Discussiones Mathematicae
General Algebra and Applications 43 (2023) 121–134
https://doi.org/10.7151/dmgaa.1417

σ-FILTERS OF COMMUTATIVE BE-ALGEBRAS

M. Sambasiva Rao

Department of Mathematics
MVGR College of Engineering
Vizianagaram, India-535005

e-mail: mssraomaths35@rediffmail.com

Abstract

The concept of σ-filters is introduced in commutative BE-algebras and some properties of these classes of filters are studied. Some equivalent conditions are derived for every filter of a commutative BE-algebra to become a σ-filter. Some necessary and sufficient conditions are given for every regular filter of a commutative BE-algebra to become a σ-filter. A set of equivalent conditions is given for the class of all σ-filters of a commutative BE-algebra to become a sublattice to the lattice of all filters.

Keywords: commutative BE-algebra, dual annihilator filter, prime filter, σ-filter, regular filter, O-filter.

2020 Mathematics Subject Classification: 03G25.

Introduction

The notion of BE-algebras was introduced and extensively studied by Kim and Kim in [5]. These classes of BE-algebras were introduced as a generalization of the class of BCK-algebras of Iseki and Tanaka [4]. Some properties of filters of BE-algebras were studied by Ahn and Kim in [1] and by Meng in [6]. In [12], Walendziak discussed some significant properties of commutative BE-algebras. He also investigated the relationship between BE-algebras, implicative algebras and J-algebras. In [6], Meng introduced the notion of prime filters in BCK-algebras, and then gave a description of the filter generated by a set, and obtained some of fundamental properties of prime filters. In [4], some properties of prime ideals are investigated in BCK-algebras. In [8], the author studied some properties of prime filters in BE-algebras. In this paper, the author extensively studied the algebraic as well as the topological properties of prime filters of commutative

BE-algebras. In [9], the authors introduced the notion of dual annihilators of commutative BE-algebra and studied extensively the properties of these dual annihilators. In 2020, the authors introduced the notions of regular filters [10] and O-filters [11] in commutative BE-algebras and the interconnection between those two special classes of filters is studied.

In this paper, the concept of σ-filters is introduced in commutative BE-algebras and their properties are studied analogous to that in a distributive lattice [3]. A set of equivalent conditions is given for every filter of a commutative BE-algebra to become a σ-filter. It is observed that every σ-filter of a commutative BE-algebra is a regular filter but not the converse in general. However, some equivalent conditions are proved for every regular filter of a commutative BE-algebra to become a σ-filter. It is also observed that every O-filter of a commutative BE-algebra is a σ-filter but not the converse in general. Some necessary and sufficient conditions are given for every σ-filter of a commutative BE-algebra to become an O-filter. Some equivalent conditions are given to prove that the class of all σ-filters of a commutative BE-algebra to become a sublattice to the lattice of all filters of a commutative BE-algebra.

1. Preliminaries

In this section, we present certain definitions and results which are taken mostly from the papers [1, 5, 9, 10], and [11] for the ready reference of the reader.

Definition 1.1 [5]**.** An algebra $(X, *, 1)$ of type $(2, 0)$ is called a BE-algebra if it satisfies the following properties:

(1) $x * x = 1$,

(2) $x * 1 = 1$,

(3) $1 * x = x$,

(4) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$.

A BE-algebra X is called self-distributive if $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$. A BE-algebra X is called transitive if $y * z \leq (x * y) * (x * z)$ for all $x, y, z \in X$. A BE-algebra X is called commutative if $(x * y) * y = (y * x) * x$ for all $x, y \in X$. Every commutative BE-algebra is transitive. For any $x, y \in X$, define $x \vee y = (y * x) * x$. If X is commutative then $(X, \vee)$ is a semilattice [12]. We introduce a relation $\leq$ on a BE-algebra X by $x \leq y$ if and only if $x * y = 1$ for all $x, y \in X$. Clearly $\leq$ is reflexive. If X is commutative, then $\leq$ is transitive, anti-symmetric and hence a partial order on X.

Theorem 1.2 [5]**.** *Let X be a transitive BE-algebra and $x, y, z \in X$. Then*

(1) $1 \leq x$ *implies* $x = 1$,

(2) $y \leq z$ *implies* $x * y \leq x * z$ *and* $z * x \leq y * x$.

Definition 1.3 [1]**.** A non-empty subset F of a BE-algebra X is called a filter of X if, for all $x, y \in X$, it satisfies the following properties:

(1) $1 \in F$,

(2) $x \in F$ and $x * y \in F$ imply that $y \in F$.

For any non-empty subset A of a transitive BE-algebra X, the set $\langle A \rangle = \{x \in X \mid a_1 * (a_2 * (\cdots * (a_n * x) \cdots)) = 1$ for some $a_1, a_2, \ldots a_n \in A\}$ is the smallest filter containing A. For any $a \in X, \langle a \rangle = \{x \in X \mid a^n * x = 1$ for some $n \in \mathbb{N}\}$, where $a^n * x = a*(a*(\cdots*(a*x)\cdots))$ with the repetition of a is n times, is called the principal filter generated a. Let F be a filter of a transitive BE-algebra and $a \in X$, then $\langle F \cup \{a\} \rangle = \{x \in X \mid a^n * x = 1$ for some $n \in \mathbb{N}\}$. A proper filter P of a BE-algebra is called prime [8] if $F \cap G \subseteq P$ implies $F \in P$ or $G \in P$ for any two proper filters F, G of X. A proper filter P of a BE-algebra is called prime [8] if $\langle x \rangle \cap \langle y \rangle \subseteq P$ implies $x \in P$ or $y \in P$ for any $x, y \in X$. A proper filter M of a transitive BE-algebra X is called maximal if there exist no proper filters Q such that $M \subset Q$. "Every maximal filter of a commutative BE-algebra is prime".

