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# FUZZY IDEALS AND FUZZY CONGRUENCES ON MENGER ALGEBRAS WITH THEIR HOMOMORPHISM PROPERTIES

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**Abstract**

It is well known that Menger algebras, sometime called superassociative algebras, play a major role in both mathematical sciences and related areas. The notion of fuzzy sets was initiated by L.A. Zadeh as a general mathematical machinery of classical sets. The present paper establishes a strong interaction between fuzzy sets and Menger algebras. We show that the set of all fuzzy subsets on $G$ together with one $(n+1)$-ary operation forms a Menger algebra. The concepts of several kinds of fuzzy ideals in Menger algebras are introduced and some related properties are investigated. Furthermore, we provide a construction of quotient Menger algebras via fuzzy congruence relations. Finally, homomorphism theorems in terms of fuzzy congruences are studied. Our results can be considered as a generalization in the study of semigroup theory too.

**Keywords:** Menger algebra, fuzzy ideal, fuzzy congruence relation, quotient Menger algebra.

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## 1. Introduction and preliminaries

The algebraic properties of the composition of multiplace functions were introduced and studied by Menger [20] in 1964. The multiplace function is one of the major concepts in the investigation of mathematics. It also has various applications in numerous areas of sciences, for instance, in automata theory and programming, theory of multi-valued logics, cybernetics, and multivariable calculas. The essential property of composition, which is called *superassociative law* was investigated in both elementary and advanced methods. Following the suggestion of Menger, the notion of superassociative systems is provided. Recall from [5, 14] that a *Menger algebra of rank n* (Menger algebra, for short) for a fixed positive integer $n$ is a pair of a nonempty set $G$ with an $(n+1)$-ary operation $\circ$ satisfying the superassociative law:

$$\circ(\circ(x, y_1, \ldots, y_n), z_1, \ldots, z_n) = \circ(x, \circ(y_1, z_1, \ldots, z_n), \ldots, \circ(y_n, z_1, \ldots, z_n)),$$

for all $x, y_1, \ldots, y_n, z_1, \ldots, z_n \in G$. Structural properties of this algebra and applications in other areas can be found in the work of Dudek and others [5].

**Example 1** [5]**.** Some fundamental examples of Menger algebras are presented.

(1) The set $\mathbb{R}^+$ of all positive real numbers with the operation $* : (\mathbb{R}^+)^{n+1} \to \mathbb{R}^+$, defined by

$$*(x_0, \ldots, x_n) = x_0 \sqrt[n]{x_1 \cdots x_n},$$

forms a Menger algebra.

(2) Let $A^n$ be the $n$-th Cartesian product of a nonempty set $A$. Any mapping from $A^n$ to $A$ is called a *full n-ary function* or an *n-ary operation* if it is defined for all elements of $A^n$. The set of all such mappings is denoted by $T(A^n, A)$. One can consider the *Menger's superposition* on the set $T(A^n, A)$, i.e., an $(n+1)$-ary operation $\mathcal{O} : T(A^n, A)^{n+1} \to T(A^n, A)$ defined by

$$\mathcal{O}(f, g_1, \ldots, g_n)(a_1, \ldots, a_n) = f(g_1(a_1, \ldots, a_n), \ldots, g_n(a_1, \ldots, a_n)), \tag{1}$$

where $f, g_1, \ldots, g_n \in T(A^n, A), a_1, \ldots, a_n \in A$. A Menger algebra of all full $n$-ary functions or a Menger algebra of all $n$-ary operations is a pair of the set $T(A^n, A)$ of all full $n$-ary functions defined on $A$ and the Menger composition of full $n$-ary functions satisfying the superassociative law.

We can remark here that Example 1 collects two powerful examples of Menger algebras that can be reduced to well-known concrete examples in studying of semigroup theory. Namely, if we set $n = 1$, then Example 1 (1) is a semigroup of positive real numbers with the usual multiplication and Example 1 (2) is a full transformation semigroup.

If there exist elements $e_1, \ldots, e_n \in G$, called *selectors*, such that

$$\circ(x, e_1, \ldots, e_n) = x \text{ and } \circ(e_i, x_1 \ldots, x_n) = x_i$$

for all $x, x_1, \ldots, x_n \in G, i = 1, \ldots, n$, then a Menger algebra $(G, \circ)$ is called *unitary*. It is obviously evident that the definition of selectors is an extension of an identity element in general semigroups by considering $n = 1$ so that $e_1 \in G$ and hence $\circ(x, e_1) = x$ and $\circ(e_1, x_1) = x_1$. Thus $e_1$ acts as a right identity and a left identity, respectively.

One of the reasons for our interest in Menger algebras is their extensive connection with the notion of semigroup theory. Obviously, a semigroup is a Menger algebra of rank $n$ if and only if $n = 1$. In a study concerning Menger algebras, the following two interesting questions arise. Firstly, which algebraic properties of arbitrary semigroups can be generalized to abstract Menger algebras? Secondly, how these properties can be described in the case of $n > 1$? Descriptions of basic relations between functions and of $(2, n)$-semigroups of functions are given in [5] and [6], respectively. In addition, a superassociative system of $n$-ary operation is one of the popular subject for studying the algebraic structural properties of $n$-ary functions in this recently decade. For further results on this direction, see [7, 15, 26].

Characterizations of different kinds of ideals of Menger algebras can be found in [5]. Actually, a nonempty subset $H$ of a Menger algebra $(G, \circ)$ is called a *v-ideal* if for all $g, h_1, \ldots, h_n \in G$, form $h_1, \ldots, h_n \in H$, it follows that $\circ(g, h_1, \ldots, h_n) \in H$. On the other hand, a nonempty subset $H$ of a Menger algebra $(G, \circ)$ is called a *s-ideal* if for all $h, g_1, \ldots, g_n \in G$, from $h \in H$, it follows that $\circ(h, g_1, \ldots, g_n) \in H$. If $A$ is both $v$-ideal and $s$-ideal, then $A$ is called a *vs-ideal* of $G$. It is observed that, in the study of Menger algebras, the concept of $v$-ideals, $s$-ideals and $vs$-ideals are natural generalizations of left ideals, right ideals and ideals in semigroup theory, respectively, if $n = 1$.

The theory of fuzzy set was initiated by Zadeh in 1965. It is a powerful mathematical concept having several applications in numerous scientific areas, for instance, artificial intelligence and machine learning. Fuzzy set theory is also connected with many other computing models, for example, soft sets and rough sets. From a classical algebra viewpoint, fuzzy sets have been extensively applied to study the fundamental properties. The investigation of fuzzy algebraic structures successfully began with the work of Rosenfeld [22] in 1971 for introducing the notion of fuzzy subgroup. Nowadays, fuzzy algebras play a significant role in the combinations of two framwork, algebras and fuzzy theory. There are several algebraic structures to investigate fuzzy sets such as groups, rings, modoles. Kuroki [16] has studied the fuzzy ideals in semigroups. Due to the great importance of fuzzy equivalence relations, there are many researchers focused in this flow, see [12, 17]. Fuzzy congruences in semigroups were developed by Kuroki [16] and Tan [24]. Kim and his colleagues studied fuzzy congruences in groups [10]. While fuzzy isomorphism theorems of soft rings were explored by

Lie and his groups [18, 19]. A strong connection of fuzzy congruence relations with quotient rings can be found in [28]. In 2000, Dudek [4] was first combined the theory of fuzzy sets in $n$-ary systems, espectially in $n$-ary semigroups. In particular, Davvaz and Dudek [3] defined fuzzy $n$-ary groups and provided their properties. In recent years, Zhou *et al.* [27] applied fuzzy congruences to the study of $n$-ary semigroups. For more significant results about fuzzy sets in other algebraic structures, the reader is refered to [1, 2, 8, 9, 11, 13, 23, 25].

