Discussiones Mathematicae
General Algebra and Applications 43 (2023) 223–231
https://doi.org/10.7151/dmgaa.1420

THE PRE-PERIOD OF THE GLUED SUM OF FINITE MODULAR LATTICES

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Abstract

The notion of a pre-period of an algebra $\mathbf{A}$ is defined by means of the notion of the pre-period $\lambda(f)$ of a monounary algebra $\langle A; f\rangle$: it is determined by $\sup\{\lambda(f) \mid f$ is an endomorphism of $\mathbf{A}\}$. In this paper we focus on the pre-period of a finite modular lattice. The main result is that the pre-period of any finite modular lattice is less than or equal to the length of the lattice; also, necessary and sufficient conditions under which the pre-period of the glued sum is equal to the length of the lattice, are shown. Moreover, we show the triangle inequality of the pre-period of the glued sum.

Keywords: ordinal sum, glued sum, modular lattice, endomorphism, pre-period, connected unary operation.

2020 Mathematics Subject Classification: 18B35, 08A30, 06C05, 08A60, 08A35.

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1. INTRODUCTION

One of the most important tools in studying universal algebra is the notion of endomorphism. An endomorphism f of a structure A can be considered as a unary operation and $\langle A; f\rangle$ is a *monounary algebra*. Some properties of monounary algebras connected with the notion of homomorphism were studied, e.g., in [3, 4, 7, 11, 12].

The importance of the theory of unary and monounary algebras is pointed out for example in the monographs [6, 8, 9, 10]. The advantage of monounary algebras is their relatively easy visualization as they can be represented as planar directed graphs. If the graph of a monounary algebra is connected, then it is called a *connected monounary algebra*. As every graph is a sum of connected components, every monounary algebra is a sum of connected monounary algebras.

Let $f : A \to A$ be a unary operation on a set A. Let f^0 be the identity map on A and $\mathrm{Im}(f) := \{f(a) \mid a \in A\}$. A *pre-period* (or *stabilizer*) of f is the least nonnegative integer n satisfying $\mathrm{Im}f^n = \mathrm{Im}f^{n+1}$ and is denoted by $\lambda(f)$ (see e.g.[15]). An operation f on A is *connected* if for each $a, b \in A$, there exist nonnegative integers n, m such that $f^n(a) = f^m(b)$. Some results from [2] and [13] imply that $\lambda(f) \leq |A|-1$ and the authors characterized f with $\lambda(f) = |A|-1$; moreover, if $\lambda(f) = |A| - 1$, then f is connected.

Several authors focus specially on connected monounary algebras (see e.g., [14, 5]). We saw in [1] that if f is a connected order-preserving map on a bounded poset $\mathbf{P}$, then f has a unique fixed point α ($f(\alpha) = \alpha$); moreover, $\lambda(f) \leq \ell(\mathbf{P})$ where the *length* $\ell(\mathbf{P})$ of $\mathbf{P}$ is defined by $|C| - 1$ for the longest chain C in $\mathbf{P}$. The *pre-period* of a finite lattice $\mathbf{A}$ *fixing* α is the maximum of $\lambda(f)$ whose f is a connected endomorphism on $\mathbf{A}$ and α is the fixed point and it is denoted by $\lambda_\alpha(\mathbf{A})$ which was studied in the case $\alpha = 0$. They showed that if $\mathbf{A}$ is distributive, then $\lambda_0(\mathbf{A}) \leq \ell(\mathbf{A})$ and the authors of [1] characterized $\mathbf{A}$ with $\lambda_0(\mathbf{A}) = \ell(\mathbf{A})$.

In this work, we generalize some notions and facts in [1] to an endomorphism without the connectivity. The supremum of $\lambda(f)$ whose f is an endomorphism of a lattice $\mathbf{A}$ is called the *pre-period* of $\mathbf{A}$ denoted by $\lambda(\mathbf{A})$ which is shown to be less than or equal to the length of $\mathbf{A}$ if it is finite modular. A finite modular lattice $\mathbf{A}$ is said to have the maximum pre-period property (briefly MPP) if $\lambda(\mathbf{A}) = \ell(\mathbf{A})$. We characterize them via the concept of the connectivity. However when $\mathbf{A}$ is complicated, it is not easy to study $\lambda(\mathbf{A})$. One of the ways to determine it is to consider $\mathbf{A}$ as built up from simpler components.

