S, \cdot)$ is a rectangular band. Conversely, if S is a semiring such that $(S, \cdot)$ is a rectangular band, then S is a strongly E-inversive semiring such that $e \in W(e)$, for all $e \in E^+(S)$.*

Proof. Since $E^+(S)$ is an ideal of $(S, \cdot)$ and $(S, \cdot)$ is simple, it follows that $S = E^+(S)$. Let $a \in S$. Then $a \in E^+(S)$ and hence by the given condition, it follows that $a \in W(a)$. This implies $a(a + a) = a + a$, $a^2 = a$. Thus $(S, \cdot)$ is a band and hence by Lemma 9, it follows that $(S, \cdot)$ is a rectangular band.

Conversely, if S is a semiring whose multiplicative reduct $(S, \cdot)$ is a rectangular band, then by Example 3, it at once follows that S is a strongly E-inversive semiring such that $e \in W(e)$, for all $e \in E^+(S)$. ■

Proposition 11. *If for every element a in a semiring S there is exactly one $x \in S$ such that $a + x \in E^+(S)$, then S is a skew-ring.*

Proof. From [9, Proposition 2], it follows that $(S, +)$ is a group and hence S is a skew-ring. ■

Proposition 12. *Let S be a strongly E-inversive semiring without zero. Then the following conditions are equivalent:*

(i) *S is weakly additive cancellative (i.e., $a + x = b + x$ and $x + a = x + b$ imply that $a = b$) and for each $a \in S$ there exists some $x \in W(a)$ satisfying $x(a + x) = a + x, x(x + a) = x + a$;*

(ii) *S is a completely simple semiring.*

Proof. (i)$\Longrightarrow$(ii) Let $a \in S$. Then there exists an element $x \in W(a)$ such that $x + a + x = x, ax = xa, a(a + x) = a + x = x(a + x), a(x + a) = x + a = x(x + a)$. Now, clearly $(a + x + a) + x = a + x$ and $x + (a + x + a) = x + a$. Since S is weakly additive cancellative, we must have $a+x+a=a$ and thus S is additively regular. Now $(a+x)(x+a)=a(x+a)+x(x+a)=x+a+x+a=x+a$. Again, $(a+x)(x+a)=(a+x)x+(a+x)a=x(a+x)+a(a+x)=a+x+a+x=a+x$. Hence $a+x=x+a$. Therefore, S is a completely regular semiring. By [5, Theorem 3.3.3], it follows that $(S,+)$ is completely simple, so $\mathcal{J}^+=S\times S$. Hence S is a completely simple semiring.

(ii)$\Longrightarrow$(i) Since S is a completely simple semiring, so $(S,+)$ is a completely simple semigroup. Then by [5, Theorem 3.3.3], we must have S is weakly additive cancellative. Again, since S is completely simple, it follows that S is a completely regular semiring and hence by Theorem 1, it follows that S is union of skew-rings. Let $a\in S$. Then H_a, the $\mathcal{H}^+$-class containing the element a is a skew-ring. Let $y\in H_a$ be the inverse of a in the group $(H_a,+)$. Then clearly $y\in W(a)$ such that $y(a+y)=a+y, y(y+a)=y+a$. ■

Proposition 13. *Let S be a strongly E-inversive semiring. Then the following are equivalent:*

(i) *S is left additive cancellative;*

(ii) *S is a right skew-ring.*

Proof. Since S is strongly E-inversive semiring, so its additive reduct is an E-inversive semigroup. Again, a semiring is called right skew-ring if its additive reduct is a right group. Hence the result holds from [9, Proposition 4]. ■

Definition. A nonempty subset I of a semiring S is said to be a bi-ideal of S if $a\in I$ and $x\in S$ imply that $a+x, x+a, ax, xa\in I$. A semiring S is said to be an ideal extension of a semiring T if T is a bi-ideal of S.

