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A NEW CHARACTERIZATION OF PROJECTIVE SPECIAL UNITARY GROUPS $U_5(Q)$

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Abstract

In this paper, we prove that projective special unitary groups $U_5(q)$, where q is prime number, can be uniquely determined by the largest elements order and the order of the group.

Keywords: element order, largest element order, projective special unitary groups, prime graph.

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1. Introduction

Let G be a finite group, we denote the set of prime divisors of the order of G and the set of element orders of G by $\pi(G)$ and $\pi_e(G)$, respectively. Also we define the largest element order of G by $k(G)$. Morever, we denote a set of primes by π. Also we denote a sylow p-subgroup of G by G_p and the number of sylow p-subgroups of G by $n_p(G)$. The prime graph $\Gamma(G)$ of group G is a graph whose

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vertex set is $\pi(G)$, and two distinct vertices u and v are adjacent if and only if $uv \in \pi_e(G)$. Moreover, assume that $\Gamma(G)$ has $t(G)$ connected components π_i, for $i = 1, 2, \ldots, t(G)$. In the case where G is of even order, we always assume that $2 \in \pi_1$.

One of the important problems in finite groups theory is, group characterization by specific property. Properties, such as, elements order, the elements with the same order, graphs, etc. One of the methods, is group characterization by using the order of group and the largest element orders. In fact, we say the group G is characterizable by using the order of group and the largest element order if there is the group H, so that, $k(G) = k(H)$ and $|G| = |H|$, then $G \cong H$. However, the authors proved that some of groups by this method be characterized. For example, the authors in ([2, 3, 4, 6, 9]) proved that the simple K_3-groups, the projective special linear group $PSL_2(q)$, $PSL_3(q)$ and $PSU_3(q)$ where q is some special power of prime, the simple K_4-group of type $PSL_2(p)$, where p is a prime but not $2^n - 1$, the suzuki groups $Sz(q)$, where $q - 1$ and $q \pm \sqrt{2q} + 1$ are prime number and the sporadic simple groups are characterizable by using the largest element orders and order of the group.

In fact, we prove the following main theorem.

Main Theorem. Let G be a group with $|G| = |U_5(q)|$ and $k(G) = k(U_5(q))$, where q is a prime number. Then $G \cong U_5(q)$.

2. Notation and preliminaries

In this section we provide several lemmas and definitions that we need further for the proof of the main theorem.

Lemma 2.1 [8]. *Let G be a Frobenius group of even order with kernel K and complement H. Then*

(a) *$t(G) = 2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$;*

(b) *$|H|$ divides $|K| - 1$;*

(c) *K is nilpotent.*

Definition 2.2. A group G is called a 2-Frobenius group if there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernels K/H and H respectively.

Lemma 2.3 [1]. *Let G be a 2-Frobenius group of even order. Then*

(a) *$t(G) = 2$, $\pi(H) \cup \pi(G/K) = \pi_1$ and $\pi(K/H) = \pi_2$;*

(b) *G/K and K/H are cyclic groups satisfying $|G/K|$ divides $|Aut(K/H)|$.*

Lemma 2.4 [14]**.** *Let G be a finite group with $t(G) \geq 2$. Then one of the following statements holds:*

(a) *G is a Frobenius group;*

(b) *G is a 2-Frobenius group. In particular, a 2-Frobenius group is soluble.*

(c) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, H is a nilpotent group and $|G/K|$ divides $|Out(K/H)|$.*

Lemma 2.5 [15]**.** Let q, k, l be natural numbers. Then

1. $(q^k - 1, q^l - 1) = q^{(k,l)} - 1$.
2. $(q^k + 1, q^l + 1) = \begin{cases} q^{(k,l)} + 1 & \text{if both } \frac{k}{(k,l)} \text{ and } \frac{l}{(k,l)} \text{ are odd,} \\ (2, q+1) & \text{otherwise.} \end{cases}$
3. $(q^k - 1, q^l + 1) = \begin{cases} q^{(k,l)} + 1 & \text{if } \frac{k}{(k,l)} \text{ is even and } \frac{l}{(k,l)} \text{ is odd,} \\ (2, q+1) & \text{otherwise.} \end{cases}$

In particular, for every $q \geq 2$ and $k \geq 1$, the inequality $(q^k - 1, q^k + 1) \leq 2$ holds.