Let $\mathbf{A}$ and $\mathbf{B}$ be (disjoint) ordered sets. The *ordinal sum* $\mathbf{A} \oplus \mathbf{B}$ is defined by taking the following order relation on $A \cup B$: $a \leq b$ if and only if

(i) $a, b \in A$ and $a \leq b$ in A,

(ii) $a, b \in B$ and $a \leq b$ in B,

(iii) $a \in A$ and $b \in B$.

If $\mathbf{A}$ has the top $1_{\mathbf{A}}$ and $\mathbf{B}$ has the bottom $0_{\mathbf{B}}$, the *glued sum* of $\mathbf{A}$ and $\mathbf{B}$, denoted $\mathbf{A} \dot{+} \mathbf{B}$, is obtained from the ordinal sum by identifying $1_{\mathbf{A}}$ with $0_{\mathbf{B}}$. We will show necessary and sufficient conditions of finite modular lattices $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{A} \dot{+} \mathbf{B}$ has the MPP. Also, we prove the triangle inequality: $\lambda(\mathbf{A} \dot{+} \mathbf{B}) \leq \lambda(\mathbf{A}) + \lambda(\mathbf{B})$ for finite lattices $\mathbf{A}$ and $\mathbf{B}$.

2. PRELIMINARIES

A unary operation f on a lattice $\mathbf{A} = \langle A; \vee, \wedge \rangle$ is said to be an *endomorphism* on $\mathbf{A}$ if $f(a \vee b) = f(a) \vee f(b)$ and $f(a \wedge b) = f(a) \wedge f(b)$ for all $a, b \in A$. One can see for a finite lattice $\mathbf{A}$ that there exists the top 1 and the bottom 0; moreover for an endomorphism f on $\mathbf{A}$, f is connected fixing 0 if and only if $f^n(1) = 0$ for some non-negative integer n. This implies that the pre-period $\lambda(f)$ of a connected endomorphism f on a finite lattice $\mathbf{A}$ fixing 0 is the least non-negative integer with $f^{\lambda(f)}(1) = 0$. It was showed in [1] for a finite distributive lattice $\mathbf{A}$ that $\lambda_0(\mathbf{A}) \leq \ell(\mathbf{A})$. A condition on $\mathbf{A}$ for $\lambda_0(\mathbf{A}) = \ell(\mathbf{A})$ is shown in the following theorem. Moreover, it can be stated for any finite modular lattice.

Theorem 1 [1]**.** *Let $\mathbf{A}$ be a finite modular lattice. Then $\lambda_0(\mathbf{A}) = \ell(\mathbf{A})$ if and only if there is an endomorphism f on $\mathbf{A}$ such that $0 = f^{\lambda(\mathbf{A})}(1) \prec f^{\lambda(\mathbf{A})-1}(1) \prec \cdots \prec f(1) \prec 1$; moreover, f is connected.*

3. A PRE-PERIOD OF THE GLUED SUM

In this section, we will start from obvious basic properties of a lattice.

Lemma 2. *Let Θ be a congruence on a lattice $\mathbf{L}$ and let $\mathbf{L}^\partial$ be the dual of $\mathbf{L}$.*

1. *If $\mathbf{L}$ is bounded, then $\lambda_0(\mathbf{L}) = \lambda_1(\mathbf{L}^\partial)$.*
2. *If $x \prec y$ in $\mathbf{L}$, then $x/\Theta \preceq y/\Theta$ in $\mathbf{L}/\Theta$.*

Theorem 3. *Let $\mathbf{L}$ be a finite modular lattice. Then*

$$\lambda_\alpha(\mathbf{L}) \leq \lambda(\mathbf{L}) \leq \ell(\mathbf{L})$$

for all $\alpha \in L$.

