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THE PRE-PERIOD OF THE GLUED SUM OF FINITE MODULAR LATTICES

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Abstract

The notion of a pre-period of an algebra $\mathbf{A}$ is defined by means of the notion of the pre-period $\lambda(f)$ of a monounary algebra $\langle A; f\rangle$: it is determined by $\sup\{\lambda(f) \mid f \text{ is an endomorphism of } \mathbf{A}\}$. In this paper we focus on the pre-period of a finite modular lattice. The main result is that the pre-period of any finite modular lattice is less than or equal to the length of the lattice; also, necessary and sufficient conditions under which the pre-period of the glued sum is equal to the length of the lattice, are shown. Moreover, we show the triangle inequality of the pre-period of the glued sum.

Keywords: ordinal sum, glued sum, modular lattice, endomorphism, pre-period, connected unary operation.

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1. INTRODUCTION

One of the most important tools in studying universal algebra is the notion of endomorphism. An endomorphism f of a structure A can be considered as a unary operation and $\langle A; f\rangle$ is a *monounary algebra*. Some properties of monounary algebras connected with the notion of homomorphism were studied, e.g., in [3, 4, 7, 11, 12].

The importance of the theory of unary and monounary algebras is pointed out for example in the monographs [6, 8, 9, 10]. The advantage of monounary algebras is their relatively easy visualization as they can be represented as planar directed graphs. If the graph of a monounary algebra is connected, then it is called a *connected monounary algebra*. As every graph is a sum of connected components, every monounary algebra is a sum of connected monounary algebras.

Let $f : A \to A$ be a unary operation on a set A. Let f^0 be the identity map on A and $\mathrm{Im}(f) := \{f(a) \mid a \in A\}$. A *pre-period* (or *stabilizer*) of f is the least nonnegative integer n satisfying $\mathrm{Im} f^n = \mathrm{Im} f^{n+1}$ and is denoted by $\lambda(f)$ (see e.g.[15]). An operation f on A is *connected* if for each $a, b \in A$, there exist nonnegative integers n, m such that $f^n(a) = f^m(b)$. Some results from [2] and [13] imply that $\lambda(f) \leq |A|-1$ and the authors characterized f with $\lambda(f) = |A|-1$; moreover, if $\lambda(f) = |A| - 1$, then f is connected.

Several authors focus specially on connected monounary algebras (see e.g., [14, 5]). We saw in [1] that if f is a connected order-preserving map on a bounded poset $\mathbf{P}$, then f has a unique fixed point α ($f(\alpha) = \alpha$); moreover, $\lambda(f) \leq \ell(\mathbf{P})$ where the *length* $\ell(\mathbf{P})$ of $\mathbf{P}$ is defined by $|C| - 1$ for the longest chain C in $\mathbf{P}$. The *pre-period* of a finite lattice $\mathbf{A}$ *fixing* α is the maximum of $\lambda(f)$ whose f is a connected endomorphism on $\mathbf{A}$ and α is the fixed point and it is denoted by $\lambda_\alpha(\mathbf{A})$ which was studied in the case $\alpha = 0$. They showed that if $\mathbf{A}$ is distributive, then $\lambda_0(\mathbf{A}) \leq \ell(\mathbf{A})$ and the authors of [1] characterized $\mathbf{A}$ with $\lambda_0(\mathbf{A}) = \ell(\mathbf{A})$.

In this work, we generalize some notions and facts in [1] to an endomorphism without the connectivity. The supremum of $\lambda(f)$ whose f is an endomorphism of a lattice $\mathbf{A}$ is called the *pre-period* of $\mathbf{A}$ denoted by $\lambda(\mathbf{A})$ which is shown to be less than or equal to the length of $\mathbf{A}$ if it is finite modular. A finite modular lattice $\mathbf{A}$ is said to have the maximum pre-period property (briefly MPP) if $\lambda(\mathbf{A}) = \ell(\mathbf{A})$. We characterize them via the concept of the connectivity. However when $\mathbf{A}$ is complicated, it is not easy to study $\lambda(\mathbf{A})$. One of the ways to determine it is to consider $\mathbf{A}$ as built up from simpler components.