The main purpose of this paper is two folds. Firstly, to introduce a novel concept of various types of fuzzy ideals in Menger algebras based on fuzzy ideals in classical algebraic structures, to study some of their related properties and a strong connection between these notions and the original ideal one and then to present a characterization of fuzzy Menger subalgebras and fuzzy ideals via some specific sets. Secondly, to propose the idea of fuzzy congruences on Menger algebras and to establish homomorphism theorems for Menger algebras based on fuzzy congruence relations.

## 2. Fuzzy ideals in Menger algebras

We assume that the reader is familiar with the fundamental of fuzzy ideals in semigroup theory. Formally, a fuzzy set in a nonempty set $G$ (sometimes called a fuzzy subset of $G$) is an arbitrary function $\mu$ from $G$ into $[0,1]$. By $F(G)$ we denote the family of all fuzzy subsets in $G$. For $A \subseteq G$, the symbol $C_A$ we mean the characteristic function of $G$. The complement of $\mu$ in $A$ is denoted by $\overline{\mu}$. For any two fuzzy subsets $\mu$ and $\nu$ of $G$, $\mu \subseteq \nu$ if $\mu(x) \leq \nu(x)$ for every $x \in G$.

We begin this section with defining the $(n+1)$-ary operation on the set of all fuzzy subsets in $G$.

**Definition.** Let $\circ$ be an $(n+1)$-ary operation on $G$, and $\mu_1, \ldots, \mu_{n+1}$ fuzzy subsets of $G$. The $(n+1)$-ary operation $\mathcal{O}$ on the set of all fuzzy subsets of $G$ is defined by

(1) $\mathcal{O}(\mu_1, \ldots, \mu_{n+1})(x) = \sup\limits_{x=\circ(y_1,\ldots,y_{n+1})} (\min(\mu_1(x_1), \ldots, \mu_{n+1}(x_{n+1})))$ if $x$ can be expressed as $x = \circ(y_1, \ldots, y_{n+1})$ for any $y_1, \ldots, y_{n+1} \in G$,

(2) in all other cases, $\mathcal{O}(\mu_1, \ldots, \mu_{n+1})(x) = 0$.

The first theorem of this study concerning a construction of Menger algebras of all fuzzy subsets.

**Theorem 2.** *The $(n+1)$-ary operation $\mathcal{O}$ on $F(G)$ is superassociative if the $(n+1)$-ary operation $\circ$ on $G$ is superassociative.*

***Proof.*** Let $\mu, \nu_1, \dots, \nu_n, \rho_1, \dots, \rho_n$ be fuzzy setsets on $G$ and let $x$ be an arbitrary element in $G$. Clearly, the case when $x$ can not expressible as product of $n+1$ elements in $G$. Otherwise, by the superassociativity of $\circ$ on $G$, we obtain

$$\begin{aligned}
&\mathcal{O}(\mathcal{O}(\mu, \nu_1, \dots, \nu_n), \rho_1, \dots, \rho_n)(x)\\
&= \sup_{x=\circ(d,c_1\dots,c_n)} (\min(\mathcal{O}(\mu, \nu_1, \dots, \nu_n)(d), \rho_1(c_1), \dots, \rho_n(c_n)))\\
&= \sup_{x=\circ(d,c_1\dots,c_n)} (\min( \sup_{d=\circ(a,b_1\dots,b_n)} (\min(\mu(a), \nu_1(b_1), \dots, \nu_n(b_n))), \rho_1(c_1), \dots, \rho_n(c_n)))\\
&= \sup_{x=\circ(d,c_1\dots,c_n)} ( \sup_{d=\circ(a,b_1\dots,b_n)} (\min(\mu(a), \nu_1(b_1), \dots, \nu_n(b_n), \rho_1(c_1), \dots, \rho_n(c_n))))\\
&= \sup_{x=\circ(\circ(a,b_1\dots,b_n),c_1\dots,c_n)} (\min(\mu(a), \nu_1(b_1), \dots, \nu_n(b_n), \rho_1(c_1), \dots, \rho_n(c_n)))\\
&= \sup_{x=\circ(a,\circ(b_1,c_1\dots,c_n),\dots,\circ(b_n,c_1\dots,c_n))} (\min(\mu(a), \nu_1(b_1), \dots, \nu_n(b_n), \rho_1(c_1), \dots, \rho_n(c_n)))\\
&= \sup_{x=\circ(a,f_1,\dots,f_n)} ( \sup_{\substack{f_i=\circ(b_i,c_1,\dots,c_n)\\ i\in\{1,\dots,n\}}} (\min(\mu(a), \nu_1(b_1), \dots, \nu_n(b_n), \rho_1(c_1), \dots, \rho_n(c_n))))\\
&= \sup_{x=\circ(a,f_1,\dots,f_n)} (\min(\mu(a), \sup_{f_1=\circ(b_1,c_1,\dots,c_n)} (\min(\nu_1(b_1), \rho_1(c_1), \dots, \rho_n(c_n))), \dots,\\
&\quad \sup_{f_n=\circ(b_n,c_1,\dots,c_n)} (\min(\nu_1(b_1), \rho_1(c_1), \dots, \rho_n(c_n)))))\\
&= \sup_{x=\circ(a,f_1,\dots,f_n)} (\min(\mu(a), \mathcal{O}(\nu_1, \rho_1, \dots, \rho_n)(f_1), \dots, \mathcal{O}(\nu_n, \rho_1, \dots, \rho_n)(f_n)))\\
&= \mathcal{O}(\mu, \mathcal{O}(\nu_1, \rho_1, \dots, \rho_n), \dots, \mathcal{O}(\nu_n, \rho_1, \dots, \rho_n))(x).
\end{aligned}$$

■

As a consequence of Theorem 2, we have that the set $F(G)$ of all fuzzy subsets on $G$ forms a Menger algebra with respect to the $(n+1)$-ary operation $\mathcal{O}$. Further, it can be considered as a canonical generalization of the semigroup of fuzzy subsets under the compisition of fuzzy subsets if we put $n = 1$.

Following the suggestion of Liu [17], an identity for a fuzzy subsets with respect to the binary composition was given. In a Menger algebra, we also have the following.

**Theorem 3.** *If an $(n+1)$-ary operation $\circ$ on $G$ has selectors, then the fuzzy singleton $e_1, \dots, e_n \in F(G)$ are selectors of an $(n+1)$-ary operation $\mathcal{O}$, i.e.,*

$$\mathcal{O}(\mu, e_1, \dots, e_n) = \mu \text{ and } \mathcal{O}(e_i, \mu_1, \dots, \mu_n) = \mu_i$$

*for all $\mu, \mu_1, \dots, \mu_n \in F(G)$ and $1 \leq i \leq n$.*

***Proof.*** The statement follows immediately from Definition 2. ■

By Theorem 2, we can establish a strong relationship between the $(n+1)$-ary operation $\mathcal{O}$ on $F(G)$ and the $(n+1)$-ary operation $\circ$ on $G$ via the characteristic function in the following theorem.

**Theorem 4.** *Let $A_1, \ldots, A_{n+1}$ be nonempty subsets of a Menger algebra $(G, \circ)$. Then*

$$\mathcal{O}(C_{A_1}, \ldots, C_{A_{n+1}}) = C_{\circ(A_1,\ldots,A_{n+1})}.$$

Now we define fuzzy Menger subalgebras, fuzzy $v$-ideals, fuzzy $s$-ideals and fuzzy $vs$-ideals of Menger algebras.