Theorem 1.4 [8]**.** *Let F and G be two filters of a transitive BE-algebra X. Then*

$$F \vee G = \{x \in X \mid a * (b * x) = 1 \text{ for some } a \in F, b \in G\}$$

is the supremum of F and G. Hence the set $\mathcal{F}(X)$ of all filters of X is a lattice.

Lemma 1.5 [9]**.** *Let X be a commutative BE-algebra. Then for any $x, y, a \in X$*

(1) $y * z \leq (z * x) * (y * x)$,

(2) $(x * y) \vee a \leq (x \vee a) * (y \vee a)$.

For any non-empty subset A of a BE-algebra X, the *dual annihilator* [9] of A is defined as $A^+ = \{x \in X \mid x \vee a = 1$ for all $a \in A\}$. In a commutative BE-algebra X, the set A^+ forms a filter of X such that $A \cap A^+ = \{1\}$. In case of $A = \{a\}$, we have $(a)^+ = \{x \in X \mid a \vee x = 1\}$. For $a \in X$, the set $(a)^+$ is called the *dual annulet* of a. Clearly $X^+ = \{1\}$ and $\{1\}^+ = X$.

Proposition 1.6 [9]**.** *Let X be a commutative BE-algebra and $\emptyset \neq A, B \subseteq X$. Then*

(1) *if $A \subseteq B$, then $B^+ \subseteq A^+$,*

(2) $A \subseteq A^{++}$,

(3) $A^+ = A^{+++}$.

Proposition 1.7 [9]**.** *Let F and G be two filters of a commutative BE-algebra X. Then*

(1) $F \cap G = \{1\}$ *if and only if* $F \subseteq G^+$,

(2) $(F \vee G)^+ = F^+ \cap G^+$,

(3) $(F \cap G)^{++} = F^{++} \cap G^{++}$.

Proposition 1.8 [9]**.** *Let X be a commutative BE-algebra and $a, b \in X$. Then we have*

(1) $\langle a \rangle \subseteq (a)^{++}$,

(2) $a \leq b$ *implies* $(a)^+ \subseteq (b)^+$,

(3) $a \in (b)^{++}$ *implies* $(b)^+ \subseteq (a)^+$.

A filter F of a commutative BE-algebra X is called a *dual annihilator filter* [9] if $F = F^{++}$. A filter F of a commutative BE-algebra X is called a *regular filter* [10] if $(x)^{++} \subseteq F$ whenever $x \in F$. A filter F of a commutative BE-algebra X is called an *O-filter* [11] if $F = O(S)$ for some $\vee$-closed subset S of X, where $O(S) = \{x \in X \mid x \vee s = 1 \text{ for some } s \in S\}$. Every O-filter of a commutative BE-algebra is a regular filter.

2. σ-filters of BE-algebras

In this section, the concept of σ-filters is introduced in commutative BE-algebras. Some properties of σ-filters are proved. A set of equivalent conditions is given for every prime filter of a commutative BE-algebra to become a σ-filter. Interconnections among σ-filters, regular filters, O-filters of commutative BE-algebras are established.

Lemma 2.1. *Let X be a commutative BE-algebra. For any $x, y \in X$, we have*

(1) $(x)^+ \cap (x * y)^+ \subseteq (y)^+$,

(2) $(x \vee y)^{++} = (x)^{++} \cap (y)^{++}$,

(3) $(x)^+ \cap (y)^+ = \{1\}$ *if and only if* $(x)^+ \subseteq (y)^{++}$,

(4) $x \in (y)^+$ *if and only if* $(x)^{++} \subseteq (y)^+$.

Proof. (1) Let $a \in (x)^+ \cap (x * y)^+$. Then $x \vee a = 1$ and $(x * y) \vee a = 1$. Hence

$$\begin{aligned} 1 &= (x * y) \vee a \\ &\leq (x \vee a) * (y \vee a) && \text{by Lemma 1.5(2)} \\ &= 1 * (y \vee a) \\ &= y \vee a \end{aligned}$$

which means $y \vee a = 1$. Hence $a \in (y)^+$. Therefore $(x)^+ \cap (x * y) \subseteq (y)^+$.

(2) Let $x, y \in X$. Since $x, y \leq x \vee y$, we get $(x)^+, (y)^+ \subseteq (x \vee y)^+$. Hence $(x \vee y)^{++} \subseteq (x)^{++}, (y)^{++}$. Thus $(x \vee y)^{++} \subseteq (x)^{++} \cap (y)^{++}$. Conversely, let $a \in (x)^{++} \cap (y)^{++}$. Suppose $b \in (x \vee y)^+$ be an arbitrary element. Since $b \in (x \vee y)^+$, we get

$$\begin{aligned} b \vee (x \vee y) = 1 &\Rightarrow b \vee x \in (y)^+ \\ &\Rightarrow a \vee b \vee x = 1 \qquad \text{since } a \in (y)^{++} \\ &\Rightarrow a \vee b \in (x)^+ \\ &\Rightarrow a \vee (a \vee b) = 1 \qquad \text{since } a \in (x)^{++} \\ &\Rightarrow a \vee b = 1 \quad \text{for all } b \in (x \vee y)^+ \end{aligned}$$

which means that $a \in (x \vee y)^{++}$. Therefore $(x)^{++} \cap (y)^{++} \subseteq (x \vee y)^{++}$.

