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A NOTE ON WEAK-INTERIOR AND QUASI-INTERIOR IDEALS IN QUASI-ORDERED SEMIGROUPS

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Abstract

This short note introduces the concepts of (left, right) weak-interior ideals and (left, right) quasi-interior ideals in quasi-ordered semigroups and analyzes the relationships between (left, right) ideals, interior ideals and these two newly introduced classes of ideals in quasi-ordered semigroups.

Keywords: quasi-ordered semigroup, ideal, interior ideal, weak-interior ideal and quasi-interior ideal.

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1. Introduction

The concept of interior ideals of a semigroup S has been introduced by Lajos in [4] as a subsemigroup J of S such that $SJS \subseteq J$. The interior ideals of semigroups have been also studied by Szász in [8, 9]. In [2, 3] Kehayopulu and Tsingelis introduced the concepts of interior ideals in ordered semigroups. In [2] it is shown that ideals and interior ideals coincide in a regular ordered semigroup. Jantanan, Johdee and Praththong in [1] and Krishna Rao in [5] also wrote about interior ideals in ordered semigroup. The concept of weak-interior ideals was introduced in articles [6, 7] by Krishna Rao. A non-empty subset J of a semigroup S is called a left (right) weak-interior ideal of S if J is a sub-semigroup of S and holds $SJJ \subseteq J$ (respectively $JJS \subseteq J$). Also, the concept of quasi-interior ideals was introduced in articles [6, 7] by Krishna Rao: A non-empty subset J of a semigroup S is said to be left (right) quasi-interior ideal of S, if J is a sub-semigroup of S and holds $SJSJ \subseteq J$ (respectively $JSJS \subseteq J$).

A relation $\preceq$ on a semigroup S is a quasi-order on S if holds

(1) $(\forall x, y \in S)(x \preceq x)$,

(2) $(\forall x, y, z \in S)((x \preceq y \wedge y \preceq z \Longrightarrow x \preceq z)$ and

(3) $(\forall x, y, u \in S)(x \preceq y \Longrightarrow (xu \preceq yu \wedge ux \preceq uy))$.

A quasi-order $\preceq$ on a semigroup S is an order on S if the following holds

(4) $(\forall x, y \in S)((x \preceq y \wedge y \preceq x) \Longrightarrow x = y)$.

If a semigroup is ordered by a quasi-order relation (by an order relation), then it is said to be quasi-ordered semigroup (res. ordered semigroup).

This short note introduces the concepts of (left, right) weak-interior ideals and (left, right) quasi-interior ideals in quasi-ordered semigroups and analyzes the relationships between (left, right) ideals, interior ideals and these two newly introduced classes of ideals in quasi-ordered semigroups.

2. Preliminaries: Ideals

In a quasi-ordered semigroup S, four classes of substructures can be identified:
– A subset A od a semigroup S is a sub-semigroup of S if the following holds

(11) $(\forall x, y \in S)((x \in A \wedge y \in A) \Longrightarrow xy \in A)$;

– A subset J of a semigroup ordered under a quasi-order $\preceq$ is a left ideal of S if holds

(12) $(\forall x, y \in S)(y \in J \Longrightarrow xy \in J)$ and

(13) $(\forall x, y \in S)((y \in J \wedge x \preceq y) \Longrightarrow x \in J)$;

– A subset J of a semigroup ordered under a quasi-order $\preceq$ is a right ideal of S if holds

(14) $(\forall x, y \in S)(x \in J \Longrightarrow xy \in J)$ and

(13) $(\forall x, y \in S)((y \in J \wedge x \preceq y) \Longrightarrow x \in J)$;

– A subset J of a semigroup ordered under a quasi-order $\preceq$ is an ideal of S if holds

(15) $(\forall x, y \in S)((x \in J \vee y \in J) \Longrightarrow xy \in J)$ and

(13) $(\forall x, y \in S)((y \in J \wedge x \preceq y) \Longrightarrow x \in J)$.

Lajos [4] defined the concept of an interior ideal in a semigroup. Interior ideal in a semigroup also was studied by Szasz [8, 9]: A non-empty subset J of a semigroup S is an interior ideal in S if it is a sub-semigroup of S and holds $SJS \subseteq J$. This means

(16) $J \neq \emptyset$,

(17) $(\forall x, y \in S)((x \in J \land y \in J) \Longrightarrow, xy \in J)$,

(18) $(\forall x, u, v \in S)(x \in J \Longrightarrow uxv \in J)$.

Let a semigroup $(S, \cdot)$ is ordered by a quasi-order relation $\preceq$. Kehayopulu in [2], Definition 1, in determining of interior ideals in such a semigroup adds the requirement (13).