Theorem 14. *Let S be an ideal extension of a semiring T. If S is strongly E-inversive, then T is also strongly E-inversive.*

Proof. Let $t\in T$. Then $t\in S$. Since S is strongly E-inversive, there exists an element $x\in S$ such that $x+t+x=x, xt=tx, t(t+x)=t+x$ and $t(x+t)=x+t$. Since T is a bi-ideal of S and $t\in T$, $x\in S$, it follows that $x=x+t+x\in T$. Hence T is strongly E-inversive. ■

The converse of Theorem 14 can be proved by taking additional condition as follows.

Theorem 15. *Let S be an ideal extension of a strongly E-inversive semiring T such that $se_t=e_ts=e_t$ for all $s\in S\setminus T, e_t\in E^+(T)$ and S is weakly additive cancellative. Then S is also strongly E-inversive.*

Proof. Let $s \in S$. If $s \in T$ then obviously s is a strongly E-inversive element. Let $s \notin T$, so $s \in S \setminus T$. Then $s+t \in T$ for all $t \in T$ as T is a bi-ideal of S. Since T is strongly E-inversive, there exists an element $x \in T$ such that $x+(s+t)+x = x$. Then $t+x+s+t+x = t+x$ which implies that $s+t+x, t+x+s \in E^+(T)$. Hence $s(s+t+x), (s+t+x)s, s(t+x+s), (t+x+s)s \in E^+(T)$. By given condition, we have $s(s+t+x) = (s+t+x)s = s+t+x$ and $s(t+x+s) = (t+x+s)s = t+x+s$ which imply that $s^2 + s(t+x) = s^2 + (t+x)s$ and $s(t+x) + s^2 = (t+x)s + s^2$. Hence $s(t+x) = (t+x)s$ as S is weakly additive cancellative. Now if we choose $t+x = x_s$, then $x_s + s + x_s = x_s, sx_s = x_s s, s(s+x_s) = s+x_s, s(x_s+s) = x_s+s$. Hence S is strongly E-inversive. ■

Corollary 16. *Let S be an ideal extension of a strongly E-inversive semiring T such that $|E^+(T)| = 1$ and S is weakly additive cancellative. Then S is strongly E-inversive semiring such that for all $s \in S$ there exists $x \in S$ satisfying (i) $s+x = x+s \in E^+(S)$, (ii) $sx = xs$ and (iii) $s(s+x) = s+x = x(s+x)$.*

4. E-unitary covers of strongly E-inversive semirings

It is easy to verify that the direct product of strongly E-inversive semirings is again a strongly E-inversive semiring. But the subdirect product of even two strongly E-inversive semirings may not be strongly E-inversive. In this section, we characterize all those subdirect products of two strongly E-inversive semirings which are again strongly E-inversive. Also, we study E-unitary covers of strongly E-inversive semirings.

Proposition 17. *Homomorphic image of a strongly E-inversive semiring is strongly E-inversive.*

Proof. Let $\varphi : S \longrightarrow T$ be a semiring epimorphism between two semirings S and T such that S is strongly E-inversive. Let $b \in T$ be arbitrary. Then there exists an element $a \in S$ such that $b = a\varphi$. Now, for the element $a \in S$, there exists an element $x \in S$ such that $x \in W(a)$. Let $y = x\varphi$. Then one can easily verify that $y \in W(b)$. Since $b \in T$ is arbitrary, it follows that T is strongly E-inversive. ■

Corollary 18. *If S is a strongly E-inversive semiring such that it is a subdirect product of two semirings A and B, then both A and B are strongly E-inversive.*

Remark 19. Subdirect product of two strongly E-inversive semirings may not be strongly E-inversive. This follows from the following example.

Example 20. Let $A = B = (\mathbb{Z}, +, \cdot)$ and S be the subsemiring of $A \times B$ generated by $\{(1,1), (-1,-3)\}$. Then S is a subdirect product of strongly E-inversive semirings A and B, but S is not strongly E-inversive, since $E^+(S) = \{(0,0)\}$ and for the element $(1,1) \in S$, there is no element $(x,y) \in S$ such that $(1,1)+(x,y) = (0,0) \in E^+(S)$.