Let $\mathbf{A}$ and $\mathbf{B}$ be (disjoint) ordered sets. The *ordinal sum* $\mathbf{A} \oplus \mathbf{B}$ is defined by taking the following order relation on $A \cup B$: $a \leq b$ if and only if

(i) $a, b \in A$ and $a \leq b$ in A,

(ii) $a, b \in B$ and $a \leq b$ in B,

(iii) $a \in A$ and $b \in B$.

If $\mathbf{A}$ has the top $1_{\mathbf{A}}$ and $\mathbf{B}$ has the bottom $0_{\mathbf{B}}$, the *glued sum* of $\mathbf{A}$ and $\mathbf{B}$, denoted $\mathbf{A} \dotplus \mathbf{B}$, is obtained from the ordinal sum by identifying $1_{\mathbf{A}}$ with $0_{\mathbf{B}}$. We will show necessary and sufficient conditions of finite modular lattices $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{A} \dotplus \mathbf{B}$ has the MPP. Also, we prove the triangle inequality: $\lambda(\mathbf{A} \dotplus \mathbf{B}) \leq \lambda(\mathbf{A}) + \lambda(\mathbf{B})$ for finite lattices $\mathbf{A}$ and $\mathbf{B}$.

2. PRELIMINARIES

A unary operation f on a lattice $\mathbf{A} = \langle A; \vee, \wedge \rangle$ is said to be an *endomorphism* on $\mathbf{A}$ if $f(a \vee b) = f(a) \vee f(b)$ and $f(a \wedge b) = f(a) \wedge f(b)$ for all $a, b \in A$. One can see for a finite lattice $\mathbf{A}$ that there exists the top 1 and the bottom 0; moreover for an endomorphism f on $\mathbf{A}$, f is connected fixing 0 if and only if $f^n(1) = 0$ for some non-negative integer n. This implies that the pre-period $\lambda(f)$ of a connected endomorphism f on a finite lattice $\mathbf{A}$ fixing 0 is the least non-negative integer with $f^{\lambda(f)}(1) = 0$. It was showed in [1] for a finite distributive lattice $\mathbf{A}$ that $\lambda_0(\mathbf{A}) \leq \ell(\mathbf{A})$. A condition on $\mathbf{A}$ for $\lambda_0(\mathbf{A}) = \ell(\mathbf{A})$ is shown in the following theorem. Moreover, it can be stated for any finite modular lattice.

Theorem 1 [1]**.** *Let $\mathbf{A}$ be a finite modular lattice. Then $\lambda_0(\mathbf{A}) = \ell(\mathbf{A})$ if and only if there is an endomorphism f on $\mathbf{A}$ such that $0 = f^{\lambda(\mathbf{A})}(1) \prec f^{\lambda(\mathbf{A})-1}(1) \prec \cdots \prec f(1) \prec 1$; moreover, f is connected.*

3. A PRE-PERIOD OF THE GLUED SUM

In this section, we will start from obvious basic properties of a lattice.

Lemma 2. *Let Θ be a congruence on a lattice $\mathbf{L}$ and let $\mathbf{L}^\partial$ be the dual of $\mathbf{L}$.*

1. *If $\mathbf{L}$ is bounded, then $\lambda_0(\mathbf{L}) = \lambda_1(\mathbf{L}^\partial)$.*
2. *If $x \prec y$ in $\mathbf{L}$, then $x/\Theta \preceq y/\Theta$ in $\mathbf{L}/\Theta$.*

Theorem 3. *Let $\mathbf{L}$ be a finite modular lattice. Then*

$$\lambda_\alpha(\mathbf{L}) \leq \lambda(\mathbf{L}) \leq \ell(\mathbf{L})$$

for all $\alpha \in L$.