**Definition.** Let $(G, \circ)$ be a Menger algebra. A fuzzy subset $\mu$ of $G$ is called

(1) a *fuzzy Menger subalgebra* of $G$ if

$$\mu(\circ(g_1, \ldots, g_{n+1})) \geq \min\{\mu(g_1), \ldots, \mu(g_{n+1})\}$$

for all $g_1, \ldots, g_{n+1} \in G$,

(2) a *fuzzy $v$-ideal* of $G$ if

$$\mu(\circ(g, h_1, \ldots, h_n)) \geq \min\{\mu(h_1), \ldots, \mu(h_n)\}$$

for all $g, h_1, \ldots, h_n \in G$,

(3) a *fuzzy $s$-ideal* of $G$ if

$$\mu(\circ(h, g_1, \ldots, g_n)) \geq \mu(h)$$

for all $h, g_1, \ldots, g_n \in G$,

(4) a *fuzzy $vs$-ideal* of $G$ it is both a fuzzy $v$-ideal and fuzzy $s$-ideal of $G$.

The following theorem provides a characterization of Menger subalgebras, $v$-ideals, $s$-ideals and $vs$-ideals using Definition 2.

**Theorem 5.** *Let $(G, \circ)$ be a Menger algebra and $\emptyset \neq A \subseteq G$. Then the following assertions hold.*

(1) *$A$ is a Menger subalgebra of $G$ if and only if the characteristic function $\mu_A$ is a fuzzy Menger subalgebra of $G$.*

(2) *$A$ is a $v$-ideal ($s$-ideal, $vs$-ideal) of $G$ if and only if the characteristic function $\mu_A$ is a fuzzy $v$-ideal (fuzzy $s$-ideal, fuzzy $vs$-ideal) of $G$.*

***Proof.*** (1) Assume that $A$ is a Menger subalgebra of $G$. Let $g_1, \ldots, g_{n+1}$ be elements in $G$. We consider the case when $g_1, \ldots, g_{n+1} \in A$. Then $\mu_A(\circ(g_1, \ldots, g_{n+1})) = 1 \geq \min\{\mu_A(g_1), \ldots, \mu_A(g_{n+1})\}$. If there exists $1 \leq j \leq n+1$ such that $g_j \notin A$, then we have $\min\{\mu_A(g_1), \ldots, \mu_A(g_{n+1})\} = 1 \leq \mu_A(\circ(g_1, \ldots, g_{n+1}))$. For the converse, let $g_1, \ldots, g_{n+1} \in A$. Then $\mu_A(g_j) = 1$ for all $1 \leq j \leq n+1$. It follows from the hypothesis that $\mu_A(\circ(g_1, \ldots, g_{n+1})) \geq \min\{\mu_A(g_1), \ldots, \mu_A(g_{n+1})\} = 1$. So, $\circ(g_1, \ldots, g_{n+1}) \in A$ and thus $A$ is a Menger subalgebra of $G$.

To prove (2) holds, suppose first that $A$ is a $v$-ideal of $G$ and $g, h_1, \ldots, h_n \in G$. If $h_1, \ldots, h_n \in A$, then we have $\circ(g, h_1, \ldots, h_n) \in A$ by the assumption.

This implies that $\mu_A(\circ(g,h_1,\ldots,h_n)) = 1 \geq \min\{\mu(h_1),\ldots,\mu(h_n)\}$. Thus, $\mu_A$ is a fuzzy $v$-ideal of $G$. For the case $h_j \notin A$ for some $1 \leq j \leq n$, we have $\min\{\mu(h_1),\ldots,\mu(h_n)\} = 0$, and so $\mu_A(\circ(g,h_1,\ldots,h_n)) \geq \min\{\mu(h_1),\ldots,\mu(h_n)\}$. Conversely, let $h_1,\ldots,h_n \in A$ and $g \in G$. Then $\mu_A(h_j) = 1$ for all $1 \leq j \leq n$ and thus $\mu_A(\circ(g,h_1,\ldots,h_n)) \geq \min\{\mu_A(h_1),\ldots,\mu_A(h_n)\} = 1$ since $\mu_A$ is a fuzzy $v$-ideal of $G$. This shows that $A$ is a $v$-ideal of $G$. ∎

Necessary and sufficient conditions for a fuzzy subset to be a fuzzy Menger subalgebra and a fuzzy $v$-ideal, fuzzy $s$-ideal and fuzzy $vs$-ideal through the $(n+1)$-ary operation $\mathcal{O}$ on $F(G)$ are presented below.

**Theorem 6.** *Let $\mu$ and $\nu$ be two fuzzy subsets of a Menger algebra $(G,\circ)$. Then*

(1) *$\mu$ is a fuzzy Menger subalgebra of G if and only if $\mathcal{O}(\mu,\underbrace{\mu,\ldots,\mu}_{n\ times}) \subseteq \mu$.*

(2) *$\mu$ is a fuzzy $s$-ideal of G if and only if $\mathcal{O}(\mu,\underbrace{G,\ldots,G}_{n\ times}) \subseteq \mu$.*

(3) *$\mu$ is a fuzzy $v$-ideal of G if and only if $\mathcal{O}(G,\underbrace{\mu,\ldots,\mu}_{n\ times}) \subseteq \mu$.*

(4) *$\mu$ is a fuzzy $vs$-ideal of G if and only if $\mathcal{O}(G,\underbrace{\mu,\ldots,\mu}_{n\ times}) \subseteq \mu$ and*

*$\mathcal{O}(\mu,\underbrace{G,\ldots,G}_{n\ times}) \subseteq \mu$.*

***Proof.*** Firstly, we prove (1). Let $\mu$ be a fuzzy Menger subalgebra of $G$ and $a \in G$. If $a$ can not be expressed in the form $\circ(x,y_1,\ldots,y_n)$ for any $x,y_1\ldots,y_n \in G$, then $\mathcal{O}(\mu,G,\ldots,G)(a) = 0 \leq \mu(a)$. If there exist elelments $x,y_1,\ldots,y_n \in G$ such that $a = \circ(x,y_1,\ldots,y_n)$, then, according to defining the $(n+1)$-ary operation $\mathcal{O}$ and the fact that $\mu$ is a fuzzy Menger subalgebra of $G$, we have

$$\begin{aligned}
\mathcal{O}(\mu,\mu,\ldots,\mu)(a) &= \sup_{a=\circ(x,y_1\ldots,y_n)} [\min\{\mu(x),\mu(y_1),\ldots,\mu(y_n)\}] \\
&\leq \sup_{a=\circ(x,y_1\ldots,y_n)} [\min\{\mu(\circ(x,y_1,\ldots,y_n))\}] \\
&= \mu(a).
\end{aligned}$$

It implies that $\mathcal{O}(\mu,\mu,\ldots,\mu) \subseteq \mu$. For the opposite inclusion, assume that $\mathcal{O}(\mu,G,\ldots,G) \subseteq \mu$. Let $x,y_1,\ldots,y_n$ be elements in $G$. Then $\circ(x,y_1,\ldots,y_n) \in G$. Let $a = \circ(x,y_1,\ldots,y_n)$. Our assumption implies that

$$\begin{aligned}
\mu(\circ(x,y_1,\ldots,y_n)) = \mu(a) &\geq \mathcal{O}(\mu,\mu,\ldots,\mu)(a) \\
&= \sup_{a=\circ(b,c_1\ldots,c_n)} [\min\{\mu(b),\mu(c_1),\ldots,\mu(c_n)\}] \\
&\geq \min\{\mu(x),\mu(y_1),\ldots,\mu(y_n)\}.
\end{aligned}$$

Similarly, we can prove the other statements. ∎

In order to give another characterization for a fuzzy subset $\mu$ to be a fuzzy Menger subalgebra, fuzzy $s$-ideal, fuzzy $v$-ideal and fuzzy $vs$-ideal, we propose the following essential sets.

Let $\mu$ be a fuzzy subset of a Menger algebra $(G, \circ)$. For any $t \in [0,1]$, the sets $U(\mu;t) = \{x \in G \mid \mu(x) \geq t\}$ and $L(\mu;t) = \{x \in G \mid \mu(x) \leq t\}$ are called an upper $t$-level subset of $\mu$ and an lower $t$-level subset of $\mu$, respectively.