3. Proof of the main theorem

In this section, we prove that the projective special unitary groups $U_5(q)$ are characterizable by using the order of the group and the largest element orders. In fact, we prove that if G is a group with $|G| = |U_5(q)|$ and $k(G) = k(U_5(q))$, where q is a prime number, then $G \cong U_5(q)$. We divide the proof to several lemmas. From now on, we denote the projective special unitary group $U_5(q)$ and also the odd component as $q^4 - q^3 + q^2 - q + 1$ by U and p, respectively. Recall that [5, 10] G is a group with $|G| = |U| = q^{10}(q^5+1)(q^4-1)(q^3+1)(q^2-1)$ and $k(G) = k(U) = q^4 + q$.

Lemma 3.1. *p is an isolated vertex of $\Gamma(G)$, whenever p is prime.*

Proof. We prove that p is an isolated vertex of Γ(G). Assume the contrary. Then there is $t \in \pi(G) - \{p\}$ such that $tp \in \pi_e(G)$. So $tp \geq 2p = 2(q^4 - q^3 + q^2 - q + 1) > q^4 + q$. As a result $k(G) > q^4 + q$, which is a contradiction. So $t(G) \geq 2$. ∎

Lemma 3.2. *The group G is not a Frobenius group.*

Proof. Let G be a Frobenius group with kernel K and complement H. Then by Lemma 2.1, $t(G) = 2$ and $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$ and $|H|$ divides $|K| - 1$. Now by Lemma 3.1, p is an isolated vertex of $\Gamma(G)$. Thus we deduce that (i) $|H| = p$ and $|K| = |G|/p$ or (ii) $|H| =$

$|G|/p$ and $|K| = p$. Since $|H|$ divides $|K| - 1$, we conclude that the last case can not occur. So $|H| = p$ and $|K| = |G|/p$, hence $q^4 - q^3 + q^2 - q + 1 \mid \frac{q^{10}(q^5+1)(q^4-1)(q^3+1)(q^2-1)}{q^4-q^3+q^2-q+1} - 1$. So we conclude that $q^4 - q^3 + q^2 - q + 1 \mid (q^4 - q^3 + q^2 - q + 1)(q^{16} + 2q^{15} - q^{13} - 3q^{11} - 4q^{10} + 5q^6 + 5q^5 - 5q - 5) + 4$ it follows that $q^4 - q^3 + q^2 - q + 1 \mid 4$ which is impossible. Hence, G is not a Frobenius group. ■

Lemma 3.3. *The group G is not a 2-Frobenius group.*

Proof. We prove that that G is not a 2-Frobenius group.Opposite, assume G be a 2-Frobenius group.Thus by Lemma2.3 we have $t(G) = 2$, $\pi(H) \cup \pi(G/K) = \pi_1$ and $\pi(K/H) = \pi_2$ and also G/K and K/H are cyclic groups satisfying $|G/K|$ divides $|Aut(K/H)|$. Now, since that p is an isolated vertex in $\Gamma(G)$. It follows that $\pi_2 = p$, so $|K/H| = p$. On the other hand $|G/K|$ divides $|Aut(K/H)|$. So $|G/K| \mid p - 1$, thus $|G/K| \mid q^4 - q^3 + q^2 - q$. It follows that $|H| \mid \frac{|G|}{|G/K||K/H|}$. As a result $|H| \mid q^{15}(q-1)(3q^{14} + 3q^{13} + 2q^{12} + 3q^{11} + 3q^{10} + q^9)$. Since that H is nilpotent so $H \cong H_l \times H_m \times H_n$, where l, m, n divisors of q^{15}, $q - 1$ $3q^{14} + 3q^{13} + 2q^{12} + 3q^{11} + 3q^{10} + q^9$. So G must be have the element of order $q^{15}(q-1)(3q^{14} + 3q^{13} + 2q^{12} + 3q^{11} + 3q^{10} + q^9)$, where this is impossible, because $k(G) = k(U) = q^4 + q$. ■

Lemma 3.4. The group G is isomorphic to the group U.