In [9], Mitsch established necessary and sufficient condition for subdirect products of two E-inversive semigroups to be again E-inversive. Now, we establish a necessary and sufficient condition for subdirect products of two strongly E-inversive semirings to be again strongly E-inversive. For this purpose, let us first define the following definition.

Definition (surjective subhomomorphism)**.** Suppose S, T be two strongly E-inversive semirings and $\psi : S \longrightarrow \mathcal{P}(T)$ (the power set of T) is a mapping. Then ψ is called surjective subhomomorphism of S onto T if the following conditions are satisfied:

(1) $s\psi \neq \emptyset$ for all $s \in S$,

(2) $s_1\psi + s_2\psi \subseteq (s_1+s_2)\psi$ and $(s_1\psi)(s_2\psi) \subseteq (s_1s_2)\psi$ for all $s_1, s_2 \in S$,

(3) $\bigcup_{s\in S} s\psi = T$,

(4) for every $t \in s\psi$, there exist $x \in W(s)$ and $y \in W(t)$ such that $y \in x\psi$.

Theorem 21. *Let S, T be two strongly E-inversive semirings and ψ be a surjective subhomomorphism of S onto T. Then $\pi(S,T,\psi) = \{(s,t) \in S \times T : t \in s\psi\}$ is a strongly E-inversive semiring which is a subdirect product of S and T. Conversely, every strongly E-inversive semiring which is a subdirect product of two strongly E-inversive semirings can be obtained in this way.*

Proof. Let $A = \pi(S,T,\psi)$. Then by condition (2) of the definition of surjective subhomomorphism, it follows that A is a semiring. Again, following [9, Theorem 7], it follows that $(A,+)$ is an E-inversive semigroup which is a subdirect product of S and T. To show A is strongly E-inversive, let $(s,t) \in A$. Then $t \in s\psi$ and hence by condition (4), there exist $x \in W(s)$ and $y \in W(t)$ such that $y \in x\psi$. This implies $(x,y) \in A$. Moreover, it is easy to verify that $(x,y) \in W((s,t))$. Hence A is a strongly E-inversive semiring.

Conversely, let B be a strongly E-inversive semiring which is a subdirect product of two strongly E-inversive semirings S and T. Then $(B,+)$ is an E-inversive semigroup. Let $\alpha : S \longrightarrow \mathcal{P}(T)$ be defined by $s\alpha = \{t \in T : (s,t) \in B\}$ for every $s \in S$. Since B is a subsemiring of $S \times T$, we can easily verify that the condition (2) holds. Let $t \in s\alpha$, where $s \in S$ and $t \in T$. Then $(s,t) \in B$, hence there exists $(x,y) \in B$ such that $(x,y) \in W((s,t))$ as B is a strongly E-inversive semiring. Therefore, $(x,y)+(s,t)+(x,y) = (x,y)$, $(s,t)(x,y) = (x,y)(s,t)$, $(s,t)((s,t)+(x,y)) = (s,t)+(x,y)$, $(s,t)((x,y)+(s,t)) = (x,y)+(s,t)$. Then we get, $x+s+x = x, sx = xs, s(s+x) = s+x, s(x+s) = x+s$. Hence $x \in W(s)$ and similarly we get $y \in W(t)$. As $(x,y) \in B$, so $y \in x\alpha$. Therefore, the condition (4) holds. Finally, since $(B,+)$ is an E-inversive semigroup, by [9, Theorem 7], it follows that the conditions (1), (3) hold and $B = \pi(S,T,\alpha)$. ∎

Definition. A subset T of a strongly E-inversive semiring S is called an ideal of S if all $a, b \in T$ and all $s \in S$ imply that $a+b, sa, as \in T$. An ideal T of S is said to be full if $E^+(S) \subseteq T$. An ideal T of S is called a normal ideal if $s+T+s' \subseteq T$ and $s'+T+s \subseteq T$ for all $s \in S$ and $s' \in I(s)$.