(3) Let $x, y \in X$. Assume that $(x)^+ \cap (y)^+ = \{1\}$. Let $a \in (x)^+$. Let $b \in (y)^+$ be any element. Then, we get that $a \vee b \in (x)^+ \cap (y)^+ = \{1\}$. Hence $a \in (b)^+$ for all $b \in (y)^+$. Therefore $a \in (y)^{++}$, which gives that $(x)^+ \subseteq (y)^{++}$. Conversely, suppose that $(x)^+ \subseteq (y)^{++}$. Then $(x)^+ \cap (y)^+ \subseteq (y)^{++} \cap (y)^+ = \{1\}$. Therefore $(x)^+ \cap (y)^+ = \{1\}$.

(4) Let $x, y \in X$. Suppose $x \in (y)^+$. Then $x \vee y = 1$. Hence $(x)^{++} \cap (y)^{++} = (x \vee y)^{++} = (1)^{++} = \{1\}$. Thus by (3), we get $(x)^{++} \subseteq (y)^{+++} = (y)^+$. Converse is clear. ■

Definition 2.2. For any prime filter P of a commutative BE-algebra X, define $O(P) = \{x \in X \mid (x)^+ \nsubseteq P\}$.

Proposition 2.3. *For any prime filter P of a commutative BE-algebra X, the set $O(P)$ is a filter of X such that $O(P) \subseteq P$.*

Proof. Clearly $1 \in O(P)$. Suppose $x, x * y \in O(P)$. Then $(x)^+ \nsubseteq P$ and $(x * y)^+ \nsubseteq P$. Since P is prime, we get $(x)^+ \cap (x * y)^+ \nsubseteq P$. By Lemma 2.1(1), we get $(y)^+ \nsubseteq P$. Hence $y \in O(P)$. Therefore $O(P)$ is a filter of X. Again, let $x \in O(P)$. Then $(x)^+ \nsubseteq P$. Then there exists $y \in (x)^+$ such that $y \notin P$. Since $y \in (x)^+$, we get $x \vee y = 1$. Hence $(x)^{++} \cap (y)^{++} = (x \vee y)^{++} = \{1\}^{++} = \{1\} \subseteq P$. Since P is prime, we get $(x)^{++} \subseteq P$ or $(y)^{++} \subseteq P$. Suppose $(y)^{++} \subseteq P$. Since $y \in (y)^{++}$, we get $y \in P$ which is a contradiction. Hence $(x)^{++} \subseteq P$, which means $x \in P$. Therefore $O(P) \subseteq P$. ■

Definition 2.4. Let X be a commutative BE-algebra. For any filter F of X, define

$$\sigma(F) = \{x \in X \mid (x)^+ \vee F = X\}.$$

Clearly $\sigma(X) = X$. For $F = \{1\}$, obviously we get $\sigma(\{1\}) = \{1\}$.

Lemma 2.5. *For any filter F of a commutative BE-algebra X, $\sigma(F)$ is a filter of X.*

Proof. Clearly $1 \in \sigma(F)$. Let $x, x * y \in \sigma(F)$. Then $(x)^+ \vee F = X$ and $(x*y)^+ \vee F = X$. Hence

$$\begin{aligned} X &= X \cap X \\ &= \{(x)^+ \vee F\} \cap \{(x*y)^+ \vee F\} \\ &= \{(x)^+ \cap (x*y)^+\} \vee F \\ &\subseteq (y)^+ \vee F. \end{aligned}$$

which gives $(y)^+ \vee F = X$. Hence $y \in \sigma(F)$. Therefore $\sigma(F)$ is a filter of X. ∎

In the following result, some elementary properties of $\sigma(F)$ are derived.

Lemma 2.6. *For any two filters F, G of a commutative BE-algebra X, we have*

(1) $\sigma(F) \subseteq F$,

(2) $F \subseteq G$ *implies* $\sigma(F) \subseteq \sigma(G)$,

(3) $\sigma(F \cap G) = \sigma(F) \cap \sigma(G)$,

(4) $\sigma(F) \vee \sigma(G) \subseteq \sigma(F \vee G)$.

Proof. (1) Let $x \in \sigma(F)$. Then $(x)^+ \vee F = X$. Hence $a * (b * x) = 1$ for some $a \in (x)^+$ and $b \in F$. Since $a \in (x)^+$, we get $(a * x) * x = a \vee x = 1$. Since X is commutative, we get $1 = a*(b*x) = b*(a*x) \leq ((a*x)*x)*(b*x) = 1*(b*x) = b*x$. Hence $b * x = 1$, which gives $b \leq x$. Since $b \in F$ and F is a filter, it concludes that $x \in F$. Therefore $\sigma(F) \subseteq F$.

(2) Suppose $F \subseteq G$. Let $x \in \sigma(F)$. Then $X = (x)^+ \vee F \subseteq (x)^+ \vee G$. Therefore $x \in \sigma(G)$.