Proof. By Lemma 3.2, and Lemma 3.3, we have that third case of Lemma 2.4 is satisfies, as G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups and also K/H is a non-abelian simple group. On the other hand every odd order components of G are the odd order components of K/H. Since $p \mid K/H$ so $t(K/H) \geq 2$. So according to the classification of the finite simple groups we know that the possibilites for K/H are alternating group A_m, $m \geq 5$, 26 sporadic groups, simple groups Lie types. First, we consider the following isomorphic.

Step 1. Let $K/H \cong A_m$, where $m \geq 5$ and $m = r$, $r+1$, $r+2$. Then by [14] $\pi(A_m) = r$, $r-2$ and $|A_m| \mid |G|$. So we consider $q^4 - q^3 + q^2 - q + 1 = r$, $r - 2$. Now, if $q^4 - q^3 + q^2 - q + 1 = r$, then $q^4 - q^3 + q^2 - q + 2 = r + 1$. Since that $r + 1 \mid |A_m||G|$, hence $q^4 - q^3 + q^2 - q + 2 \mid |G|$, which is a contradiction. The another case, we consider $q^4 - q^3 + q^2 - q + 1 = r - 2$, then $q^4 - q^3 + q^2 - q + 3 = r$. Since that $r \mid |A_m||G|$, hence $q^4 - q^3 + q^2 - q + 3 \mid |G|$, which is a contradiction.

Step 2. If K/H be isomorphic sporadic groups, then by [10], $k(S) = \{11, 15, 19, 20, 23, 24, 28, 29, 30, 31, 39, 40, 60, 66, 67, 70, 119\}$, where S be a sporadic groups. Now for example if $q^4 + q = 11$, then $q^4 + q - 11 = 0$, which this is impossible. If $q^4 + q = 19$, then we can see easily a contradiction. For other groups we have a contradiction, similarily.

Step 3. Here, we consider K/H is isomorphic to a the group of Lie-type.

3.1. Suppose that $K/H \cong {}^2G_2(3^{2m+1})$, where $m \geq 1$. Now, by [10], $k({}^2G_2(3^{2m+1})) = 3^{2m+1} + 3^{m+1} + 1$, also $q'^3(q'^3 + 1)(q' - 1) \mid q^{10}(q^5 + 1)(q^4 - 1)(q^3 + 1)(q^2 - 1)$. Next, we consider $q^4 + q = 3^{2m+1} + 3^{m+1} + 1$, so $q^4 + q - 2 = 3^{2m+1} + 3^{m+1} - 1$. Hence $(q - 1)(q^3 + q^2 + q + 2) = 3^{2m+1} + 3^{m+1} - 1$. Now, assume $3^m = x$, so we deduce $(q - 1)(q^3 + q^2 + q + 2) = (x - \frac{-3+\sqrt{21}}{6})(x + \frac{-3+\sqrt{21}}{6})$. It follows that $q - 1 = x - \frac{-3+\sqrt{21}}{6}$ and $q^3 + q^2 + q + 2 = x + \frac{-3+\sqrt{21}}{6}$, in other words we have $q - 1 = 3^m - \frac{-3+\sqrt{21}}{6}$ and $q^3 + q^2 + q + 2 = 3^m + \frac{-3+\sqrt{21}}{6}$, then we can see easily this equation has not any solution in natural number $\mathbb{N}$, so which is contradiction.