Proof. The first inequality is trivial. Let $f : \mathbf{L} \to \mathbf{L}$ be an endomorphism and $\mathbf{L}_n = f^n(\mathbf{L})$ for all $n \geq 0$. Then the restriction function $f \restriction_{\mathbf{L}_n} : \mathbf{L}_n \to \mathbf{L}_{n+1}$ is an onto homomorphism. By the Homomorphism Theorem, $\mathbf{L}_{n+1} \cong \mathbf{L}_n/\Theta_n$ where $\Theta_n = \ker(f \restriction_{\mathbf{L}_n})$. So, if $\Theta_n \neq \triangle_{\mathbf{L}_n}$, then $|\mathbf{L}_{n+1}| < |\mathbf{L}_n|$ by finiteness which implies that $n < \lambda(f)$. Thus,

(1) $$\lambda(f) \text{ is the smallest } n \text{ such that } \Theta_n = \triangle_{\mathbf{L}_n}.$$

We will show that

$$\text{if } \Theta_n \neq \triangle_{\mathbf{L}_n}, \text{ then } \ell(\mathbf{L}_n) > \ell(\mathbf{L}_{n+1}). \tag{2}$$

Suppose that $(a, b) \in \Theta_n$ with $a \prec b$. Then a and b are in a maximal chain $C = \{0_{\mathbf{L}_n} = c_0 \prec c_1 \prec \cdots \prec c_k = 1_{\mathbf{L}_n}\}$ where $k = \ell(\mathbf{L}_n)$ since any two maximal chains of a finite modular lattice have the same cardinality. Lemma 2 and $a/\Theta_n = b/\Theta_n$ imply that

$$C' = \{c_0/\Theta_n \preceq c_1/\Theta_n \preceq \cdots \preceq c_k/\Theta_n\}$$

is a maximal chain in $\mathbf{L}_{n+1}$ with $\ell(\mathbf{L}_{n+1}) = \ell(C') < k = \ell(\mathbf{L}_n)$. By (1) and (2),

$$\text{if } \lambda(f) = n, \text{ then } \ell(\mathbf{L}) > \ell(\mathbf{L}_1) > \cdots > \ell(\mathbf{L}_{n-1}) > \ell(\mathbf{L}_n) \geq 0 \tag{3}$$

which implies that $n \leq \ell(\mathbf{L}_0)$. Hence, $\lambda(\mathbf{L}) \leq \ell(\mathbf{L})$. ■

Remark 4. Theorem 3 is not true for some infinite modular lattices; for example, $\lambda(\mathbf{1} \oplus \mathbb{N} \oplus \mathbf{1}) = \infty$ but $\ell(\mathbf{1} \oplus \mathbb{N} \oplus \mathbf{1}) = 2$.

Example 5. Let $\mathbf{L}$ be the lattice which is shown in Figure 1. One can see that the map $f : L \to L$ defined by $f(0) = 0$, $f(1) = 1$, $f(b_i) = a_i$ and $f(a_i) = b_{i-1}$ for $1 \leq i \leq 4$ where $b_0 = b_1$ is an endomorphism of $\mathbf{L}$. Moreover,

$$\lambda(\mathbf{L}) \geq \lambda(f) = 6 > 5 = \ell(\mathbf{L}).$$

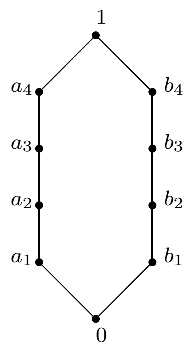

Figure 1. A non-modular lattice $\mathbf{L}$ with $\lambda(\mathbf{L}) > \ell(\mathbf{L})$.

Corollary 6. *Let* $\mathbf{L}$ *be a finite modular lattice. Then*

$\mathbf{L}$ *has the MPP if and only if either* $\lambda_0(\mathbf{L}) = \ell(\mathbf{L})$ *or* $\lambda_1(\mathbf{L}) = \ell(\mathbf{L})$.

Proof. Suppose that $f : \mathbf{L} \to \mathbf{L}$ is an endomorphism with $\lambda(f) = \ell(\mathbf{L}) := n$. By (3), we get $\ell(\mathbf{L}_n) = 0$; that is, $f^n(\mathbf{L}) = \mathbf{L}_n = \{\alpha\}$ for some $\alpha \in \mathbf{L}$. So, $f^n(x) = \alpha = f^n(y)$ for all $x, y \in \mathbf{L}$. Hence, f is connected with $f(\alpha) = \alpha$. For $i \in \{0, 1\}$, let $k_i = \min\{m \in \mathbb{N} \cup \{0\} \mid f^m(i) = \alpha\}$ and $k = \max\{k_0, k_1\}$. Then

$$0 < f(0) < \cdots < f^{k_0}(0) = \alpha = f^{k_1}(1) < \cdots < f(1) < 1$$

and

$$\alpha = f^k(0) \leq f^k(x) \leq f^k(1) = \alpha$$

for all $x \in \mathbf{L}$. So, $\lambda(f) = k$ and $k_0 + k_1 \leq \ell(\mathbf{L})$.