**Example 7.** On the Menger algebra $\mathbb{R}^+$ of all positive real numbers under the $(n+1)$-ary operation, defined by $*(x_0, \ldots, x_n) = x_0 \sqrt[n]{x_1 \cdots x_n}$, define a fuzzy subset $\mu$ on $(\mathbb{R}^+, *)$ by

$$\mu(x) = \begin{cases} 0 & \text{if } 0 < x \leq 100, \\ 0.75 & \text{if } 100 < x \leq 1000, \\ 1 & \text{otherwise.} \end{cases}$$

Then $U(\mu; 0.75) = [100, \infty)$ and $L(\mu; 0.75) = (0, 100]$.

We can use these sets to characterize a fuzzification as follows.

**Theorem 8.** *Let $\mu$ be a fuzzy subset of a Menger algebra $(G, \circ)$. Then*

(1) *$\mu$ is a fuzzy Menger subalgebra of $G$ if and only if for all $0 \leq t \leq 1$, if each its nonempty upper $t$-level subset is a Menger subalgebra of $G$.*

(2) *$\mu$ is a fuzzy $v$-ideal (fuzzy $s$-ideal, fuzzy $vs$-ideal) of $G$ if and only if for all $0 \leq t \leq 1$, if each its nonempty upper $t$-level subset is a $v$-ideal ($s$-ideal, $vs$-ideal) of $G$.*

***Proof.*** To prove (1), assume that $\mu$ is a fuzzy Menger subalgebra of $G$ Let $t \in [0,1]$ be such that $U(\mu;t) \neq \emptyset$. If $x_1, \ldots, x_{n+1} \in G$, then $\mu(x_j) \geq t$ for all $1 \leq j \leq n+1$. By the assumption, we have $\mu(\circ(x_1, \ldots, x_{n+1})) \geq \min_{1 \leq j \leq n+1}(\mu(x_j)) \geq t$ and then $\circ(x_1, \ldots, x_{n+1}) \in U(\mu;t)$. Conversely, let $x_1, \ldots, x_{n+1} \in G$. Choose $t = \min_{1 \leq j \leq n+1}(\mu(x_j))$. Then for each $1 \leq j \leq n+1$, we obtain $\mu(x_j) \geq t$, which implies that $x_j \in U(\mu;t)$ for all $1 \leq j \leq n+1$. Since $U(\mu;t)$ is a Menger subalgebra of $G$, $\circ(x_1, \ldots, x_{n+1}) \in U(\mu;t)$. So $\mu(\circ(x_1, \ldots, x_{n+1})) \geq t = \min_{1 \leq j \leq n+1}(\mu(x_j))$. As a result, $\mu$ is a fuzzy Menger subalgebra of $G$. The proof of statement (2) is omitted. ∎

To present sufficient and necessary conditions for the complement of a fuzzy subset $\mu$ to be a fuzzy Menger subalgebra of $G$ and other fuzzy ideals, we need the following lemma.

**Lemma 9.** *Let $\mu$ be a fuzzy subset in a Menger algebra $(G, \circ)$. For any positive integer $1 \leq i \leq n+1$, the following assertions are valid.*

(1) $1 - \min\limits_{1 \leq i \leq n+1}(\mu(x_i)) = \max\limits_{1 \leq i \leq n+1}(1 - \mu(x_i))$.

(2) $1 - \max_{1 \leq i \leq n+1}(\mu(x_i)) = \min_{1 \leq i \leq n+1}(1 - \mu(x_i))$.

***Proof.*** We first show that (1) holds. Assume that $\min_{1 \leq i \leq n+1}(\mu(x_i)) = \mu(x_j)$ for some $1 \leq j \leq n+1$. Then $\mu(x_j) \leq \mu(x_k)$ for all $1 \leq k \leq n+1$. Thus $1-\mu(x_j) \geq 1-\mu(x_k)$. So $\max_{1 \leq i \leq n+1}(1-\mu(x_i)) = 1-\mu(x_j) = 1-\min_{1 \leq i \leq n+1}(\mu(x_i))$. In order to prove (2), we suppose that there is $1 \leq j \leq n+1$ such that $\max_{1 \leq i \leq n+1}(\mu(x_i)) = \mu(x_j)$. Then we have $\mu(x_k) \leq \mu(x_j)$ for all $1 \leq k \leq n+1$. Hence $1 - \mu(x_k) \geq 1 - \mu(x_j)$. So $\min_{1 \leq i \leq n+1}(1 - \mu(x_i)) = 1 - \mu(x_j) = 1 - \max_{1 \leq i \leq n+1}(\mu(x_i))$. The proof is actually finished. ■

**Theorem 10.** *Let $\mu$ be a fuzzy subset of a Menger algebra $(G, \circ)$. Then $\overline{\mu}$ is a fuzzy Menger subalgebra of $G$ if and only if for all $t \in [0,1]$, $L(\mu; t)$ is a Menger subalgebra of $G$, if $L(\mu; t) \neq \emptyset$.*

***Proof.*** The proof follows immediately from Lemma 9. ■

**Theorem 11.** *Let $\mu$ be a fuzzy subset of a Menger algebra $(G, \circ)$. Then $\overline{\mu}$ is a fuzzy $v$-ideal (fuzzy $s$-ideal, fuzzy $vs$-ideal) of $G$ if and only if for all $t \in [0,1]$, $L(\mu; t)$ is a $v$-ideal ($s$-ideal, $vs$-ideal) of $G$, if $L(\mu; t) \neq \emptyset$.*

***Proof.*** Applying Lemma 9, the proof is obtained. ■

## 3. FUZZY CONGRUENCES ON MENGER ALGEBRAS

In this section the notion of a fuzzy congruence relation on Menger algebras is introduced and their properties are dealt with in detail.

Before we begin the results, we will use the following notation: for nonnegative integers $i, j$, the sequence $x_i, \ldots, x_j$ is well defined if $i < j$. Otherwise, if $i > j$, $x_i, \ldots, x_j$ is the empty symbol. For convention, we sometime write $x_i^j$ instead of a sequence of the form $x_i, \ldots, x_j$.

A fuzzy subset $\mu$ of $G \times G$ is called a *fuzzy relation* on $G$. A fuzzy equivalence relation is a fuzzy relation satisfying the conditions:

(1) (fuzzy reflexive) $\mu(x, x) = 1$ for all $x \in G$,

(2) (fuzzy symmetric) $\mu(x, y) = \mu(y, x)$,

(3) (fuzzy transitive) $\mu(x, y) \geq \sup_{z \in G}(\min(\mu(x, z), \mu(z, y)))$,

for all $x, y \in G$. We note that $\mu$ is fuzzy transitive if and only if $\mu \circ \mu \subseteq \mu$.

**Definition.** A fuzzy relation on a Menger algebra $(G, \circ)$ is called a *fuzzy $i$-compatible* relation where $1 \leq i \leq n+1$ if

$$\mu\left(\circ\left(x_1^{i-1}, a, x_{i+1}^{n+1}\right), \circ\left(x_1^{i-1}, b, x_{i+1}^{n+1}\right)\right) \geq \mu(a, b)$$

for all $x_1^{i-1}, x_{i+1}^{n+1}, a, b \in G$. A fuzzy relation on $G$ is called a *fuzzy compatible* relation if it is a fuzzy $i$-compatible relation for every $1 \leq i \leq n+1$.

**Definition.** A *fuzzy $i$-congruence* relation on a Menger algebra $(G, \circ)$ is a fuzzy equivalence relation on $G$ and a fuzzy $i$-compatible where $1 \leq i \leq n+1$. A fuzzy equivalence relation on $G$ which is compatible is called a *fuzzy congruence* relation on $G$.