Proof. The first inequality is trivial. Let $f : \mathbf{L} \to \mathbf{L}$ be an endomorphism and $\mathbf{L}_n = f^n(\mathbf{L})$ for all $n \geq 0$. Then the restriction function $f\restriction_{\mathbf{L}_n} : \mathbf{L}_n \to \mathbf{L}_{n+1}$ is an onto homomorphism. By the Homomorphism Theorem, $\mathbf{L}_{n+1} \cong \mathbf{L}_n/\Theta_n$ where $\Theta_n = \ker(f\restriction_{\mathbf{L}_n})$. So, if $\Theta_n \neq \triangle_{\mathbf{L}_n}$, then $|\mathbf{L}_{n+1}| < |\mathbf{L}_n|$ by finiteness which implies that $n < \lambda(f)$. Thus,

$$\lambda(f) \text{ is the smallest } n \text{ such that } \Theta_n = \triangle_{\mathbf{L}_n}. \tag{1}$$

We will show that

$$\text{if } \Theta_n \neq \triangle_{\mathbf{L}_n}, \text{ then } \ell(\mathbf{L}_n) > \ell(\mathbf{L}_{n+1}). \tag{2}$$

Suppose that $(a, b) \in \Theta_n$ with $a \prec b$. Then a and b are in a maximal chain $C = \{0_{\mathbf{L}_n} = c_0 \prec c_1 \prec \cdots \prec c_k = 1_{\mathbf{L}_n}\}$ where $k = \ell(\mathbf{L}_n)$ since any two maximal chains of a finite modular lattice have the same cardinality. Lemma 2 and $a/\Theta_n = b/\Theta_n$ imply that

$$C' = \{c_0/\Theta_n \preceq c_1/\Theta_n \preceq \cdots \preceq c_k/\Theta_n\}$$

is a maximal chain in $\mathbf{L}_{n+1}$ with $\ell(\mathbf{L}_{n+1}) = \ell(C') < k = \ell(\mathbf{L}_n)$. By (1) and (2),

$$\text{if } \lambda(f) = n, \text{ then } \ell(\mathbf{L}) > \ell(\mathbf{L}_1) > \cdots > \ell(\mathbf{L}_{n-1}) > \ell(\mathbf{L}_n) \geq 0 \tag{3}$$

which implies that $n \leq \ell(\mathbf{L}_0)$. Hence, $\lambda(\mathbf{L}) \leq \ell(\mathbf{L})$. ∎

Remark 4. Theorem 3 is not true for some infinite modular lattices; for example, $\lambda(\mathbf{1} \oplus \mathbb{N} \oplus \mathbf{1}) = \infty$ but $\ell(\mathbf{1} \oplus \mathbb{N} \oplus \mathbf{1}) = 2$.

Example 5. Let $\mathbf{L}$ be the lattice which is shown in Figure 1. One can see that the map $f : L \to L$ defined by $f(0) = 0$, $f(1) = 1$, $f(b_i) = a_i$ and $f(a_i) = b_{i-1}$ for $1 \leq i \leq 4$ where $b_0 = b_1$ is an endomorphism of $\mathbf{L}$. Moreover,

$$\lambda(\mathbf{L}) \geq \lambda(f) = 6 > 5 = \ell(\mathbf{L}).$$

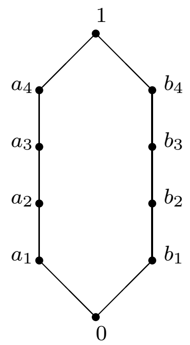

Figure 1. A non-modular lattice $\mathbf{L}$ with $\lambda(\mathbf{L}) > \ell(\mathbf{L})$.

Corollary 6. *Let* $\mathbf{L}$ *be a finite modular lattice. Then*

$\mathbf{L}$ *has the MPP if and only if either* $\lambda_0(\mathbf{L}) = \ell(\mathbf{L})$ *or* $\lambda_1(\mathbf{L}) = \ell(\mathbf{L})$.

Proof. Suppose that $f : \mathbf{L} \to \mathbf{L}$ is an endomorphism with $\lambda(f) = \ell(\mathbf{L}) := n$. By (3), we get $\ell(\mathbf{L}_n) = 0$; that is, $f^n(\mathbf{L}) = \mathbf{L}_n = \{\alpha\}$ for some $\alpha \in \mathbf{L}$. So, $f^n(x) = \alpha = f^n(y)$ for all $x, y \in \mathbf{L}$. Hence, f is connected with $f(\alpha) = \alpha$. For $i \in \{0, 1\}$, let $k_i = \min\{m \in \mathbb{N} \cup \{0\} \mid f^m(i) = \alpha\}$ and $k = \max\{k_0, k_1\}$. Then

$$0 < f(0) < \cdots < f^{k_0}(0) = \alpha = f^{k_1}(1) < \cdots < f(1) < 1$$

and

$$\alpha = f^k(0) \leq f^k(x) \leq f^k(1) = \alpha$$

for all $x \in \mathbf{L}$. So, $\lambda(f) = k$ and $k_0 + k_1 \leq \ell(\mathbf{L})$.