If $k = k_1$, then $k_0 = 0$; that is, f fixes 0; and so,

$$\ell(\mathbf{L}) = \lambda(f) \leq \lambda_0(\mathbf{L}) \leq \ell(\mathbf{L}).$$

Similarly, if $k = k_0$, then $\lambda_1(\mathbf{L}) = \ell(\mathbf{L})$. The converse is clear by the fact that $\max\{\lambda_0(\mathbf{L}), \lambda_1(\mathbf{L})\} \leq \lambda(\mathbf{L}) \leq \ell(\mathbf{L})$. ■

Lemma 7. *Let $\mathbf{A}$ and $\mathbf{B}$ be finite lattices and let $\mathbf{D} = \mathbf{A} \dotplus \mathbf{B}$. If f is a connected endomorphism on $\mathbf{D}$ fixing $0_{\mathbf{D}}$, then $f_{\restriction A}$ is a connected endomorphism on $\mathbf{A}$ fixing $0_{\mathbf{A}}$. And if $\mathbf{A}$ is non-trivial, then B is not closed under f.*

Proof. It suffices to show that A is closed under f. It is clear that f preserves $\leq$. If $f(1_{\mathbf{A}}) > 1_{\mathbf{A}}$, then by uniqueness of the fix-point $0_{\mathbf{D}}$,

$$1_{\mathbf{A}} < f(1_{\mathbf{A}}) < f^2(1_{\mathbf{A}}) < \cdots < f^{\lambda(\mathbf{D})}(1_{\mathbf{A}}) \leq f^{\lambda(\mathbf{D})}(1_{\mathbf{D}}) = 0_{\mathbf{D}} = 0_{\mathbf{A}},$$

a contradiction. Hence, $f(1_{\mathbf{A}}) \leq 1_{\mathbf{A}}$ which implies that $f(x) \leq f(1_{\mathbf{A}}) \leq 1_{\mathbf{A}}$ for all $x \in A$; that is, $f(A) \subseteq A$. Moreover, if $\mathbf{A}$ is non-trivial, then $f(0_{\mathbf{B}}) = f(1_{\mathbf{A}}) < 1_{\mathbf{A}}$; thus, $f(0_{\mathbf{B}}) \notin B$. ■

Theorem 8. *Let $\mathbf{A}$ and $\mathbf{B}$ be non-trivial finite modular lattices. Then $\mathbf{A} \dotplus \mathbf{B}$ has the MPP if and only if either*

1. $\lambda_0(\mathbf{A}) = \ell(\mathbf{A})$ *and* $\mathbf{B}$ *is a chain, or*
2. $\lambda_1(\mathbf{B}) = \ell(\mathbf{B})$ *and* $\mathbf{A}$ *is a chain.*

Proof. First, we will show that

$$\lambda_0(\mathbf{A} \dotplus \mathbf{B}) = \ell(\mathbf{A} \dotplus \mathbf{B}) \text{ if and only if } \lambda_0(\mathbf{A}) = \ell(\mathbf{A}) \text{ and } \mathbf{B} \text{ is a chain.}$$

For convenience, let $\mathbf{D} = \mathbf{A} \dotplus \mathbf{B}$.