3. WEAK-INTERIOR IDEALS

Krishna Rao introduced in the paper [7] the notion of weak-interior ideal as a generalization of interior ideal of semigroup and he analyzed some features of such a newly introduced type of ideal and its connection with interior ideal:

– A non-empty subset J of a semigroup S is said to be a left weak-interior ideal of S if J is a sub-semigroup of S and holds $SJJ \subseteq J$. In other words, J is a left weak-interior ideal of a semigroup S if valid

(16) $J \neq \emptyset$,

(17) $(\forall x, y \in S)((x \in J \land y \in J) \Longrightarrow xy \in J)$,

(21) $(\forall x, u, v \in S)((u \in J \land v \in J) \Longrightarrow xuv \in J)$.

– A non-empty subset J of a semigroup S is said to be a right weak-interior ideal of S if J is a sub-semigroup of S and holds $JJS \subseteq J$. In other words, J is a right weak-interior ideal of a semigroup S if valid

(16) $J \neq \emptyset$,

(17) $(\forall x, y \in S)((x \in J \land y \in J) \Longrightarrow xy \in J)$,

(22) $(\forall x, u, v \in S)((u \in J \land v \in J) \Longrightarrow uvx \in J)$.

– A non-empty subset J of a semigroup S is said to be a weak-interior ideal of S if J is a sub-semigroup of S and J is left and right weak-interior ideal of S.

If we add the requirement (13) to conditions (16), (17) and (21) in the definition of the concept of left weak-interior ideals of a semigroup, we get the determination of left weak-interior ideals of quasi-ordered semigroup. Analogously to the previous, if we add the requirement (13) to conditions (16), (17) and (22) in the definition of the concept of right weak-interior ideals of a semigroup, we get the determination of right weak-interior ideals of quasi-ordered semigroup. Finally, a non-empty subset J of a quasi-ordered semigroup S is said to be a weak-interior ideal of S if J is a left and right weak-interior ideal of S.

Theorem 1. *Any right ideal of a quasi-ordered semigroup S is a right weak-interior ideal of S.*

Proof. Let J be a right ideal of S. This means that J satisfies the conditions (11), (14) and (13). It is clear that J is a sub-semigroup of S, i.e., J satisfies the condition (17). Let us prove (22). Let $u, v, x \in S$ be such that $u \in J \wedge v \in J$. Then $uv \in J$ by (11). Thus $uvx \in J$ by (14). This proves that J is a right weak-interior ideal of S. ■

The reverse of the previous theorem can be demonstrated in one special case.

Theorem 2. *Let S be a quasi-ordered semigroup which satisfies the condition*

(A) $(\forall x \in S)(x \preceq x^2)$.

Then the right ideals and the right weak-interior ideals in S coincide.

Proof. Suppose that S is a quasi-ordered semigroup which satisfies the condition (A) and J is a right weak-interior ideal of S. Let $x, y \in S$ be arbitrary elements such that $x \in J \vee y \in J$. For definiteness, suppose $x \in J$. Then $x^2y \in J$ by (22). Since $x \preceq x^2$ holds for each element $x \in S$, we have $xy \preceq x^2y$ by (3) and $xy \in J$ by (13). This means that J is a right ideal od S. ■

Analogous to the previous one, the following two theorems can be proved.

Theorem 3. *Let S be a quasi-ordered semigroup which satisfies the condition* (A). *Then the left ideals and the left weak-interior ideals in S coincide.*

Theorem 4. *Let S be a quasi-ordered semigroup which satisfies the condition* (A). *Then the ordered ideals and the ordered weak-interior ideals in S coincide.*

Any interior ideal J of a quasi-ordered semigroup S is a weak interior ideal of S. In fact: $SJJ \subseteq SJS \subseteq J$ and $JJS \subseteq SJS \subseteq J$. Also, it can be shown that if the quasi-ordered semigroup S satisfies the condition (A), then any weak-interior ideal is an interior ideal.

Theorem 5. *Let the quasi-ordered semigroup S satisfy condition* (A). *Then any weak-interior ideal of S is an interior ideal of S.*

Proof. Let J be a weak-interior ideal of a quasi-ordered semigroup S. This means that the following conditions (16), (17), (21), (22) and (13) are valid. Let us prove (18).

Let $x, y, v \in S$ be such that $x \in J$. Then $xxv \in J$ by (22). On the other hand, as $x \preceq x^2$, we have $xv \preceq xxv \in J$ by (3). Therefore, we have $xv \in J$ by (13). Now, from $xv \in J$ it follows $u(xv)(xv) \in J$ by (21). Again, from $xv \preceq (xv)(xv)$, it follows $uxv \preceq u(xv)(xv)$ by (3). So, we have $uxv \preceq u(xv)(xv)$ and $u(xv)(xv) \in J$. Hence $uxv \in J$ by (13).

This shows that J is an interior ideal of S. ■

4. Quasi-interior ideals

Determining the concept of quasi-interior ideals of a semigroup is somewhat different from the description of the concept of ordered quasi-interior ideals of a quasi-ordered semigroup. This is the reason why the definitions and analyses of these two classes of concepts we consider separately.