(3) Clearly $\sigma(F \cap G) \subseteq \sigma(F) \cap \sigma(G)$. Conversely, let $x \in \sigma(F) \cap \sigma(G)$. Then $(x)^+ \vee F = (x)^+ \vee G = X$. Now $(x)^+ \vee (F \cap G) = \{(x)^+ \vee F\} \cap \{(x)^+ \vee G\} = X \cap X = X$. Hence $x \in \sigma(F \cap G)$. Thus $\sigma(F) \cap \sigma(G) \subseteq \sigma(F \cap G)$. Therefore $\sigma(F \cap G) = \sigma(F) \cap \sigma(G)$.

(4) By (2), it is obvious. ∎

Proposition 2.7. *Let P be a proper filter of a commutative BE-algebra X. Then*

(1) *if P is prime, then $\sigma(P) \subseteq O(P)$,*

(2) *if P is maximal, then $\sigma(P) = O(P)$.*

Proof. (1) Let $x \in \sigma(P)$. Then $(x)^+ \vee P = X$. Suppose that $(x)^+ \subseteq P$. Then we get $P = X$, which is a contradiction. Hence $(x)^+ \nsubseteq P$. Thus $x \in O(P)$. Therefore $\sigma(P) \subseteq O(P)$.

(2) Since every maximal filter is prime, we get $\sigma(P) \subseteq O(P)$. Conversely, let $x \in O(P)$. Then $a \vee x = 1$ for some $a \notin P$. Thus there exists $a \in (x)^+$ and $a \notin P$. Hence $(x)^+ \nsubseteq P$. Since P is maximal, we get $(x)^+ \vee P = X$. Thus $x \in \sigma(P)$. Therefore $\sigma(P) = O(P)$. ■

Definition 2.8. A filter F of a BE-algebra X is called a σ- filter if $F = \sigma(F)$.

Clearly the improper filters $\{1\}$ and X are trivial σ-filters of X. In the following, we observe a non-trivial example for σ-filters of a BE-algebra.

Example 2.9. Let $X = \{a, b, c, d, 1\}$ be a set. Define a binary operation $*$ on X as

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	1	d
b	1	c	1	c	d
c	1	b	b	1	d
d	1	a	b	c	1

$\vee$	1	a	b	c	d
1	1	1	1	1	1
a	1	a	b	c	1
b	1	b	b	1	1
c	1	c	1	c	1
d	1	1	1	c	1

Clearly $(X, *, \vee, 1)$ is a commutative BE-algebra. Consider the filter $F = \{1, a, b, c\}$. It can be easily verified that $(a)^+ = \{1, d\}$, $(b)^+ = \{1, c, d\}$, $(c)^+ = \{1, b, d\}$ and $(d)^+ = \{1, a, b, c\}$. Clearly $(1)^+ \vee F = X$. Observe that $(a)^+ \vee F = (b)^+ \vee F = (c)^+ \vee F = X$. Thus $\sigma(F) = \{1, a, b, c\} = F$. Therefore F is a σ-filter of X.

It is observed that a proper σ-filter of a commutative BE-algebra contains no dual dense elements (an element x of a commutative BE-algebra is called *dual dense* if $(x)^+ = \{1\}$) and the converse is not true. For this, consider the following example.

Example 2.10. Let $X = \{1, a, b, c, d\}$ be a set. Define a binary operation $*$ on X as

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	a	c	d
b	1	1	1	c	d
c	1	a	b	1	d
d	1	a	b	c	1

$*$	1	a	b	c	d
1	1	1	1	1	1
a	1	a	a	1	1
b	1	a	b	1	1
c	1	1	1	c	1
d	1	1	1	1	d

Clearly $(X, *, \vee, 1)$ is a commutative BE-algebra. Now $(a)^+ = \{1, c, d\}$; $(b)^+ = \{1, c, d\}$; $(c)^+ = \{1, a, b, d\}$ and $(d)^+ = \{1, a, b, d\}$. Consider the filter $F = \{1, d\}$ of X which is not containing dual dense elements. Hence $(a)^+ \vee F = (b)^+ \vee F = \{1, c, d\}$, $(c)^+ \vee F = F$ and $(d)^+ \vee F = F$. Thus $\sigma(F) = \{1\}$. Therefore F is not a σ-filter of X.

Theorem 2.11. *Following assertions are equivalent in a commutative BE-algebra X:*

(1) *every filter is a σ-filter;*

(2) *every prime filter is a σ-filter;*

(3) *for every prime filter P, $O(P) = P$.*

Proof. (1) $\Rightarrow$ (2): It is clear.

(2) $\Rightarrow$ (3): Assume that every prime filter is a σ-filter. Let P be a prime filter of X. Since P is proper, there exists $c \in X$ such that $c \notin P$. Since by (2), P is a σ-filter of X, we have $\sigma(P) = P$. Clearly $O(P) \subseteq P$. Conversely, let $x \in P = \sigma(P)$. Then $(x)^+ \vee P = X$. Since $c \in X$, we get $c \in (x)^+ \vee P$. Then $a * (b * c) = 1$ for some $a \in (x)^+$ and $b \in P$. Hence $a \leq b * c$. Suppose $a \in P$. Then $b * c \in P$. Since $b \in P$, we get $c \in P$, which is a contradiction. Thus $a \notin P$. Hence $a \vee x = 1$ for some $a \notin P$. Therefore $x \in O(P)$, which gives that $P = O(P)$.