Lemma 22. *Let S be a strongly E-inversive semiring. Then the least skew-ring congruence σ on S is given by : for $a, b \in S$,*

$$a\,\sigma\,b \text{ if and only if } x+a=b+y \text{ for some } x, y \in N,$$

where N is the intersection of all full normal ideals of S.

Proof. Following the proof of [9, Proposition 9], it is easy to verify that for every full normal ideal T, the relation σ_T on S given by : for $a, b \in S$, $a\,\sigma_T\,b$ if and only if $x+a=b+y$ for some $x, y \in T$ is a group congruence on $(S,+)$. Let $a\,\sigma_T\,b$ and $c \in S$. This implies $xc+ac=bc+yc$ with $xc, yc \in T$ and thus $ac\,\sigma_T\,bc$. Similarly, we can show that $ca\,\sigma_T\,cb$. Therefore, σ_T is a semiring congruence on S such that $(S/\sigma_T,+)$ is a group. Consequently, S/σ_T is a skew-ring and hence σ_T is a skew-ring congruence on S.

Conversely, let δ be a skew-ring congruence on S and let $T=\{t \in S : t_\delta = 0_\delta$, where 0_δ is the zero element of the skew-ring $S/\delta\}$. Obviously, T is an ideal of S such that $E^+(S) \subseteq T$. Now for each $s \in S$ and $s' \in I(s)$, s'_δ is the additive inverse of $s_\delta \in S/\delta$. Again, for any $t \in T$, $(s+t+s')_\delta = (s+s')_\delta = 0_\delta$. Hence $s+t+s' \in T$. Similarly, it can be verified that $s'+t+s \in T$. Therefore, T is a full normal ideal of S. It is easy to show that $\sigma_T \leq \delta$. Let $a\delta b$, then $(b'+a)\delta(b'+b)$, where $b' \in I(b)$ and hence $b'+a \in T$. Now $(b+b')+a = b+(b'+a)$ implies that $a\sigma_T b$, so $\sigma_T = \delta$. Now it is clear that $\sigma = \sigma_N$, where N is the intersection of all full normal ideals of S. ∎

Definition. A semiring S with additive idempotents is said to be E-unitary if its additive semigroup reduct $(S,+)$ is an E-unitary semigroup, i.e., $e+s, e \in E^+(S), s \in S$ imply $s \in E^+(S)$. One can easily verify that this condition is equivalent to : $s+e, e \in E^+(S)$, $s \in S$ imply $s \in E^+(S)$. A semiring T is called E-unitary cover of a semiring S if T is E-unitary and if there is an epimorphism $\varphi : T \longrightarrow S$ such that φ is additive idempotent separating, i.e., for any two elements $e, f \in E^+(T)$, $e\varphi = f\varphi$ implies $e=f$. In addition, if there is a least skew-ring congruence δ on T and a skew-ring R such that $T/\delta \cong R$, then T is called E-unitary cover of S through R.

Proposition 23. *If σ is the least skew-ring congruence on a strongly E-inversive, E-unitary semiring S, then* $\ker\sigma = N = E^+(S)$, *where N is the intersection of all full normal ideals of S.*

Proof. Using Lemma 22 and following the proof of [9, Corollary 10], in a similar way it can be easily verified that $E^+(S)$ is a full normal ideal of S and $E^+(S) = N = ker\sigma$. ∎

Definition. Suppose S be a strongly E-inversive semiring and R be a skew-ring; a subhomomorphism ψ of S into R is called nullary if $0 \in s\psi$ implies that $s \in E^+(S)$, where $s \in S$ and 0 is the zero element of the skew-ring R.