If $k = k_1$, then $k_0 = 0$; that is, f fixes 0; and so,

$$\ell(\mathbf{L}) = \lambda(f) \leq \lambda_0(\mathbf{L}) \leq \ell(\mathbf{L}).$$

Similarly, if $k = k_0$, then $\lambda_1(\mathbf{L}) = \ell(\mathbf{L})$. The converse is clear by the fact that $\max\{\lambda_0(\mathbf{L}), \lambda_1(\mathbf{L})\} \leq \lambda(\mathbf{L}) \leq \ell(\mathbf{L})$. ■

Lemma 7. *Let $\mathbf{A}$ and $\mathbf{B}$ be finite lattices and let $\mathbf{D} = \mathbf{A} \dotplus \mathbf{B}$. If f is a connected endomorphism on $\mathbf{D}$ fixing $0_{\mathbf{D}}$, then $f_{\restriction A}$ is a connected endomorphism on $\mathbf{A}$ fixing $0_{\mathbf{A}}$. And if $\mathbf{A}$ is non-trivial, then B is not closed under f.*

Proof. It suffices to show that A is closed under f. It is clear that f preserves $\leq$. If $f(1_{\mathbf{A}}) > 1_{\mathbf{A}}$, then by uniqueness of the fix-point $0_{\mathbf{D}}$,

$$1_{\mathbf{A}} < f(1_{\mathbf{A}}) < f^2(1_{\mathbf{A}}) < \cdots < f^{\lambda(\mathbf{D})}(1_{\mathbf{A}}) \leq f^{\lambda(\mathbf{D})}(1_{\mathbf{D}}) = 0_{\mathbf{D}} = 0_{\mathbf{A}},$$

a contradiction. Hence, $f(1_{\mathbf{A}}) \leq 1_{\mathbf{A}}$ which implies that $f(x) \leq f(1_{\mathbf{A}}) \leq 1_{\mathbf{A}}$ for all $x \in A$; that is, $f(A) \subseteq A$. Moreover, if $\mathbf{A}$ is non-trivial, then $f(0_{\mathbf{B}}) = f(1_{\mathbf{A}}) < 1_{\mathbf{A}}$; thus, $f(0_{\mathbf{B}}) \notin B$. ■

Theorem 8. *Let $\mathbf{A}$ and $\mathbf{B}$ be non-trivial finite modular lattices. Then $\mathbf{A} \dotplus \mathbf{B}$ has the MPP if and only if either*

1. $\lambda_0(\mathbf{A}) = \ell(\mathbf{A})$ *and $\mathbf{B}$ is a chain, or*
2. $\lambda_1(\mathbf{B}) = \ell(\mathbf{B})$ *and $\mathbf{A}$ is a chain.*

Proof. First, we will show that

$$\lambda_0(\mathbf{A} \dotplus \mathbf{B}) = \ell(\mathbf{A} \dotplus \mathbf{B}) \text{ if and only if } \lambda_0(\mathbf{A}) = \ell(\mathbf{A}) \text{ and } \mathbf{B} \text{ is a chain.}$$

For convenience, let $\mathbf{D} = \mathbf{A} \dotplus \mathbf{B}$.