($\Rightarrow$) Let $\lambda_0(\mathbf{D}) = \ell(\mathbf{D})$. By Theorem 1, there is a connected endomorphism f on $\mathbf{D}$ fixing $0_{\mathbf{D}}$ such that

$$0_{\mathbf{D}} = f^{\lambda(\mathbf{D})}(1_{\mathbf{D}}) \prec f^{\lambda(\mathbf{D})-1}(1_{\mathbf{D}}) \prec \cdots \prec f(1_{\mathbf{D}}) \prec 1_{\mathbf{D}}.$$

Suppose that m is the greatest non-negative number with $f^m(1_{\mathbf{D}}) \geq 1_{\mathbf{A}}$. As $f^m(1_{\mathbf{D}}) \geq 1_{\mathbf{A}} > f^{m+1}(1_{\mathbf{D}})$ and $f^m(1_{\mathbf{D}}) \succ f^{m+1}(1_{\mathbf{D}})$, we have $f^m(1_{\mathbf{D}}) = 1_{\mathbf{A}}$. By Lemma 7, $f_{\restriction A}$ is an endomorphism on $\mathbf{A}$ fixing $0_{\mathbf{A}}$ with $f^i_{\restriction A}(1_{\mathbf{A}}) = f^i(f^m(1_{\mathbf{D}})) \prec f^{i+1}(f^m(1_{\mathbf{D}}))$ for all $0 \leq i \leq k-m-1$ which implies by Theorem 1 that $\lambda_0(\mathbf{A}) = \ell(\mathbf{A})$. Next, we will show that $f^{m-1}(1_{\mathbf{D}})$ is the unique atom of $\mathbf{B}$. Suppose that a is an atom of $\mathbf{B}$ with $a \neq f^{m-1}(1_{\mathbf{D}})$. Then $a \wedge f^{m-1}(1_{\mathbf{D}}) = f^m(1_{\mathbf{D}})$ which implies that $f(a) \wedge f^m(1_{\mathbf{D}}) = f^{m+1}(1_{\mathbf{D}})$. If $f(a) \geq f^m(1_{\mathbf{D}}) = 0_{\mathbf{B}}$, then $f^m(1_{\mathbf{D}}) = f^{m+1}(1_{\mathbf{D}})$, a contradiction. So, $f(a) < f^m(1_{\mathbf{D}})$ which implies that $f(a) = f^{m+1}(1_{\mathbf{D}})$. Let k be the greatest non-negative integer such that $a < f^k(1_{\mathbf{D}})$. Then $k \leq m-2$. Since $f^{k+1}(1_{\mathbf{D}}) \prec f^k(1_{\mathbf{D}})$, we get $a \vee f^{k+1}(1_{\mathbf{D}}) = f^k(1_{\mathbf{D}})$ which implies that

$$f^{k+1}(1_{\mathbf{D}}) = f(a) \vee f^{k+2}(1_{\mathbf{D}}) = f^{m+1}(1_{\mathbf{D}}) \vee f^{k+2}(1_{\mathbf{D}}).$$

Since $k \leq m-2$, we get $k+2 \leq m < m+1$; and so, $f^{m+1}(1_{\mathbf{D}}) \leq f^{k+2}(1_{\mathbf{D}})$. It follows that $f^{k+2}(1_{\mathbf{D}}) = f^{k+1}(1_{\mathbf{D}})$, a contradiction. So, $\mathbf{B}$ has the unique atom which implies that $\mathbf{B} = \mathbf{2} \dotplus \mathbf{B_1}$ for some lattice $\mathbf{B_1}$. Similarly, $\mathbf{B_1}$ has the unique atom. If we continue in this way, we get that $\mathbf{B}$ is a chain.

($\Leftarrow$) Assume that there is an endomorphism f_A on $\mathbf{A}$ with $f_A^{i-1}(1_{\mathbf{A}}) \succ f_A^i(1_{\mathbf{A}})$ for all $1 \leq i \leq n$ and $\mathbf{B} = \{1_{\mathbf{B}} = b_m \succ b_{m-1} \succ \cdots \succ b_0 = 0_{\mathbf{B}}\}$. Define a unary operation f_D on D by

$$f_D(x) = \begin{cases} b_{i-1} & \text{if } x = b_i \text{ for some } 1 \leq i \leq m, \\ f_A(x) & \text{if } x \in A. \end{cases}$$

It is clear that f_D is an endomorphism on $\mathbf{D}$ such that $f_D^{i-1}(1_{\mathbf{D}}) \succ f_D^i(1_{\mathbf{D}})$ for all $1 \leq i \leq m+n$. Hence, $\lambda_0(\mathbf{D}) = \ell(\mathbf{D})$.