Lemma 24. *Let S be a strongly E-inversive semiring, R be a skew-ring and ψ be a nullary surjective subhomomorphism of S onto R. Then $\pi_2 \circ {\pi_2}^{-1}$ is the least skew-ring congruence on the strongly E-inversive semiring $T = \pi(S, R, \psi)$, where $\pi_2 : T \longrightarrow R$ is the projection mapping defined by $\pi_2(s, r) = r$ for all $(s, r) \in T$. Hence $T/\sigma = T/\pi_2 \circ {\pi_2}^{-1} \cong R$, where σ is the least skew-ring congruence on T.*

Proof. From Theorem 21, it follows that T is strongly E-inversive semiring which is a subdirect product of S and R and hence R is a homomorphic image of T under the projection mapping defined by $\pi_2(s, r) = r$ for all $(s, r) \in T$. Now, the congruence induced by π_2 on T, i.e., $\pi_2 \circ {\pi_2}^{-1}$ on T is a skew-ring congruence as R is a skew-ring. Then $\sigma \subseteq \pi_2 \circ {\pi_2}^{-1}$, where σ is the least skew-ring congruence on the strongly E-inversive semiring T.

Suppose $(s_1, r_1)\ \pi_2 \circ {\pi_2}^{-1}\ (s_2, r_2)$, where $(s_1, r_1), (s_2, r_2) \in T$. This implies $(s_1, r_1)\pi_2 = (s_2, r_2)\pi_2$, i.e., $r_1 = r_2$ and thus $(s_1, r_2) \in T$. Since T is strongly E-inversive, then there exists $(u, v) \in T$ such that $(u, v)+(s_1, r_2) \in E^+(T) = \{(e, 0) : e \in E^+(S)\}$, where 0 is the zero element of the skew-ring R. Therefore, $v = -r_2$ and so $(u, -r_2) \in T$ and $(u + s_1, 0) \in E^+(T)$. Again, $(s_2, r_2) + (u, -r_2) = (s_2 + u, 0) \in T$ implies $0 \in (s_2 + u)\psi$. Since ψ is nullary, therefore $s_2 + u \in E^+(S)$. Let N be the intersection of all full normal ideals of T. Then $(s_2 + u, 0), (u + s_1, 0) \in E^+(T) \subseteq N$ and $(s_2 + u, 0) + (s_1, r_1) = (s_2 + u + s_1, r_1) = (s_2, r_1) + (u + s_1, 0) = (s_2, r_2) + (u + s_1, 0)$ imply that $(s_1, r_1)\, \sigma\, (s_2, r_2)$. Hence $\pi_2 \circ {\pi_2}^{-1} \subseteq \sigma$ and so $\pi_2 \circ {\pi_2}^{-1} = \sigma$. Consequently, $T/\sigma = T/\pi_2 \circ {\pi_2}^{-1} \cong R$. ∎

Theorem 25. *Let S be a strongly E-inversive semiring, R be a skew-ring and ψ be a nullary surjective subhomomorphism of S onto R. Then $\pi(S, R, \psi)$ is a strongly E-inversive, E-unitary cover of S through R.*

Proof. By Theorem 21, it follows that $\pi(S, R, \psi)$ is a strongly E-inversive semiring. Similar to the proof of [9, Theorem 8], we can prove that $\pi(S, R, \psi)$ is an E-unitary cover of S. Finally, by Lemma 24, it follows that $\pi(S, R, \psi)/\sigma \cong R$, where σ is the least skew-ring congruence on $\pi(S, R, \psi)$ and hence $\pi(S, R, \psi)$ is a strongly E-inversive, E-unitary cover of S through R. ∎

A construction of all unitary, surjective subhomomorphisms of an inverse semigroup onto a group was described by Petrich and Reilly in [11]. Later on, this result was generalized by Mitsch in [9] for E-inversive semigroups. In the following theorem, we make an extension of this idea to strongly E-inversive semirings.