**Example 12.** The set $\mathbb{R}$ of all real numbers is a Menger algebra with respect to the $(n+1)$-ary operation $\circ$ defined by $\circ(a, b_1, \ldots, b_n) = a + \frac{b_1 + \cdots + b_n}{n}$ where $+$ is the usual addition. The fuzzy relation $\mu$ on $\mathbb{R}$ defined by

$$\mu(a,b) = \begin{cases} 1 & \text{if } a = b, \\ 0.5 & \text{if } a \neq b \text{ and } |a| = |b|, \\ 0 & \text{in all other cases,} \end{cases}$$

is a fuzzy congruence relation on $\mathbb{R}$.

Applying the result of Corollary 3.4 in [21], a characterization of a fuzzy congruence relation on a Menger algebra is presented as follows.

**Theorem 13.** *A fuzzy equivalence relation $\mu$ on a Menger algebra $(G, \circ)$ is a fuzzy congruence relation on $G$ if and only if*

$$\mu\left(\circ\left(x_1^{n+1}\right), \circ\left(y_1^{n+1}\right)\right) \geq \min\left(\mu(x_1, y_1), \ldots, \mu\left(x_{n+1}, y_{n+1}\right)\right).$$

There are several possibilities to provide a characterization of a fuzzy congruence relation on a Menger algebra. For these, we present the following two binary relations.

Let $\mu$ be a fuzzy relation on a Menger algebra $(G, \circ)$. For each $\lambda \in [0,1]$, the upper $\lambda$-level set of $\mu$ is a relation

$$\tilde{U}(\mu; \lambda) = \{(a,b) \in G \times G \mid \mu(a,b) \geq \lambda\}.$$

Similarly, the lower $\lambda$-level set of $\mu$ is a relation

$$\tilde{L}(\mu; \lambda) = \{(a,b) \in G \times G \mid \mu(a,b) \leq \lambda\}.$$

Now, a charaterization of fuzzy congruences using the upper $\lambda$-level set is proposed.

**Theorem 14.** *A fuzzy relation $\mu$ is fuzzy congruence on a Menger algebra $(G, \circ)$ if and only if for each $\lambda \in [0,1]$, $\tilde{U}(\mu; \lambda)$ is congruence on $G$.*

***Proof.*** It is not difficult to show that $\tilde{U}(\mu;\lambda)$ is congruence on $G$. For the converse, let $\lambda \in [0,1]$. Suppose that $\tilde{U}(\mu;\lambda)$ is congruence on $G$. Since, $\tilde{U}(\mu;\lambda)$ is reflexive, $\mu(x,x) \geq \lambda$ for all $x \in G$ and thus $\mu(x,x) = 1$. For each $x, y \in G$, suppose, to the contrary, that $\mu(x,y) \neq \mu(y,x)$. Let $\mu(x,y) = \lambda_1$ and $\mu(y,x) = \lambda_2$. If $\lambda_1 > \lambda_2$, then $(y,x) \notin \tilde{U}(\mu;\lambda)$. Since $(x,y) \in \tilde{U}(\mu;\lambda)$ and $\tilde{U}(\mu;\lambda)$ is symmetric, we have $(y,x) \in \tilde{U}(\mu;\lambda)$, which is a contradiction. Similarly, in the case when $\lambda_1 < \lambda_2$. Hence, $\mu$ is fuzzy symmetric. For any $x, y, z \in G$, let $\mu(x,z) = \beta_1$ and $\mu(z,y) = \beta_2$. We consider the first case when $\beta_1 \leq \beta_2$, then $(x,z)$ and $(z,y)$ belong to $\tilde{U}(\mu;\lambda)$ and so $(x,y) \in \tilde{U}(\mu;\lambda)$. It follows that $\mu(x,y) \geq \beta_1 = \sup_{z\in G}(\min(\mu(x,z),\mu(z,y)))$. Thus $\mu$ is a fuzzy transitive. We can prove in the same manner if $\beta_1 > \beta_2$. Hence, $\mu$ is a fuzzy equivalence relation on $G$. For each $1 \leq i \leq n+1$, let $x_i$ and $y_i$ be elements in $G$ such that $(x_i, y_i) \in \tilde{U}(\mu;\lambda)$. Assume that $\mu(x_i,y_i) = \beta_i$ for all $1 \leq i \leq n+1$. Put $\epsilon = \min_{1\leq i\leq n+1}(\beta_i)$. Then we have $\mu(x_i,y_i) \geq \epsilon$ and so $(x_i,y_i) \in \tilde{U}(\mu;\lambda)$. This implies that $\mu(\circ(x_1^{n+1}), \circ(y_1^{n+1})) \geq \epsilon = \min_{1\leq i\leq n+1}(\beta_i) = \min_{1\leq i\leq n+1}((x_i,y_i))$. Consequently, $\mu$ is a fuzzy congruence on $G$. ■

Let $\mu$ be a fuzzy relation on $G$. The fuzzy relation $\overline{\mu}$ defined by, for all $x, y \in G$, $\overline{\mu}(x,y) = 1 - \mu(x,y)$ is called the *complement* of $\mu$ in $G$.

**Theorem 15.** *A fuzzy relation $\overline{\mu}$ is fuzzy congruence on a Menger algebra $(G,\circ)$ if and only if for each $\lambda \in [0,1]$, $\tilde{L}(\mu;\lambda)$ is congruence on $G$.*

***Proof.*** We can prove in the same manner as in Theorem 14. ■

## 4. Quotient Menger algebras induced by fuzzy congruences

The main aim of this section is to apply a fuzzy congruence relation which given in Section 3 for a construction of quotient Menger algebras in a natural way. We will supplement these results by establishing further properties of their corresponding homomorphisms.

Let $\mu$ be a fuzzy congruence of a Menger algebra $(G,\circ)$. For any $x, y \in G$, we define a binary relation on $G$ by

$$x \sim y \text{ if and only if } \mu(x,y) = 1.$$

**Proposition 16.** *A binary relation $\sim$ is congruence on $G$.*

For every $x \in G$, we associate the set $\mu_x = \{y \in G \mid y \sim x\}$. Then $\mu_x$ is a congruence class that contains $x$. We now construct a quotient set $G/\mu$ for some fuzzy congruence $\mu$ as follow $G/\mu := G/\sim = \{\mu_x \mid x \in G\}$.

**Remark 17.** $\mu_x = \mu_y$ if and only if $\mu(x,y) = 1$.

**Definition.** On the quotient set $G/\mu$ for some fuzzy congruence $\mu$ on a Menger algebra $(G, \circ)$, an $(n+1)$-ary operation $\circ^{G/\mu}$ is defined by

$$\circ^{G/\mu}(\mu_{x_1}, \ldots, \mu_{x_{n+1}}) = \mu_{\circ(x_1,\ldots,x_{n+1})}.$$

**Theorem 18.** *Let $\mu$ be a fuzzy congruence relation on a Menger algebra $(G, \circ)$. The quotient set $G/\mu$ is a Menger algebra under the $(n+1)$-ary operation defined in Definition 4.*

***Proof.*** It follows directly from Remark 17 that the $(n+1)$-ary operation $\circ^{G/\mu}$ on $G/\mu$ is well-defined. The superassociativity of the fundamental operation $\circ$ on $G$ implies that the operation $\circ^{G/\mu}$ also satisfies superassociative law too. ■

This Menger algebra $(G/\mu, \circ^{G/\mu})$ is called the quotient of the Menger algebra by the fuzzy congruence $\mu$. To present several extensive connections between fuzzy congruence relations and homomorphisms on Menger algebras, some potential preparations are needed.