($\Rightarrow$) Let $\lambda_0(\mathbf{D}) = \ell(\mathbf{D})$. By Theorem 1, there is a connected endomorphism f on $\mathbf{D}$ fixing $0_{\mathbf{D}}$ such that

$$0_{\mathbf{D}} = f^{\lambda(\mathbf{D})}(1_{\mathbf{D}}) \prec f^{\lambda(\mathbf{D})-1}(1_{\mathbf{D}}) \prec \cdots \prec f(1_{\mathbf{D}}) \prec 1_{\mathbf{D}}.$$

Suppose that m is the greatest non-negative number with $f^m(1_{\mathbf{D}}) \geq 1_{\mathbf{A}}$. As $f^m(1_{\mathbf{D}}) \geq 1_{\mathbf{A}} > f^{m+1}(1_{\mathbf{D}})$ and $f^m(1_{\mathbf{D}}) \succ f^{m+1}(1_{\mathbf{D}})$, we have $f^m(1_{\mathbf{D}}) = 1_{\mathbf{A}}$. By Lemma 7, $f_{\restriction A}$ is an endomorphism on $\mathbf{A}$ fixing $0_{\mathbf{A}}$ with $f^i_{\restriction A}(1_{\mathbf{A}}) = f^i(f^m(1_{\mathbf{D}})) \prec f^{i+1}(f^m(1_{\mathbf{D}}))$ for all $0 \leq i \leq k-m-1$ which implies by Theorem 1 that $\lambda_0(\mathbf{A}) = \ell(\mathbf{A})$. Next, we will show that $f^{m-1}(1_{\mathbf{D}})$ is the unique atom of $\mathbf{B}$. Suppose that a is an atom of $\mathbf{B}$ with $a \neq f^{m-1}(1_{\mathbf{D}})$. Then $a \wedge f^{m-1}(1_{\mathbf{D}}) = f^m(1_{\mathbf{D}})$ which implies that $f(a) \wedge f^m(1_{\mathbf{D}}) = f^{m+1}(1_{\mathbf{D}})$. If $f(a) \geq f^m(1_{\mathbf{D}}) = 0_{\mathbf{B}}$, then $f^m(1_{\mathbf{D}}) = f^{m+1}(1_{\mathbf{D}})$, a contradiction. So, $f(a) < f^m(1_{\mathbf{D}})$ which implies that $f(a) = f^{m+1}(1_{\mathbf{D}})$. Let k be the greatest non-negative integer such that $a < f^k(1_{\mathbf{D}})$. Then $k \leq m-2$. Since $f^{k+1}(1_{\mathbf{D}}) \prec f^k(1_{\mathbf{D}})$, we get $a \vee f^{k+1}(1_{\mathbf{D}}) = f^k(1_{\mathbf{D}})$ which implies that

$$f^{k+1}(1_{\mathbf{D}}) = f(a) \vee f^{k+2}(1_{\mathbf{D}}) = f^{m+1}(1_{\mathbf{D}}) \vee f^{k+2}(1_{\mathbf{D}}).$$

Since $k \leq m-2$, we get $k+2 \leq m < m+1$; and so, $f^{m+1}(1_{\mathbf{D}}) \leq f^{k+2}(1_{\mathbf{D}})$. It follows that $f^{k+2}(1_{\mathbf{D}}) = f^{k+1}(1_{\mathbf{D}})$, a contradiction. So, $\mathbf{B}$ has the unique atom which implies that $\mathbf{B} = \mathbf{2} \dotplus \mathbf{B_1}$ for some lattice $\mathbf{B_1}$. Similarly, $\mathbf{B_1}$ has the unique atom. If we continue in this way, we get that $\mathbf{B}$ is a chain.

($\Leftarrow$) Assume that there is an endomorphism f_A on $\mathbf{A}$ with $f_A^{i-1}(1_{\mathbf{A}}) \succ f_A^i(1_{\mathbf{A}})$ for all $1 \leq i \leq n$ and $\mathbf{B} = \{1_{\mathbf{B}} = b_m \succ b_{m-1} \succ \cdots \succ b_0 = 0_{\mathbf{B}}\}$. Define a unary operation f_D on D by

$$f_D(x) = \begin{cases} b_{i-1} & \text{if } x = b_i \text{ for some } 1 \leq i \leq m, \\ f_A(x) & \text{if } x \in A. \end{cases}$$

It is clear that f_D is an endomorphism on $\mathbf{D}$ such that $f_D^{i-1}(1_{\mathbf{D}}) \succ f_D^i(1_{\mathbf{D}})$ for all $1 \leq i \leq m+n$. Hence, $\lambda_0(\mathbf{D}) = \ell(\mathbf{D})$.