(3) $\Rightarrow$ (1): Assume that $O(P) = P$ for every prime filter of X. Let F be an arbitrary filter of X. By Lemma 2.6(1), $\sigma(F) \subseteq F$. Conversely, let $x \in F$. Suppose $(x)^+ \vee F \neq X$. Then there exists a maximal filter P such that $(x)^+ \vee F \subseteq P$. Since every maximal filter is prime, we get that P is prime. Hence $(x)^+ \subseteq P$ and $F \subseteq P$. Since $(x)^+ \subseteq P$, we get that $x \notin O(P) = P$. Since $x \in F$, we get $x \in P$ which is a contradiction. Hence $(x)^+ \vee F = X$. Therefore F is a σ-filter of X. ∎

In [10], the class of all regular filters of a commutative BE-algebra X is characterized in terms of dual annihilators. In the following theorem, it is proved that the class of all regular filters of X contains properly the class of all σ-filters of X.

Proposition 2.12. *Every σ-filter of a commutative BE-algebra is a regular filter.*

Proof. Let F be a σ-filter of a commutative BE-algebra X. Then $\sigma(F) = F$. Let $x \in F$. Then $(x)^+ \vee F = X$. Now, let $t \in (x)^{++}$. Then, by Proposition 1.8(3), $(x)^+ \subseteq (t)^+$. Hence $X = (x)^+ \vee F \subseteq (t)^+ \vee F$. Thus $t \in \sigma(F) = F$. Thus $(x)^{++} \subseteq F$. Therefore F is a regular filter of X. ∎

The converse of the above proposition is not true, i.e., every regular filter of a commutative BE-algebra need not be a σ-filter. Indeed, consider Example 2.9. Here, $F = \{1, d\}$ is clearly a regular filter, because $(d)^{++} \subseteq F$. But F is not a σ-filter of X, because of $(d)^+ \vee F \neq X$. However, some equivalent conditions are given for every regular filter of a commutative BE-algebra to become a σ-filter.

Theorem 2.13. *Following assertions are equivalent in a commutative BE-algebra X:*

(1) *every regular filter is a σ-filter;*

(2) *every dual annihilator filter is a σ-filter;*

(3) *for each $x \in X$, $(x)^{++}$ is a σ-filter;*

(4) *for each $x \in X$, $(x)^{+} \vee (x)^{++} = X$.*

Proof. (1) $\Rightarrow$ (2): Since every dual annihilator filter is a regular filter, it is clear.

(2) $\Rightarrow$ (3): Since each $(x)^{++}$ is a dual annihilator filter, it is clear.

(3) $\Rightarrow$ (4): Assume the statement (3). Let $x \in X$. Since $(x)^{++}$ is a σ-filter of X, we get $(x)^{++} = \sigma((x)^{++})$. Clearly $x \in (x)^{++} = \sigma((x)^{++})$. Hence $(x)^{+} \vee (x)^{++} = X$.

(4) $\Rightarrow$ (1): Assume that $(x)^{+} \vee (x)^{++} = X$ for each $x \in X$. Let F be a regular filter of X. Clearly $\sigma(F) \subseteq F$. Conversely, let $x \in F$. Since F is a regular filter, we get $(x)^{++} \subseteq F$. Hence $X = (x)^{+} \vee (x)^{++} \subseteq (x)^{+} \vee F$. Thus $x \in \sigma(F)$. Therefore F is a σ-filter of X. ∎

Recall that a filter F of a commutative BE-algebra X is called an O-filter if $F = O(S)$ for some $\vee$-closed subset S of X. In [11], authors studied the properties of O-filters and proved that every O-filter of a self-distributive and commutative BE-algebra is the intersection of all minimal prime filters containing it. In the following result, it is proved that the class of all σ-filters of a commutative BE-algebra X is properly contained in the class of all O-filters of X.

Theorem 2.14. *Suppose X is a commutative BE-algebra with a dual-dense element (i.e., $(x)^{+} = \{1\}$). Then every σ-filter of X is an O-filter.*

Proof. Let F be a σ-filter of X. Then $\sigma(F) = F$. Consider the set $S = \{\ x \in X \mid (x)^{++} \vee F = X\ \}$. It can be easily verified, by using Lemma 2.1(2), that S is a $\vee$-closed subset of X. We now show that $F = O(S)$. Let $x \in O(S)$. Then $x \vee y = 1$ for some $y \in S$. Now

$$\begin{aligned} x \vee y = 1 &\Rightarrow y \in (x)^{+} \\ &\Rightarrow (y)^{++} \subseteq (x)^{+} && \text{by Lemma 2.1(4)} \\ &\Rightarrow X = (y)^{++} \vee F \subseteq (x)^{+} \vee F && \text{since } y \in S \\ &\Rightarrow x \in \sigma(F) = F && \text{since } F \text{ is a } \sigma\text{-filter} \end{aligned}$$

which concludes that $O(S) \subseteq F$. Conversely, let $x \in F = \sigma(F)$ and d a dual-dense element of X. Then $(x)^{+} \vee \sigma(F) = X$. Therefore $d \in (x)^{+} \vee \sigma(F)$. Hence $a * (b * d) = 1$ for some $a \in (x)^{+}$ and $b \in \sigma(F)$. Thus $a \vee x = 1$ and $(b)^{+} \vee F = X$. Now