Theorem 26. *Let S be a strongly E-inversive semiring and R be a skew-ring. Then ψ is a nullary surjective subhomomorphism of S onto R if and only if $\psi = \alpha^{-1} \circ \beta$ for some strongly E-inversive semiring T and some surjective homomorphisms $\alpha : T \longrightarrow S$ and $\beta : T \longrightarrow R$ such that* $\ker\beta \subseteq \ker\alpha$.

Proof. First we assume that T is a strongly E-inversive semiring and $\psi = \alpha^{-1}\circ\beta$, where $\alpha : T \longrightarrow S$ and $\beta : T \longrightarrow R$ are surjective homomorphisms such that $\ker\beta \subseteq \ker\alpha$. As α, β both are surjective, it follows that $s\psi \neq \emptyset$ and ψ is surjective, i.e., $\bigcup_{s\in S} s\psi = R$. Suppose $r_1 \in s_1\psi$ and $r_2 \in s_2\psi$, where $s_1, s_2 \in S$. Then there exist $t_1, t_2 \in T$ such that $t_1\alpha = s_1$, $t_1\beta = r_1$, $t_2\alpha = s_2$, $t_2\beta = r_2$. Now, $r_1+r_2 = (t_1+t_2)\beta$ and $(t_1+t_2)\alpha = s_1+s_2$ imply that $(r_1+r_2) \in (s_1+s_2)(\alpha^{-1}\circ\beta) = (s_1 + s_2)\psi$ and similarly $(r_1r_2) \in (s_1s_2)\psi$. Therefore, $s_1\psi + s_2\psi \subseteq (s_1 + s_2)\psi$ and $(s_1\psi)(s_2\psi) \subseteq (s_1s_2)\psi$ for all $s_1, s_2 \in S$. Again, as T is a strongly E-inversive semiring, so $W(t_1) \neq \emptyset$. Let $t_1' \in W(t_1)$. Let $t_1'\alpha = s_3$ and $t_1'\beta = r_3$. Then clearly $s_3 \in W(s_1), r_3 \in W(r_1)$ and obviously $r_3 \in s_3(\alpha^{-1} \circ \beta) = s_3\psi$ and thus ψ is a surjective subhomomorphism. Now, suppose $0 \in s\psi$. Then there exists some $t \in T$ such that $t\alpha = s$ and $t\beta = 0$. This implies $t \in \ker\beta \subseteq \ker\alpha$ and hence $s = t\alpha \in E^+(S)$. Consequently, ψ is nullary surjective subhomomorphism of S onto R.

Conversely, we assume that ψ is a nullary surjective subhomomorphism from S onto R. By Theorem 25, it follows that $\pi(S, R, \psi)$ is a strongly E-inversive, E-unitary cover of S through R and by Theorem 21, we have $\pi(S, R, \psi)$ is a subdirect product of S and R. Let $T = \pi(S, R, \psi)$ and let $\pi_1 : T \longrightarrow S$ and $\pi_2 : T \longrightarrow R$ be the projection mappings. By Lemma 24, $\pi_2 \circ {\pi_2}^{-1}$ is the least skew-ring congruence on T and so by Proposition 23, we have $\ker\pi_2 = E^+(T)$. Clearly, $E^+(T) \subseteq \ker\pi_1$ and so by the sufficiency of this theorem $\gamma = {\pi_1}^{-1} \circ \pi_2$ is a nullary surjective subhomomorphism from S onto R and it can be easily verified that $\pi(S, R, \gamma) = \{(t\pi_1, t\pi_2) : t \in T\} = T$. Hence $s\gamma = \{r \in R : (s, r) \in \pi(S, R, \gamma)\} = \{r \in R : (s, r) \in T\} = s\psi$ for all $s \in S$. Therefore, $\psi = \gamma = {\pi_1}^{-1} \circ \pi_2$. ■

We are now in a position to prove the converse of the Theorem 25.