Observe that $\lambda_1(\mathbf{A} \dotplus \mathbf{B}) = \lambda_0(\mathbf{A} \dotplus \mathbf{B})^\partial = \lambda_0(\mathbf{B}^\partial \dotplus \mathbf{A}^\partial)$. Hence,

$$\begin{aligned} \lambda_1(\mathbf{A} \dotplus \mathbf{B}) = \ell(\mathbf{A} \dotplus \mathbf{B}) &\Leftrightarrow \lambda_0(\mathbf{B}^\partial) = \ell(\mathbf{B}) \text{ and } \mathbf{A}^\partial \text{ is a chain} \\ &\Leftrightarrow \lambda_1(\mathbf{B}) = \ell(\mathbf{B}) \text{ and } \mathbf{A} \text{ is a chain.} \end{aligned}$$

By Corollary 6, we are done. ■

Example 9. The lattice $\mathbf{D} = \mathbf{2}^2 \dotplus \mathbf{2}^3$ is shown as Figure 2. Since $\mathbf{2}^2$ and $\mathbf{2}^3$ are not chains, we get $\lambda(\mathbf{2}^2 \dotplus \mathbf{2}^3) < \ell(\mathbf{2}^2 \dotplus \mathbf{2}^3) = 5$. We can find an endomorphism $f : \mathbf{D} \to \mathbf{D}$, defined by

$$f(x) = \begin{cases} (a_2, a_3, 1) & \text{if } x = (a_1, a_2, a_3) \text{ for some } a_1, a_2, a_3 \in \{0,1\}, \\ (0,0,1) & \text{if } x = (1, a) \text{ for some } a \in \{0,1\}, \\ (0,0,0) & \text{if } x = (0, a) \text{ for some } a \in \{0,1\}, \end{cases}$$

where $\mathbf{2}^3 = \{(a_1, a_2, a_3) \mid a_1, a_2, a_3 \in \{0,1\}\}$ and $\mathbf{2}^2 = \{(a_1, a_2) \mid a_1, a_2 \in \{0,1\}\}$. Hence, $\lambda(f) = 4$ which implies $\lambda(\mathbf{2}^2 \dotplus \mathbf{2}^3) = 4$.

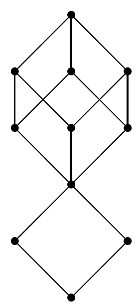

Figure 2. The glued sum $\mathbf{2}^2 \dotplus \mathbf{2}^3$.

Theorem 10. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be finite lattices. Then*

$$\lambda_i(\mathbf{A} \dotplus \mathbf{B}) \leq \lambda_i(\mathbf{A}) + \lambda_i(\mathbf{B})$$

and

$$\lambda(\mathbf{A} \dotplus \mathbf{B}) \leq \lambda(\mathbf{A}) + \lambda(\mathbf{B})$$

for $i \in \{0, 1\}$.

Proof. First, we will consider $i = 0$. Let $\mathbf{D} = \mathbf{A} \dotplus \mathbf{B}$ and $f_D : D \to D$ be an endomorphism. To prove the second inequality, we assume that k_a and k_b stands for $\lambda(\mathbf{A})$ and $\lambda(\mathbf{B})$, respectively. For the first inequality, k_a and k_b stands for $\lambda_0(\mathbf{A})$ and $\lambda_0(\mathbf{B})$, respectively. From now on, the argument for the two inequalities are (almost) the same: we need to show that $f^{k_a+k_b}(\mathbf{D}) = f^{k_a+k_b+1}(\mathbf{D})$. Let $\beta := 1_{\mathbf{A}} = 0_{\mathbf{B}}$. Note that if f is connected with $f(0_{\mathbf{D}}) = 0_{\mathbf{D}}$, then $f(\beta) < \beta$.

Case (i). $f(\beta) = \beta$. Then $f(A) \subseteq A$ and $f(B) \subseteq B$ since f is order-preserving. Thus, $f^{\max\{k_a,k_b\}}(\mathbf{D}) = f^{\max\{k_a,k_b\}+1}(\mathbf{D})$ and we are done since $\max\{k_a, k_b\} \leq k_a + k_b$.