**Definition.** Let $\alpha$ be a mapping from a Menger algebra $(G, \circ)$ to a Menger algebra $(K, *)$. Let $\mu$ and $\mu'$ be fuzzy relations of $G$ and $K$, respectively. Then, the *inverse image* $f^{-1}(\lambda)$ of $\lambda$ is a fuzzy subset on $G$ defined by

$$\alpha^{-1}(\mu')(x, y) = \mu'(\alpha(x), \alpha(y)),$$

for all $x \in X$. The *image* $\alpha(\mu)$ of $\mu$ is a fuzzy relation on $K$ defined by

$$\alpha(\mu)(x, y) = \begin{cases} \sup\limits_{(x_i,y_i)} (\mu(x_i, y_i)) & \text{if } \alpha^{-1}(x, y) \neq \emptyset, \\ 0 & \text{in all other cases,} \end{cases}$$

for all $x, y \in K, x_i, y_i \in G$ and $1 \leq i \leq n+1$.

It is easy to see that for any fuzzy congruence $\mu$ on a Menger algebra $(G, \circ)$ satisfying $\mu \subseteq \alpha^{-1}(\alpha(\mu))$, then $\mu = \alpha^{-1}(\alpha(\mu))$ if $\alpha$ is injective.

**Theorem 19.** *Let $\alpha$ be a homomorphism from a Menger algebra $(G, \circ)$ to a Menger algebra $(K, *)$. If $\mu'$ is a fuzzy congruence on $K$, then the inverse image $\alpha^{-1}(\mu')$ of $\mu'$ is a fuzzy congruence on $G$.*

***Proof.*** For any elements $x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1}$ of $G$, it follows from Theorem 13 that

$$\begin{aligned} \alpha^{-1}(\mu')\big(\circ\big(x_1^{n+1}\big), \circ\big(y_1^{n+1}\big)\big) &= \big(\alpha\big(\circ\big(x_1^{n+1}\big)\big), \alpha\big(\circ\big(y_1^{n+1}\big)\big)\big) \\ &= (*(\alpha(x_1), \ldots, \alpha(x_{n+1})), *(\alpha(y_1), \ldots, \alpha(y_{n+1}))) \\ &\geq \min(\mu'(\alpha(x_1), \alpha(y_1)), \ldots, \mu'(\alpha(x_{n+1}), \alpha(y_{n+1}))) \\ &= \min(\alpha^{-1}(\mu')(x_1, y_1), \ldots, \alpha^{-1}(\mu')(x_{n+1}, y_{n+1})). \end{aligned}$$

This shows that the relation $\alpha^{-1}(\mu')$ is a fuzzy congruence relation on $G$. ■

**Theorem 20.** *Let $(G,\circ)$ and $(K,*)$ be two Menger algebras, $\alpha$ an epimorphism from $G$ to $K$ and $\mu'$ a fuzzy congruence relation on $K$. Then*

$$G/\alpha^{-1}(\mu') \cong K/\mu'.$$

***Proof.*** Clearly, the sets $G/\alpha^{-1}(\mu')$ and $K/\mu'$ are both Menger algebras by Theorems 18 and 19. On these two sets, we denote the $(n+1)$-ary operations by $\circ^{G/\alpha^{-1}(\mu')}$ and $*^{K/\mu'}$, respectively. A mapping $\beta$ from $G/\alpha^{-1}(\mu')$ to $K/\mu'$ can be defined by

$$\beta(\alpha^{-1}(\mu')_x) = \mu'_{\alpha(x)}$$

for all $x \in G$. Firstly, we show that this defining is well-defined. To do this, assume that $\alpha^{-1}(\mu')_x = \alpha^{-1}(\mu')_y$. Then, according to Remark 17, we have $\alpha^{-1}(\mu')(x,y) = 1$. By the definition of inverse image, we obtain $\mu'(\alpha(x),\alpha(y)) = 1$, which means that $\mu'_{\alpha(x)} = \mu'_{\alpha(y)}$. To show that $\beta$ is a homomorphism, let $x_1,\ldots,x_{n+1}$ be any elements of $G$. Then we have

$$\begin{aligned}
&\beta\left(\circ^{G/\alpha^{-1}(\mu')}(\alpha^{-1}(\mu')_{x_1},\ldots,\alpha^{-1}(\mu')_{x_{n+1}})\right)\\
&= \beta\left(\alpha^{-1}(\mu')_{\circ(x_1,\ldots,x_{n+1})}\right)\\
&= \mu'_{\alpha(\circ(x_1,\ldots,x_{n+1}))}\\
&= \mu'_{*(\alpha(x_1),\ldots,\alpha(x_{n+1}))}\\
&= *^{K/\mu'}\left(\mu'_{\alpha(x_1)},\ldots,\mu'_{\alpha(x_{n+1})}\right)\\
&= *^{K/\mu'}\left(\beta(\alpha^{-1}(\mu')_{x_1}),\ldots,\beta(\alpha^{-1}(\mu')_{x_{n+1}})\right).
\end{aligned}$$

It is not difficult to prove that $\beta$ is surjective. In fact, for any $\mu'_y \in K/\mu'$, since $\alpha$ is surjective, then there exits $x \in G$ such that $\alpha(x) = y$. Thus $\beta(\alpha^{-1}(\mu')_x) = \mu'_{\alpha(x)} = \mu'_y$. It is actually injective, since, for all $x, y \in G$, suppose that $\beta(\alpha^{-1}(\mu')_x) = \beta(\alpha^{-1}(\mu')_y)$. Then $\mu'_{\alpha(x)} = \mu'_{\alpha(y)}$, which implies that $\mu'(\alpha(x),\alpha(y)) = 1$. By Remark 17, we have $\alpha^{-1}(\mu')(x,y) = 1$. So, $\alpha^{-1}(\mu')_x = \alpha^{-1}(\mu')_y$. Therefore, $\beta$ is an injection. We finally conclude that a mapping $\beta$ is an isomorphism from $G/\alpha^{-1}(\mu')$ to $K/\mu'$. ■

The image of a fuzzy congruence relation under a homomorphism is investigated in the next theorem.

**Theorem 21.** *Let $\alpha$ be a homomorphism from a Menger algebra $(G,\circ)$ to a Menger algebra $(K,*)$. If $\mu$ is a fuzzy congruence relation on $G$, then the image $\alpha(\mu)$ of $\mu$ is also a fuzzy congruence relation on $K$.*

***Proof.*** Let $h_1, \ldots, h_{n+1}, k_1, \ldots, k_{n+1} \in K$. Since $\alpha$ is an epimorphism, there exists $f_1, \ldots, f_{n+1}, g_1, \ldots, g_{n+1} \in G$ such that $\alpha(f_i) = h_i$ and $\alpha(g_i) = k_i$ for every $1 \le i \le n+1$. Thus $\alpha(\circ(f_1, \ldots, f_{n+1})) = *(\alpha(f_1), \ldots, \alpha(f_{n+1})) = *(h_1, \ldots, h_{n+1})$ and $\alpha(\circ(g_1, \ldots, g_{n+1})) = *(\alpha(g_1), \ldots, \alpha(g_{n+1})) = *(k_1, \ldots, k_{n+1})$. Hence,

$$\left\{(f_i, g_i)(i = 1, \ldots, n+1) \mid (f_i, g_i) \in \alpha^{-1}(*(h_1, \ldots, h_{n+1}), *(k_1, \ldots, k_{n+1}))\right\}$$
$$\supseteq \left\{(\circ(f_1, \ldots, f_{n+1}), \circ(g_1, \ldots, g_{n+1})) \mid (f_i, g_i) \in \alpha^{-1}(h_i, k_i)(i = 1, \ldots, n+1)\right\}.$$