3.2. If $K/H \cong {}^2F_4(q')$, where $q' = 2^{2m+1} > 2$ then by [10], $k({}^2F_4(q')) = 2^{4m+2} + 2^{3m+2} + 2^{2m+1} + 2^{m+1} + 1$. So, we consider $q^4 + q = 2^{4m+2} + 2^{3m+2} + 2^{2m+1} + 2^{m+1} + 1$. It follows that $q^4 + q - 2 = 2^{4m+2} + 2^{3m+2} + 2^{2m+1} + 2^{m+1} - 1$. Thus $(q-1)(q^3+q^2+q+2) = 2^{4m+2}+2^{3m+2}+2^{2m+1}+2^{m+1}-1$, which is impossible, because, $(q - 1)(q^3 + q^2 + q + 2)$ be even, and $2^{4m+2} + 2^{3m+2} + 2^{2m+1} + 2^{m+1} - 1$ be odd.

3.3. Suppose that $K/H \cong {}^2B_2(2^{2m+1})$, where $m \geq 1$. Now, by [10], $k({}^2B_2(2^{2m+1})) = 2^{2m+1} + 2^{m+1} + 1$, also $|{}^2B_2(2^{2m+1})| = q'^2(q'^2 + 1)(q' - 1) \mid q^{10}(q^5 + 1)(q^4 - 1)(q^3 + 1)(q^2 - 1)$. For this purpose, we consider $q^4 + q = 2^{2m+1} + 2^{m+1} + 1$. As a result $q^4 + q - 2 = 2^{2m+1} + 2^{m+1} - 1$. Hence, $(q - 1)(q^3 + q^2 + q + 2) = 2^{2m+1} + 2^{m+1} - 1$. Now assume $2^m = x$, so $(q - 1)(q^3 + q^2 + q + 2) = 2x^2 + 2x - 1$. It follows that $(q - 1)(q^3 + q^2 + q + 2) = (x - \frac{-2+\sqrt{12}}{4})(x + \frac{-2+\sqrt{12}}{4})$, it follows that $(q - 1) = (x - \frac{-2+\sqrt{12}}{4})$ and also $(q^3 + q^2 + q + 2) = (x + \frac{-2+\sqrt{12}}{4})$, we can see easily this equation has not any solution in natural number $\mathbb{N}$, which is a contradiction.

3.4. Suppose that $K/H \cong G_2(q')$, now by [10], $k(G_2(q')) = q'^2 + q' + 1$ and also $|G_2(q')| = q'^6(q'^6 - 1)(q'^2 - 1) \mid q^{10}(q^5 + 1)(q^4 - 1)(q^3 + 1)(q^2 - 1)$. For this purpose, we consider $q^4 + q = q'^2 + q' + 1$. As a result $q^4 + q - 2 = q'^2 + q' - 1$. Hence $(q - 1)(q^3 + q^2 + q + 2) = q'^2 + q' - 1$, it follows that $(q - 1)(q^3 + q^2 + q + 2) = (q' - \frac{-1+\sqrt{5}}{2})(x + \frac{-1+\sqrt{5}}{2})$. So that $(q - 1) = (q' - \frac{-1+\sqrt{5}}{2})$ and $(q^3 + q^2 + q + 2) = (x + \frac{-1+\sqrt{5}}{2})$, where we can see easily this equation has not any solution in natural number $\mathbb{N}$, which is a contradiction.

3.5. If $K/H \cong D_n(q')$, $C_n(q')$, where $n \geq 4$, $n \geq 3$, respectively. Then, we have a contradiction, similarly.