$$\begin{aligned} a * (b * d) = 1 &\Rightarrow a \leq b * d \\ &\Rightarrow (a)^{+} \subseteq (b * d)^{+} \end{aligned}$$

$$\begin{aligned}
&\Rightarrow (a)^+ \cap (b)^+ \subseteq (b)^+ \cap (b * d)^+ && \\
&\Rightarrow (a)^+ \cap (b)^+ \subseteq (d)^+ = \{1\} && \text{by Lemma 2.1(1)} \\
&\Rightarrow (b)^+ \subseteq (a)^{++} && \text{by Lemma 2.1(3)} \\
&\Rightarrow X = (b)^+ \vee F \subseteq (a)^{++} \vee F && \text{since } b \in \sigma(F) \\
&\Rightarrow a \in S \ \text{ and } \ a \vee x = 1 && \\
&\Rightarrow x \in O(S) &&
\end{aligned}$$

which gives $F = \sigma(F) \subseteq O(S)$. Hence $F = O(S)$. Therefore F is an O-filter of X. ∎

The converse of the above theorem is not true, i.e., every O-filter of a commutative BE-algebra need not be a σ-filter. For, consider the following example.

Example 2.15. Let $X = \{1, a, b, c\}$ be a set. Define a binary operation $*$ on X as

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	c
b	1	1	1	c
c	1	a	b	1

$\vee$	1	a	b	c
1	1	1	1	1
a	1	a	a	1
b	1	a	b	1
c	1	1	1	c

It can be routinely verified that $(X, *, \vee, 1)$ is a commutative BE-algebra. Observe that $(a)^+ = (b)^+ = \{1, c\}$, and $(c)^+ = \{1, a, b\}$. Consider the filter $F = \{1, c\}$ of X. Clearly $S = \{a, b\}$ is a $\vee$-closed subset of X. It is easy to observe that $F = O(S)$. Hence F is an O-filter of X. Now $\sigma(F) = \{1\} \subset F$. Therefore F is not a σ-filter of X.

Lemma 2.16. *In a commutative BE-algebra, every dual annulet is an O-filter.*

Proof. Let X be a commutative BE-algebra and $a \in X$. Consider $[a] = \{x \in X \mid x \leq a\}$. Let $x, y \in [a]$. Then $x \leq a$ and $y \leq a$. Since X is commutative, it is partially ordered. Hence $x \vee y \leq a$, which gives that $x \vee y \in [a]$. Therefore $[a]$ is a $\vee$-closed subset of X. We now show that $(a)^+ = O([a])$. Let $x \in (a)^+$. Then $a \vee x = 1$ and $a \in [a]$. Hence $x \in O([a])$, which gives that $(a)^+ \subseteq O([a])$. Conversely, let $x \in O([a])$. Then $x \vee y = 1$ for some $y \in [a]$. Since $y \in [a]$, we get $y \leq a$. Hence $1 = x \vee y \leq x \vee a$. Thus $x \in (a)^+$. Hence $O([a]) \subseteq (a)^+$. Therefore $(a)^+$ is an O-filter of X. ∎

Theorem 2.17. *Following assertions are equivalent in a commutative BE-algebra X:*

(1) *every O-filter is a σ-filter;*

(2) *each dual annulet is a σ-filter;*

(3) *for any* $x, y \in X$, $x \vee y = 1$ *implies* $(x)^+ \vee (y)^+ = X$.

Proof. (1) $\Rightarrow$ (2): Since each dual annulet is an O-filter, it is clear.

(2) $\Rightarrow$ (3): Assume that each dual annulet is a σ-filter of X. Let $x, y \in X$ be such that $x \vee y = 1$. Hence $x \in (y)^+$. By (2), we get that $(y)^+$ is a σ-filter of X. Hence $x \in (y)^+ = \sigma((y)^+)$. Thus we get $(x)^+ \vee (y)^+ = X$. Therefore, condition (3) is proved.

(3) $\Rightarrow$ (1): Assume that condition (3) holds. Let F be an O-filter of X. Then $F = O(S)$ for some $\vee$-closed subset S of X. Clearly $\sigma(F) \subseteq F$. We claim that $O(S) \subseteq \sigma(F)$. Now

$$\begin{aligned} x \in O(S) &\Rightarrow x \vee y = 1 \text{ for some } y \in S \\ &\Rightarrow (x)^+ \vee (y)^+ = X && \text{by (3)} \\ &\Rightarrow X = (x)^+ \vee (y)^+ \subseteq (x)^+ \vee O(S) && \text{since } y \in S \\ &\Rightarrow x \in \sigma(O(S)) = \sigma(F) \end{aligned}$$

Hence $O(S) \subseteq \sigma(F)$, which gives $F = O(S) = \sigma(F)$. Therefore F is a σ-filter of X. ∎

Theorem 2.18. *Let P be a prime filter of a commutative BE-algebra X such that $P = O(P)$. If X satisfies any one assertions of the above theorem, then P is a σ-filter.*

Proof. Assume that X satisfies condition (3) of the above theorem. Let P be a prime filter of X such that $P = O(P)$. By Proposition 2.7(1), we have $\sigma(P) \subseteq O(P) = P$. Conversely, let $x \in O(P)$. Then there exists $y \notin P$ such that $x \vee y = 1$. Since $y \notin O(P)$, we get $(y)^+ \subseteq P$. By (3) of the above theorem, we get that $(x)^+ \vee (y)^+ = X$. Hence $X = (x)^+ \vee (y)^+ \subseteq (x)^+ \vee P$. Thus $x \in \sigma(P)$. Hence P is a σ-filter of X. ∎

Let us denote by μ the set of all maximal filters of a BE-algebra X. For any filter F of a BE-algebra X, we also denote $\mu(F) = \{M \in \mu \mid F \subseteq M\}$. Since every maximal filter of a commutative BE-algebra is prime, by Proposition 2.3, we conclude that $O(M)$ is a filter such that $O(M) \subseteq M$ for every $M \in \mu$. Then we have the following result.