It implies that

$$\begin{aligned}
&\alpha(\mu)(*(h_1, \ldots, h_{n+1}), *(k_1, \ldots, k_{n+1})) \\
&= \sup_{(f_i,g_i)\in\alpha^{-1}(*(h_1,\ldots,h_{n+1}),*(k_1,\ldots,k_{n+1}))} (\mu(f_i, g_i)) \\
&\ge \sup_{(f_1,g_1)\in\alpha^{-1}(h_1,k_1),\ldots,(f_{n+1},g_{n+1})\in\alpha^{-1}(h_{n+1},k_{n+1})} (\mu(\circ(f_1, \ldots, f_{n+1}), \circ(g_1, \ldots, g_{n+1}))) \\
&\ge \sup_{(f_1,g_1)\in\alpha^{-1}(h_1,k_1),\ldots,(f_{n+1},g_{n+1})\in\alpha^{-1}(h_{n+1},k_{n+1})} (\min_{1\le i\le n+1} (\mu(f_i, g_i))) \\
&= \min_{1\le i\le n+1} (\sup_{(f_i,g_i)\in\alpha^{-1}(h_i,k_i)} (\mu(f_i, g_i))) \\
&= \min_{1\le i\le n+1} (\alpha(\mu)(h_i, k_i)).
\end{aligned}$$

This completes the proof. ■

**Theorem 22.** *Let $(G, \circ)$ and $(K, *)$ be two Menger algebras, $\alpha$ an isomorphism from $G$ to $K$ and $\mu$ a fuzzy congruence relation on $G$. Then*

$$G/\mu \cong K/\alpha(\mu).$$

***Proof.*** We first obtian immediately from Theorems 18 and 21 that $G/\mu$ and $K/\alpha(\mu)$ are Menger algebras. Suppose that $\circ^{G/\mu}$ and $*^{K/\alpha(\mu)}$ are $(n+1)$-ary superassociative operations on $G/\mu$ and $K/\alpha(\mu)$, respectively. We define a mapping $\gamma : G/\mu \to K/\alpha(\mu)$ by $\gamma(\mu_x) = \alpha(\mu)_{\alpha(x)}$ for all $x \in G$. Obviously, $\gamma$ is well-defined. In fact, let $x, y \in G$. Assume that $\mu_x = \mu_y$. Then $\mu(x, y) = 1$. By Definition 4, we have $\alpha(\mu)(\alpha(x), \alpha(y)) = \sup_{(x,y)\in\alpha^{-1}(\alpha(x),\alpha(y))} (\mu(x, y)) = 1$, which implies that $\alpha(\mu)_{\alpha(x)} = \alpha(\mu)_{\alpha(y)}$ and thus $\gamma(\mu_x) = \gamma(\mu_y)$. In order to prove the homomorphism property, let $g_1, \ldots, g_n \in G$. Then we have

$$\begin{aligned}
\gamma(\circ^{G/\mu}(\mu_{g_1}, \ldots, \mu_{g_{n+1}})) &= \gamma(\mu_{\circ(g_1^{n+1})}) \\
&= \alpha(\mu)_{\alpha(\circ(g_1^{n+1}))} \\
&= \alpha(\mu)_{*(\alpha(g_1),\ldots,\alpha(g_{n+1}))} \\
&= *^{K/\alpha(\mu)}(\alpha(\mu)_{\alpha(g_1)}, \ldots, \alpha(\mu)_{\alpha(g_{n+1})}) \\
&= *^{K/\alpha(\mu)}(\gamma(\mu_{g_1}), \ldots, \gamma(\mu_{g_{n+1}})).
\end{aligned}$$

Hence, $\gamma$ is a homomorphism. To show that $\gamma$ is surjective, let $\alpha(\mu)_y \in K/\alpha(\mu)$ and $y \in K$. Since $\alpha$ is surjective, there exists $x \in G$ such that $\alpha(x) = y$. So $\gamma(\mu_x) = \alpha(\mu)_{\alpha(x)} = \alpha(\mu)_y$. Finally, let $x, y$ be two elements in $G$. Assume that $\alpha(\mu)_{\alpha(x)} = \alpha(\mu)_{\alpha(y)}$. By the injectivity of $\alpha$, we have $\mu = \alpha^{-1}(\alpha(\mu))$. Then $\mu(x,y) = \alpha^{-1}(\alpha(\mu))(x,y) = \alpha(\mu)(\alpha(x), \alpha(y)) = 1$. It follows that $\mu_x = \mu_y$ and so $\gamma$ is injective. Therefore $\gamma$ is an isomorphism from $G/\mu$ to $K/\alpha(\mu)$. This completes the proof. ∎

**Example 23.** It is not difficult to prove that a Menger algebra $(\mathbb{R}, \circ)$ where the $(n+1)$-ary operation $\circ$ is defined by $\circ(a, b_1, \ldots, b_n) = a + \frac{b_1 + \cdots b_n}{n}$ and a Menger algebra $(\mathbb{R}^+, *)$ where the operation $\circ : (\mathbb{R}^+)^{n+1} \to \mathbb{R}^+$ is defined by $\circ(x_0, \ldots, x_n) = x_0 \sqrt[n]{x_1 \cdots x_n}$ are isomorphic under a mapping $\alpha : (\mathbb{R}, \circ) \to (\mathbb{R}^+, *)$ defined by $\alpha(x) = 2^x$ for every real number $a$. Applying the fuzzy congruence relation $\mu$ on $\mathbb{R}$ which given already in Example 12 and Theorem 22, we immediately obtain $\mathbb{R}/\mu \cong \mathbb{R}^+/\alpha(\mu)$.

One interesting application of homomorphisms is to the situation where $\mu_1$ and $\mu_2$ are two fuzzy congruences on $G$ with $\mu_1 \subseteq \mu_2$.

**Theorem 24.** *Let $\mu_1$ and $\mu_2$ be two fuzzy congruences on a Menger algebra $(G, \circ)$ with $\mu_1 \subseteq \mu_2$. Then the fuzzy relation $\mu_2/\mu_1$ on $G$, given by*

$$(\mu_2/\mu_1)((\mu_1)_x, (\mu_1)_y) = \mu_2(x, y),$$

*is a fuzzy congruence on $G$.*

***Proof.*** First of all, the fact that $\mu_2/\mu_1$ is well-defined follows immediately from Theorem 3.3 in [27]. It is obviously clear that $\mu_2/\mu_1$ is fuzzy equivalence of $G$. It is not hard to prove that $\mu_2/\mu_1$ is a fuzzy compatible relation on $G$. ∎

**Theorem 25.** *Let $\mu_1$ and $\mu_2$ be two fuzzy congruences on a Menger algebra $(G, \circ)$ with $\mu_1 \subseteq \mu_2$. Then*

$$(G/\mu_1)/(\mu_2/\mu_1) \cong G/\mu_2.$$

***Proof.*** The fact that $\mu_2/\mu_1$ is a fuzzy relation on $G$ has been shown in the proof of Theorem 24. Then, by Theorem 18, $(G/\mu_1)/(\mu_2/\mu_1)$ and $G/\mu_2$ form Menger algebras. Denote the $(n+1)$-ary superassociative operation on $(G/\mu_1)/(\mu_2/\mu_1)$ and $G/\mu_2$ by $\circ^{(G/\mu_1)/(\mu_2/\mu_1)}$ and $\circ^{G/\mu_2}$, respectively. Now define a mapping

$$\alpha : (G/\mu_1)/(\mu_2/\mu_1) \to G/\mu_2$$

by $\alpha((\mu_2/\mu_1)_{(\mu_1)_x}) = (\mu_2)_x$ for all $x \in G$. Then $\alpha$ is both well-defined and injective, since

$$\begin{aligned}(\mu_2/\mu_1)_{(\mu_1)_x} = (\mu_2/\mu_1)_{(\mu_1)_y} &\Leftrightarrow (\mu_2/\mu_1)((\mu_1)_x, (\mu_1)_y) = 1 \\ &\Leftrightarrow \mu_2(x, y) = 1 \\ &\Leftrightarrow (\mu_2)_x = (\mu_2)_y \\ &\Leftrightarrow \alpha\big((\mu_2/\mu_1)_{(\mu_1)_x}\big) = \alpha\big((\mu_2/\mu_1)_{(\mu_1)_y}\big).\end{aligned}$$

Moreover, it is not hard to show that $\alpha$ is surjective. In fact, for any $(\mu_2)_x \in G/\mu_2$, there exists $\mu_1 = \mu_2$ such that $\alpha((\mu_2/\mu_1)_{(\mu_2)_x}) = \alpha((\mu_2/\mu_1)_{(\mu_1)_x}) = (\mu_2)_x$. It is actually a homomorphism. Therefore, a mapping $\alpha$ is an isomorphism from $(G/\mu_1)/(\mu_2/\mu_1)$ to $G/\mu_2$. ■

The following generalization is a consequence of Theorems 24 and 25.