3.6. Suppose that $K/H \cong {}^3D_4(q')$. Now, by [10], $k({}^3D_4(q')) = (q'^3 - 1)(q' + 1)$. On the other hand that $|{}^3D_4(q')| \mid |G|$, so $q'^{12}(q'^8 + q'^4 + 1)(q'^6 - 1)(q'^2 - 1) \mid q^{10}(q^5 + 1)(q^4 - 1)(q^3 + 1)(q^2 - 1)$. Now we consider $(q'^3 - 1)(q' + 1) = q^4 + q$. So $q(q^3 + 1) = q'^3 - 1)(q' + 1)$. As a result $q = q' + 1$ and $q^3 + 1 = q'^3 - 1$. First if $q = q' + 1$, then $q' = q - 1$. Since $|{}^3D_4(q')| \nmid |G|$, which is impossible. Now if

$q^3+1=q'^3-1$, then we deduce $q'^3=q^3+2$, as a result $q'^{12}=(q^3+2)^4$. Since $q'^{12} \mid |G|$, but we have $(q^3+2)^4 \nmid |G|$, which is impossible.

3.7. Suppose that $K/H \cong E_6(q')$, $E_7(q')$, $E_8(q')$, $F_4(q')$. For example if $K/H \cong E_8(q')$, then by[10] $k(E_8(q')) = (q'+1)(q'^2+q'+1)(q'^5-1)$. On the other hand, $|E_8(q')| = q'^{120}(q'^{30}-1)(q'^{24}-1)(q'^{20}-1)(q'^{18}-1)(q'^{14}-1)(q'^{12}-1)(q'^8-1)(q'^2-1) \mid q^{10}(q^5+1)(q^4-1)(q^3+1)(q^2-1)$. Hence, we consider $q^4+q = (q'+1)(q'^2+q'+1)(q'^5-1)$. As a result $q(q^3+1) = (q'-1)(q'^7+3q'^6+5q'^5+6q'^4+6q'^3+6q'^5+2q'-q'^2-q'-1)$. Then $q=q'-1$, so $q^3+1 = q'^7+3q'^6+5q'^5+6q'^4+6q'^3+6q'^5+2q'-q'^2-q'-1$, since that $|E_8(q')| \nmid |G|$, which is a contradiction.

For $K/H \ncong E_6(q')$,$E_7(q')$, $F_4(q')$, we have a contradiction, similarily.

3.8. If $K/H \cong^2 E_6(q')$, then by [10] $k(^2E_6(q')) = \frac{(q'+1)(q'^2+1)(q'^3-1)}{(3,q'+1)}$. On the other hand, we know that $|^2E_6(q')| = \frac{q'^{36}(q'^2-1)(q'^5+1)(q'^6-1)}{(3,q'+1)} \mid q^{10}(q^5+1)(q^4-1)(q^3+1)(q^2-1)$. Now, we consider $q^4+q = \frac{(q'+1)(q'^2+1)(q'^3-1)}{(3,q'+1)}$. First if $(3,q'+1)=1$, then $q^4+q=(q'+1)(q'^2+1)(q'^3-1)$. As a result, $q(q^3+1)=(q'-1)(q'^5+2q'^4+3q'^3+3q'^2+2q'+1)$, hence $q=q'-1$ and $q^3+1=q'^5+2q'^4+3q'^3+3q'^2+2q'+1$, which is impossible. Another case is impossible.

3.9. Suppose that $K/H \cong L_{n+1}(q')$, where $n \geq 1$. First if $n=1$, then $K/H \cong L_2(q')$. Now by [10], $k(L_2(q')) = q'+1, q'$, where q' be even, odd respectively. Thus we consider $q^4+q=q'$ and $q^4+q=q'+1$. As a result, $|L_2(q')| \nmid |G|$, which is impossible. For $n>1$, $K/H \ncong L_{n+1}(q')$, similarily.

Hence, $K/H \cong U$. It follows that $|K/H| = |U|$ and also, we know that $H \trianglelefteq K \trianglelefteq G$, where p is an isolated vertex of $\Gamma(G)$. It follows that $k(K/H) \mid k(G)$. Hence $q^4+q=q'^4+q'$. As a result $q=q'$. Now, since that $|K/H|=|U|$ and $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, we deduce that $H=1$ and $G=K\cong U$. ∎

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