Case (ii). $f(\beta) \neq \beta$. Since β is comparable with all elements of $\mathbf{D}$, either $f(\beta) > \beta$ or $f(\beta) < \beta$. By duality, we may assume that $f(\beta) < \beta$. So, $f(A) \subseteq A$. Let $f_A = f \restriction_A$. Since f is order-preserving and β is comparable with all elements of $\mathbf{D}$, the map $g : B \to B$ defined by

$$g(x) = \begin{cases} f(x) & \text{if } f(x) > \beta, \\ \beta & \text{if } f(x) \leq \beta. \end{cases}$$

is an endomorphism of $\mathbf{B}$. Note that if f is connected with $f(0_{\mathbf{D}}) = 0_{\mathbf{D}}$, then so are f_A and g with $f_A(0_{\mathbf{A}}) = 0_{\mathbf{A}}$ and $g(0_{\mathbf{B}}) = 0_{\mathbf{B}}$. Let P be the set of elements $x \in B$ with $f^n(x) > \beta$ for all $n \in \mathbb{N}$ and $N := B \setminus P$. Clearly, $f \restriction_P = g \restriction_P$ is closed under P and $f(N) \cap P = \emptyset$ and $N \cap f^i(P) = \emptyset$ for all $i \geq 0$. We will show that

$$g^{k_b}(P) = g^{k_b+1}(P) \text{ and } g^{k_b}(N) = \{\beta\}.$$

Since $\mathbf{B}$ is finite, there exists t with $g^t(N) = \{\beta\}$. Let $T = \min\{t \in \mathbb{N} \mid g^t(N) = \{\beta\}\}$. Then for $t < T$, $g^{t+1}(B) = g^{t+1}(P) \cup g^{t+1}(N) \subsetneq g^t(P) \cup g^t(N) = g^t(B)$; and so, $T \leq \lambda(g) \leq k_b$. So, $g^{k_b}(N) = \{\beta\}$. Since

$$g^{k_b}(P) \cup g^{k_b}(N) = g^{k_b}(B) = g^{k_b+1}(B) = g^{k_b+1}(P) \cup g^{k_b+1}(N)$$

and $\beta \notin g^{k_b}(P)$, we get $g^{k_b}(P) = g^{k_b+1}(P)$. Hence, $f^{k_a+k_b}(P) = f^{k_a+k_b+1}(P)$ and $f^{k_b}(N) \subseteq A$. So,

$$\begin{aligned}
f^{k_a+k_b}(N \cup A) &= f^{k_a+k_b}(N \cup A) \\
&= f^{k_a}\big(f^{k_b}(N \cup A)\big) = f_A^{k_a}\big(f^{k_b}(N \cup A)\big) \\
&= f_A^{k_a}\big(f^{k_b}(N)\big) \cup f_A^{k_a}\big(f^{k_b}(A)\big) \\
&= f^{k_a}\big(f^{k_b+1}(N)\big) \cup f^{k_b}\big(f_A^{k_a+1}(A)\big) \\
&= f^{k_a+k_b+1}(N \cup A).
\end{aligned}$$

So, $f^{k_a+k_b}(\mathbf{D}) = f^{k_a+k_b+1}(\mathbf{D})$. This implies that $\lambda(\mathbf{A} \dotplus \mathbf{B}) \leq \lambda(\mathbf{A}) + \lambda(\mathbf{B})$ and $\lambda_0(\mathbf{A} \dotplus \mathbf{B}) \leq \lambda_0(\mathbf{A}) + \lambda_0(\mathbf{B})$; and so,

$$\lambda_1(\mathbf{A} \dotplus \mathbf{B}) = \lambda_0(\mathbf{A} \dotplus \mathbf{B})^\partial = \lambda_0(\mathbf{B}^\partial \dotplus \mathbf{A}^\partial) \leq \lambda_0(\mathbf{A}^\partial) + \lambda_0(\mathbf{B}^\partial) = \lambda_1(\mathbf{A}) + \lambda_1(\mathbf{B}).$$

■

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