Observe that $\lambda_1(\mathbf{A} \dotplus \mathbf{B}) = \lambda_0(\mathbf{A} \dotplus \mathbf{B})^{\partial} = \lambda_0(\mathbf{B}^{\partial} \dotplus \mathbf{A}^{\partial})$. Hence,

$$\begin{aligned} \lambda_1(\mathbf{A} \dotplus \mathbf{B}) = \ell(\mathbf{A} \dotplus \mathbf{B}) &\Leftrightarrow \lambda_0(\mathbf{B}^{\partial}) = \ell(\mathbf{B}) \text{ and } \mathbf{A}^{\partial} \text{ is a chain} \\ &\Leftrightarrow \lambda_1(\mathbf{B}) = \ell(\mathbf{B}) \text{ and } \mathbf{A} \text{ is a chain.} \end{aligned}$$

By Corollary 6, we are done. ∎

Example 9. The lattice $\mathbf{D} = \mathbf{2}^2 \dotplus \mathbf{2}^3$ is shown as Figure 2. Since $\mathbf{2}^2$ and $\mathbf{2}^3$ are not chains, we get $\lambda(\mathbf{2}^2 \dotplus \mathbf{2}^3) < \ell(\mathbf{2}^2 \dotplus \mathbf{2}^3) = 5$. We can find an endomorphism $f : \mathbf{D} \to \mathbf{D}$, defined by

$$f(x) = \begin{cases} (a_2, a_3, 1) & \text{if } x = (a_1, a_2, a_3) \text{ for some } a_1, a_2, a_3 \in \{0,1\}, \\ (0,0,1) & \text{if } x = (1, a) \text{ for some } a \in \{0,1\}, \\ (0,0,0) & \text{if } x = (0, a) \text{ for some } a \in \{0,1\}, \end{cases}$$

where $\mathbf{2}^3 = \{(a_1,a_2,a_3) \mid a_1,a_2,a_3 \in \{0,1\}\}$ and $\mathbf{2}^2 = \{(a_1,a_2) \mid a_1,a_2 \in \{0,1\}\}$. Hence, $\lambda(f) = 4$ which implies $\lambda(\mathbf{2}^2 \dotplus \mathbf{2}^3) = 4$.

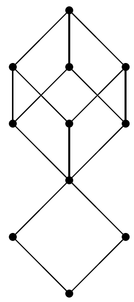

Figure 2. The glued sum $\mathbf{2}^2 \dot{+} \mathbf{2}^3$.

Theorem 10. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be finite lattices. Then*

$$\lambda_i(\mathbf{A} \dot{+} \mathbf{B}) \leq \lambda_i(\mathbf{A}) + \lambda_i(\mathbf{B})$$

and

$$\lambda(\mathbf{A} \dot{+} \mathbf{B}) \leq \lambda(\mathbf{A}) + \lambda(\mathbf{B})$$

for $i \in \{0, 1\}$.

Proof. First, we will consider $i = 0$. Let $\mathbf{D} = \mathbf{A} \dot{+} \mathbf{B}$ and $f_D : D \to D$ be an endomorphism. To prove the second inequality, we assume that k_a and k_b stands for $\lambda(\mathbf{A})$ and $\lambda(\mathbf{B})$, respectively. For the first inequality, k_a and k_b stands for $\lambda_0(\mathbf{A})$ and $\lambda_0(\mathbf{B})$, respectively. From now on, the argument for the two inequalities are (almost) the same: we need to show that $f^{k_a+k_b}(\mathbf{D}) = f^{k_a+k_b+1}(\mathbf{D})$. Let $\beta := 1_{\mathbf{A}} = 0_{\mathbf{B}}$. Note that if f is connected with $f(0_{\mathbf{D}}) = 0_{\mathbf{D}}$, then $f(\beta) < \beta$.

Case (i). $f(\beta) = \beta$. Then $f(A) \subseteq A$ and $f(B) \subseteq B$ since f is order-preserving. Thus, $f^{\max\{k_a,k_b\}}(\mathbf{D}) = f^{\max\{k_a,k_b\}+1}(\mathbf{D})$ and we are done since $\max\{k_a, k_b\} \leq k_a + k_b$.