Theorem 2.19. *For any filter F of a commutative BE-algebra $X, \sigma(F) = \bigcap_{M \in \mu(F)} O(M)$.*

Proof. Let $x \in \sigma(F)$ and $F \subseteq M$ where $M \in \mu$. Then $X = (x)^+ \vee F \subseteq (x)^+ \vee M$. Suppose $(x)^+ \subseteq M$, then $M = X$, which is a contradiction. Hence $(x)^+ \nsubseteq M$. Thus $x \in O(M)$ for all $M \in \mu(F)$. Therefore $\sigma(F) \subseteq \bigcap_{M \in \mu(F)} O(M)$. Conversely, let $x \in \bigcap_{M \in \mu(F)} O(M)$. Then $x \in O(M)$ for all $M \in \mu(F)$. Suppose

$(x)^+ \vee F \neq X$. Then there exists a maximal filter M_0 such that $(x)^+ \vee F \subseteq M_0$. Hence $(x)^+ \subseteq M_0$ and $F \subseteq M$. Since $F \subseteq M_0$, by hypothesis, we get $x \in O(M_0)$. Hence $(x)^+ \nsubseteq M_0$, which is a contradiction. Therefore $(x)^+ \vee F = X$. Thus $x \in \sigma(F)$. Hence $\bigcap_{M \in \mu(F)} O(M) \subseteq \sigma(F)$. ■

From the above theorem, it can be easily observed that $\sigma(F) \subseteq O(M)$ for every $M \in \mu(F)$. Now, in the following, a set of equivalent conditions is given for the class of all σ-filters of a commutative BE-algebra to become a sublattice to the lattice $\mathcal{F}(X)$ of all filters of the commutative BE-algebra X.

Theorem 2.20. *The following assertions are equivalent in a commutative BE-algebra X:*

(1) *for any $M \in \mu$, $O(M)$ is maximal;*

(2) *for any $F, G \in \mathcal{F}(X)$, $F \vee G = X$ implies $\sigma(F) \vee \sigma(G) = X$;*

(3) *for any $F, G \in \mathcal{F}(X)$, $\sigma(F) \vee \sigma(G) = \sigma(F \vee G)$;*

(4) *for any two distinct maximal filters M and N, $O(M) \vee O(N) = X$;*

(5) *for any $M \in \mu$, M is the unique member of μ such that $O(M) \subseteq M$.*

Proof. (1) $\Rightarrow$ (2): Assume the condition (1). Then clearly $O(M) = M$ for all $M \in \mu$. Let $F, G \in \mathcal{F}(X)$ be such that $F \vee G = X$. Suppose $\sigma(F) \vee \sigma(G) \neq X$. Then there exists a maximal filter M such that $\sigma(F) \vee \sigma(G) \subseteq M$. Hence $\sigma(F) \subseteq M$ and $\sigma(G) \subseteq M$. Now

$$\begin{aligned}
\sigma(F) \subseteq M &\Rightarrow \bigcap_{M_i \in \mu(F)} O(M_i) \subseteq M \\
&\Rightarrow O(M_i) \subseteq M \quad \text{for some } M_i \in \mu(F) \text{ (since } M \text{ is prime)} \\
&\Rightarrow M_i \subseteq M \qquad \text{by condition (1)} \\
&\Rightarrow F \subseteq M \qquad \text{since } F \subseteq M_i.
\end{aligned}$$

Similarly, we can obtain that $G \subseteq M$. Hence $X = F \vee G \subseteq M$, which is a contradiction to the maximality of M. Therefore $\sigma(F) \vee \sigma(G) = X$.

(2) $\Rightarrow$ (3): Assume the condition (2). Let $F, G \in \mathcal{F}(X)$. Clearly $\sigma(F) \vee \sigma(G) \subseteq \sigma(F \vee G)$. Let $x \in \sigma(F \vee G)$. Then $\{(x)^+ \vee F\} \vee \{(x)^+ \vee G\} = (x)^+ \vee F \vee G = X$. Hence by condition (2), we get $\sigma((x)^+ \vee F) \vee \sigma((x)^+ \vee G) = X$. Thus $x \in \sigma((x)^+ \vee F) \vee \sigma((x)^+ \vee G)$. Hence $r * (s * x) = 1$ for some $r \in \sigma((x)^+ \vee F)$ and $s \in \sigma((x)^+ \vee G)$. Now

$$\begin{aligned}
r \in \sigma((x)^+ \vee F) &\Rightarrow (r)^+ \vee \{(x)^+ \vee F\} = X \\
&\Rightarrow X = \{(r)^+ \vee (x)^+\} \vee F \subseteq (r \vee x)^+ \vee F \\
&\Rightarrow (r \vee x)^+ \vee F = X \\
&\Rightarrow r \vee x \in \sigma(F).
\end{aligned}$$