**Theorem 26.** *Let $G$ be a Menger algebra and let $\mu_1, \mu_2, \ldots, \mu_{m+1}$ be fuzzy congruences on $G$ such that $\mu_1 \subseteq \mu_2 \subseteq \ldots \subseteq \mu_{m+1}$. Then for each $i = 1, \ldots, m$, the fuzzy relation $\mu_{i+1}/\mu_i$ defined by*

$$(\mu_{i+1}/\mu_i)((\mu_i)_x, (\mu_i)_y) = (\mu_{i+1})(x, y)$$

*is a fuzzy congruence on $G/\mu_i$ and*

$$(G/\mu_i)/(\mu_{i+1}/\mu_i) \cong G/\mu_{i+1}.$$

## 5. Conclusion

This paper was contributed to the discussion of the combination among fuzzy sets and Menger algebras. We defined the concepts of various types of fuzzy ideals in Menger algebras. Some fundamental notions, the $(n+1)$-ary superasscociative operation on the set of all fuzzy subsets, characterizations and related properties concerning fuzzy Menger subalgebras, fuzzy $v$-ideals, fuzzy $s$-ideals and fuzzy $vs$-ideals were given. We further proposed fuzzy congruence relations over Menger algebras and obtained certain quotient structures related to them. Finally, we established several homomorphism and isomorphism theorems via fuzzy congruence relations. It turned out that our results are also noticeable extensions of semigroups if we set an arbitrary fixed natural number $n = 1$.

There are two potential types of continuation of this research. Firstly, it is possible to change the kind of ideals in Menger algebras to which fuzzy ideals are considered, for example, $l$-ideals and $i$-ideals. Secondly, hyperideals in hypercompositional algebras can be applied to study in this direction.

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## References


[1] S.S. Ahn and C. Kim, *Rough set theory applied to fuzzy filters in $BE$-algebras*, Commun. Korean Math. Soc. **31** (3) (2016) 451–460. https://doi.org/10.4134/CKMS.c150168

[2] M. Demirci and J. Recasens, *Fuzzy groups, fuzzy functions and fuzzy equivalence relations*, Fuzzy Sets Syst. **144** (2004) 441–458. https://doi.org/10.1016/S0165-0114(03)00301-4

[3] B. Davvaz and W.A. Dudek, *Fuzzy $n$-ary groups as a generalizationof Rosenfeld's fuzzy groups*, J. Mult-Valued Log. S. **15** (2009) 451–469.

[4] W.A. Dudek, *Fuzzication of $n$-ary groupoids*, Quasigroups Relat. Syst. **7** (2000) 45–66.

[5] W.A. Dudek and V.S. Trokhimenko, Algebras of Multiplace Functions (De Gruyter, Berlin, 2012).

[6] W.A. Dudek and V.S. Trokhimenko, *De Morgan $(2,n)$-semigroups of $n$-place functions*. Commun. Algebra **44** (2016) 4430–4437. https://doi.org/10.1080/00927872.2015.1094486

[7] W.A. Dudek and V.S. Trokhimenko, *Menger algebras of idempotent $n$-ary operations*, Stud. Sci. Math. Hung. **55** (2) (2019) 260–269. https://doi.org/10.1556/012.2018.55.2.1396

[8] Y.B. Jun, *Fuzzy BCK/BCI-algebras with operators*, Sci. Math. **3** (2) (2000) 283–287.

[9] S. Kar and P. Sarkar, *Fuzzy ideals of ternary semigroups*, Fuzzy Inf. Eng. **2** (2012) 181–193. https://doi.org/10.1007/s12543-012-0110-4

[10] J.P. Kim and D.R. Bae, *Fuzzy congruences in groups*, Fuzzy Sets Syst. **85** (1997) 115–120.

[11] H.S. Kim, J. Neggers and S.S. Ahn, *Almost $\varphi$-fuzzy semi-ideals in groupoids*, J. Intell. Fuzzy Syst. **36** (2019) 5361–5368. https://doi.org/10.3233/JIFS-181220

[12] M. Kondo, *Fuzzy congruences on groups*, Quasigroups Relat. Syst. **11** (2004) 59–70.

[13] S. Kuka, K. Hila and K. Naka, *Application of L-fuzzy sets in $m$-ary semigroups*, J. Intell. Fuzzy Syst. **34** (6) (2018) 4031–4040. https://doi.org/10.3233/JIFS-171365

[14] T. Kumduang and S. Leeratanavalee, *Left translations and isomorphism theorems for Menger algebras of rank $n$*, Kyungpook Math. J. **61** (2021) 223–237. https://doi.org/10.5666/KMJ.2021.61.2.223

[15] T. Kumduang and S. Leeratanavalee, *Semigroups of terms, tree languages, Menger algebra of $n$-ary functions and their embedding theorems*, Symmetry, **13** (4) (2021) Article 558. https://doi.org/10.3390/sym13040558

[16] N. Kuroki, *Fuzzy congruences on $T^*$-pure semigroups*, Inf. Sci. **84** (1995), 239–24.

[17] W.J. Liu, *Fuzzy invariant subgroups and fuzzy ideals*, Fuzzy Sets Syst. **21** (1982) 133–139.

[18] X.P. Liu, D.J. Xiang and J.M. Zhan, *Isomorphism theorems for soft ring*, Algebra Colloq. **19** (2012) 649–656.
https://doi.org/10.1142/S100538671200051X

[19] X.P. Liu, D.J. Xiang and J.M. Zhan, *Fuzzy isomorphism theorems of soft rings*, Neural Comput. Appl. **21** (2012) 391–397.
https://doi.org/10.1007/s00521-010-0439-8

[20] K. Menger, *Superassociative systems and logical functors*, Math. Ann. **157** (1964) 278–295.

[21] V. Murali, *Fuzzy congruence relations*, Fuzzy Sets Syst. **41** (1991) 359–369.

[22] A. Rosenfeld, *Fuzzy groups*, J. Math. Anal. Appl. **35** (1971) 512–517.

[23] J. Somjanta, N. Thuekaew, P. Kumpeangkeaw and A. Iampan, *Fuzzy sets in UP-algebras*, Ann. Fuzzy Math. Inform. **12** (6) (2016) 739–756.

[24] Y.J. Tan, *Fuzzy congruences on a regular semigroup*, Fuzzy Sets Syst. **117** (2001) 447–453.
https://doi.org/10.1016/S0165-0114(98)00275-9

[25] A. Ullah, A. Khan, A. Ahmadian, N. Senu, F. Karaaslan and I. Ahmad, *Fuzzy congruences on $AG$-group*, AIMS Mathematics, **6** (2) (2021) 1754–1768.
https://doi.org/10.3934/math.2021105

[26] K. Wattanatripop, T. Kumduang, T. Changphas and S. Leeratanavalee, *Power Menger algebras of terms induced by order-decreasing transformations and superpositions*, Int. J. Math. Comput. Sci. **16** (4) (2021) 1697–1707.

[27] X. Zhou, D. Xiang and J. Zhan, *A novel study on fuzzy congruence on $n$-ary semigroup*, U.P.B. Sci. Bull., Ser. A **78** (30 (2016) 19–30.

[28] X. Zhou, D. Xiang and J. Zhan, *Quotient rings via fuzzy congruence relations*, Ital. J. Pure Appl. Math. **33** (2014) 411–424.

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