Theorem 27. *Let S be a strongly E-inversive semiring, R be a skew-ring and T be a strongly E-inversive, E-unitary cover of S through R. Then there is a nullary surjective subhomomorphism ψ of S onto R such that $\pi(S, R, \psi)$ is a homomorphic image of T and $\pi(S, R, \psi)$ is a strongly E-inversive, E-unitary cover of S through R. Moreover, if T is a subdirect product of S and R, then $T \cong \pi(S, R, \psi)$.*

Proof. Using Theorem 21, Proposition 23, Theorem 25 and Theorem 26, the proof follows similar to the proof of [9, Theorem 13]. ∎

Following [9], a subdirect product H of a semigroup K and a group G is called full if $(e, 1) \in H$ for every $e \in E(K)$, where 1 is the identity element of the group G. A subdirect product T of a semiring S and a skew-ring R is called full if $(e, 0) \in T$ for every $e \in E^+(S)$, where 0 is the zero element of the skew-ring R.

In [9], Mitsch gave the construction of all E-inversive semigroups which are full subdirect products of a semilattice and a group. Similar to semigroup, using the concept of surjective subhomomorphism, we here describe the construction of all strongly E-inversive semirings which are full subdirect products of an idempotent semiring and a skew-ring.

Theorem 28. *Let I be an idempotent semiring, R be a skew-ring and $\mathcal{H}(R)$ be the collection of all subhemirings of R. Suppose $\theta : I \longrightarrow \mathcal{H}(R) \subseteq \mathcal{P}(R)$ is a mapping such that θ is a surjective subhomomorphism of I onto R. Then $S = \{(\alpha, r) \in I \times R : r \in \alpha\theta\}$ is a strongly E-inversive semiring which is a full subdirect product of I and R. Conversely, every such semiring can be constructed in this manner.*

Proof. Suppose $S = \{(\alpha, r) \in I \times R : r \in \alpha\theta\}$, where θ is a surjective subhomomorphism of I onto R. Then by Theorem 21, it follows that S is a strongly E-inversive semiring which is a subdirect product of I and R. Also, S is full as $\alpha\theta$ is a subhemiring of R for all $\alpha \in I$.

Conversely, suppose that S is a strongly E-inversive semiring which is a full subdirect product of an idempotent semiring I and a skew-ring R. We define $\theta : I \longrightarrow \mathcal{H}(R) \subseteq \mathcal{P}(R)$ by $\alpha\theta = \{r \in R : (\alpha, r) \in S\}$ for all $\alpha \in I$. Since $(\alpha, 0) \in S$ for all $\alpha \in I$, it follows that $0 \in \alpha\theta$ and thus $\alpha\theta \neq \emptyset$. Moreover, $\bigcup_{\alpha \in I} \alpha\theta = R$, since S is a subdirect product of I and R. It is easy to verify that $\alpha\theta$ is a subhemiring of R for all $\alpha \in I$. Suppose $r_1 \in \alpha\theta$ and $r_2 \in \beta\theta$. Then $(\alpha, r_1), (\beta, r_2) \in S$. Since S is a semiring, we must have $(\alpha + \beta, r_1 + r_2) = (\alpha, r_1) + (\beta, r_2) \in S$ and $(\alpha\beta, r_1 r_2) = (\alpha, r_1)(\beta, r_2) \in S$ and thus $\alpha\theta + \beta\theta \subseteq (\alpha + \beta)\theta$ and $(\alpha\theta)(\beta\theta) \subseteq (\alpha\beta)\theta$. Let $r \in \alpha\theta$. Then $(\alpha, r) \in S$. As S is strongly E-inversive, so there exists $(\beta, p) \in S$ such that $(\beta, p) \in W((\alpha, r))$. It is easy to verify that $\beta \in W(\alpha)$ and $p \in W(r)$. Obviously, $p \in \beta\theta$ as $(\beta, p) \in S$. Therefore, θ is a surjective subhomomorphism of I onto R and finally, one can easily verify that $S \cong \{(\alpha, r) \in I \times R : r \in \alpha\theta\}$. ∎

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