Similarly, we can get $s \vee x \in \sigma(G)$. Now, we have the following consequence:

$$\begin{aligned} r * (s * x) = 1 &\Rightarrow r \leq s * x \\ &\Rightarrow r \vee x \leq (s * x) \vee x \leq (s \vee x) * (x \vee x) \\ &\Rightarrow r \vee x \leq (s \vee x) * (x \vee x) \\ &\Rightarrow r \vee x \leq (s \vee x) * x \\ &\Rightarrow (r \vee x) * ((s \vee x) * x) = 1 \end{aligned}$$

where $r \vee x \in \sigma(F)$ and $s \vee x \in \sigma(G)$. Hence $x \in \sigma(F) \vee \sigma(G)$. Thus $\sigma(F \vee G) \subseteq \sigma(F) \vee \sigma(G)$. Therefore $\sigma(F) \vee \sigma(G) = \sigma(F \vee G)$.

(3) $\Rightarrow$ (4): Assume the condition (3). Let M, N be two distinct maximal filters of X. Choose $x \in M - N$ and $y \in N - M$. Since $x \notin N$, we get $N \vee \langle x \rangle = X$. Since $y \notin M$, we get $M \vee \langle y \rangle = X$. Now

$$\begin{aligned} X &= \sigma(X) \\ &= \sigma(X \vee X) \\ &= \sigma(\{N \vee \langle x \rangle\} \vee \{M \vee \langle y \rangle\}) \\ &= \sigma(\{M \vee \langle x \rangle\} \vee \{N \vee \langle y \rangle\}) \\ &= \sigma(M \vee N) && \text{since } x \in M \text{ and } y \in N \\ &= \sigma(M) \vee \sigma(N) && \text{by condition (3)} \\ &\subseteq O(M) \vee O(N) && \text{by Proposition 2.7(1).} \end{aligned}$$

Therefore $O(M) \vee O(N) = X$.

(4) $\Rightarrow$ (5): Assume condition (4). Let $M \in \mu$. Suppose $N \in \mu$ such that $N \neq M$ and $O(N) \subseteq M$. Since $O(M) \subseteq M$, by hypothesis, we get $X = O(M) \vee O(N) = M$, which is a contradiction. Hence M is the unique maximal filter such that $O(M) \subseteq M$.

(5) $\Rightarrow$ (1): Let $M \in \mu$. Suppose $O(M)$ is not maximal. Let M_0 be a maximal filter of X such that $O(M) \subseteq M_0$. We have always $O(M_0) \subseteq M_0$, which is a contradiction. ∎

Acknowledgements

The author would like to thank the referee for his valuable suggestions and comments that improved the presentation of this article.

References

[1] S.S. Ahn, Y.H. Kim and J.M. Ko, *Filters in commutative BE-algebras*, Commun. Korean. Math. Soc. **27** (2) (2012) 233–242.
https://doi.org/10.4134/CKMS.2012.27.2.233

[2] A. Borumand Saeid, A. Rezaei and R.A. Borzooei, *Some types of filters in BE-algebras*, Math. Comput. Sci. **7** (2013) 341–352.
https://doi.org/10.1007/s11786-013-0157-6

[3] W.H. Cornish, *O-ideals, congruences, sheaf representation of distributive lattices*, Rev. Roum. Math. Pures et Appl. **22** (8) (1977) 1059–1067.

[4] K. Iseki and S. Tanaka, *An introduction to the theory of BCK-algebras*, Math. Japon. **23** (1) (1979) 1–6.

[5] H.S. Kim and Y.H. Kim, *On BE-algebras*, Sci. Math. Jpn. **66** (1) (2007) 113–116.

[6] B.L. Meng, *On filters in BE-algebras*, Sci. Math. Japon. (2010) 105–111.
https://doi.org/10.32219/isms.71.2-201

[7] S. Rasouli, *Generalized co-annihilators in residuated lattices*, Annals of the University of Craiova, Mathematics and Computer Science Series **45** (2) (2019) 190–207.

[8] M. Sambasiva Rao, *Prime filters of commutative BE-algebras*, J. Appl. Math. Inf. **33** (5–6) (2015) 579–591.
https://doi.org/10.14317/jami.2015.579

[9] V.V. Kumar and M.S. Rao, *Dual annihilator filters of commutative BE-algebras*, Asian-European J. Math. **10** (1) (2017) 1750013(11 pages).
https://doi.org/10.1142/s1793557117500139

[10] V.V. Kumar and M.S. Rao and S.K. Vali, *Regular filters of commutative BE-algebras*, TWMS J. Appl. & Engg. Math. **11** (4) (2021) 1023–1035.

[11] V.V. Kumar and M.S. Rao, and S.K. Vali, *Quasi-complemented BE-algebras*, Discuss. Math. Gen. Algebra Appl. **41** (2021) 265–281.
https://doi.org/10.2307/1996101

[12] A. Walendziak, *On commutative BE-algebras*, Sci. Math. Japon. (2008) 585–588.
https://doi.org/10.32219/isms.69.2-281

Received 13 July 2021
Revised 4 November 2021
Accepted 4 November 2021