Case (ii). $f(\beta) \neq \beta$. Since β is comparable with all elements of $\mathbf{D}$, either $f(\beta) > \beta$ or $f(\beta) < \beta$. By duality, we may assume that $f(\beta) < \beta$. So, $f(A) \subseteq A$. Let $f_A = f \restriction_A$. Since f is order-preserving and β is comparable with all elements of $\mathbf{D}$, the map $g : B \to B$ defined by

$$g(x) = \begin{cases} f(x) & \text{if } f(x) > \beta, \\ \beta & \text{if } f(x) \leq \beta. \end{cases}$$

is an endomorphism of $\mathbf{B}$. Note that if f is connected with $f(0_{\mathbf{D}}) = 0_{\mathbf{D}}$, then so are f_A and g with $f_A(0_{\mathbf{A}}) = 0_{\mathbf{A}}$ and $g(0_{\mathbf{B}}) = 0_{\mathbf{B}}$. Let P be the set of elements $x \in B$ with $f^n(x) > \beta$ for all $n \in \mathbb{N}$ and $N := B \setminus P$. Clearly, $f \restriction_P = g \restriction_P$ is closed under P and $f(N) \cap P = \emptyset$ and $N \cap f^i(P) = \emptyset$ for all $i \geq 0$. We will show that

$$g^{k_b}(P) = g^{k_b+1}(P) \text{ and } g^{k_b}(N) = \{\beta\}.$$

Since $\mathbf{B}$ is finite, there exists t with $g^t(N) = \{\beta\}$. Let $T = \min\{t \in \mathbb{N} \mid g^t(N) = \{\beta\}\}$. Then for $t < T$, $g^{t+1}(B) = g^{t+1}(P) \cup g^{t+1}(N) \subsetneq g^t(P) \cup g^t(N) = g^t(B)$; and so, $T \leq \lambda(g) \leq k_b$. So, $g^{k_b}(N) = \{\beta\}$. Since

$$g^{k_b}(P) \cup g^{k_b}(N) = g^{k_b}(B) = g^{k_b+1}(B) = g^{k_b+1}(P) \cup g^{k_b+1}(N)$$

and $\beta \notin g^{k_b}(P)$, we get $g^{k_b}(P) = g^{k_b+1}(P)$. Hence, $f^{k_a+k_b}(P) = f^{k_a+k_b+1}(P)$ and $f^{k_b}(N) \subseteq A$. So,

$$\begin{aligned}
f^{k_a+k_b}(N \cup A) &= f^{k_a+k_b}(N \cup A) \\
&= f^{k_a}\big(f^{k_b}(N \cup A)\big) = f_A^{k_a}\big(f^{k_b}(N \cup A)\big) \\
&= f_A^{k_a}\big(f^{k_b}(N)\big) \cup f_A^{k_a}\big(f^{k_b}(A)\big) \\
&= f^{k_a}\big(f^{k_b+1}(N)\big) \cup f^{k_b}\big(f_A^{k_a+1}(A)\big) \\
&= f^{k_a+k_b+1}(N \cup A).
\end{aligned}$$

So, $f^{k_a+k_b}(\mathbf{D}) = f^{k_a+k_b+1}(\mathbf{D})$. This implies that $\lambda(\mathbf{A} \dotplus \mathbf{B}) \leq \lambda(\mathbf{A}) + \lambda(\mathbf{B})$ and $\lambda_0(\mathbf{A} \dotplus \mathbf{B}) \leq \lambda_0(\mathbf{A}) + \lambda_0(\mathbf{B})$; and so,

$$\lambda_1(\mathbf{A} \dotplus \mathbf{B}) = \lambda_0(\mathbf{A} \dotplus \mathbf{B})^\partial = \lambda_0(\mathbf{B}^\partial \dotplus \mathbf{A}^\partial) \leq \lambda_0(\mathbf{A}^\partial) + \lambda_0(\mathbf{B}^\partial) = \lambda_1(\mathbf{A}) + \lambda_1(\mathbf{B}).$$

■

Acknowledgment

The authors are grateful to the referee(s) for improving the quality of the manuscript. This work was financially supported by Academic Affairs Promotion Fund, Faculty of Science, Khon Kaen University, Fiscal year 2021 (RAAPF).

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Received 30 November 2021
Revised 13 December 2021
Accepted 4 January 2022