The concept of (left, right) quasi-interior ideals was analyzed in papers [5, 6, 7] by Krishna Rao.

– A non-empty subset J of a semigroup S is said to be a left quasi-interior ideal of S if J is a sub-semigroup of S and holds $SJSJ \subseteq J$. Thus means

(16) $J \neq \emptyset$,

(17) $(\forall x, y \in S)((x \in J \wedge y \in J) \Longrightarrow xy \in J)$,

(26) $(\forall x, y, u, v \in S)((u \in J \wedge v \in J) \Longrightarrow xuyv \in J)$.

– A non-empty subset J of S is said to be a right quasi-interior ideal of S if J is a sub-semigroup of S and holds $JSJS \subseteq J$. This means

(16) $J \neq \emptyset$,

(17) $(\forall x, y \in S)((x \in J \wedge y \in J) \Longrightarrow xy \in J)$,

(27) $(\forall x, y, u, v \in S)((u \in J \wedge v \in J) \Longrightarrow uxvy \in J)$.

– A non-empty subset J of a semigroup S is said to be a quasi-interior ideal of S if it is both a left quasi-interior ideal and a right quasi-interior ideal of S.

If in the determination of previous substructures in a semigroup ordered under a quasi-order we add requirement (13), we get concepts of (left, right) quasi-interior ideals of quasi-ordered semigroups.

Theorem 6. *Let S be a quasi-ordered semigroup. Every left (right) ideal is a left (right) quasi-interior ideal of S.*

Proof. Let J be a left ideal of S. Thus, (11), (12) and (13) are valid formulas. Then J is a sub-semigroup of S, ie, (17) is valid. Let us prove (26).

Let $x, y, y, v \in S$ be arbitrary elements such that $u \in J \wedge v \in J$. Then $xu \in J \wedge yv \in J$ by (12). Thus $xuyv \in J$ by (11). So, J is a left quasi-interior ideal of S. ■

An analogous procedure can be demonstrated for right ideals. Hence

Theorem 7. *Let S be a quasi-ordered semigroup. Every ideal is a quasi-interior ideal of S.*

The inverse of the Theorem 6 can be demonstrated if S is a quasi-ordered regular semigroup.

Theorem 8. *Let S be a regular semigroup ordered under a quasi-order. Then the left quasi-interior ideals and the left ideals in S coincide.*

Proof. Let S be a regular semigroup ordered under a quasi-order and suppose that J is a left quasi-interior ideal of S. This means that the conditions (16), (17), (26) and (13) are valid. Let us prove (12).

Let $x, y \in S$ be arbitrary elements such that $y \in J$. Since for y there exists an element $u \in S$ such that the following holds $y \preceq yuy$, because S is a regular quasi-ordered semigroup, then $xy \preceq xyuy$. On the other hand, we have $xyuy \in J$ by (26). Thus $xy \in J$ by (13). Therefore, J is a left ideal of S. ■

Also, we have:

Theorem 9. *Let S be a regular semigroup ordered under a quasi-order. Then the right quasi-interior ideals and the right ideals in S coincide.*

One interesting consequence of the previous Theorem 8 and Theorem 9 is obtained if the following lemma is taken into account.

Lemma 10 ([2], Proposition 1)**.** *In regular ordered semigroups, the ideals and the interior ideals coincide.*

Corollary 11. *Let S be a regular semigroup ordered under a quasi-order. Then the quasi-interior ideals and the interior ideals in S coincide.*

In addition to the previous, we have:

Theorem 12. *Every interior ideal of a quasi-ordered semigroup S is a left* (*right*) *quasi-interior ideal of S.*

Proof. Let J be an interior ideal of S. This means that (16), (17), (18) and (13) are valid formulas. Let us prove (26).

Let $x, y, u, v \in S$ be such that $u \in J \wedge y \in J$. Then $xuy \in J$ by (18). Thus $xuyv \in J$ by (11). Hence J is a left quasi-interior ideal of S.

An analogous procedure for right quasi-interior ideals of a quasi-ordered semigroup also can be demonstrated. ■

The reverse of the previous theorem can be proved if the quasi-ordered semigroup S satisfies one additional condition.

Theorem 13. *Suppose that a quasi-ordered semigroup S satisfies one additional condition.*

(C) *For every elements $a, b \in S$ the following holds $a \preceq ab$.*

Then the interior ideals and the left quasi-interior ideals in S coincide.

Proof. Suppose that a quasi-ordered semigroup S satisfies the condition (C) and let J be a left quasi-interior ideal of S. This means that J satisfies the conditions (16), (17), (26) and (13). Let us prove (18).

Let $u, x, y \in S$ be arbitrary elements such that $x \in J$. Then $xuyu \in J$ by (26). Since $xuy \preceq xuyu$ by (C), it follows from here $xuy \in J$ by (13). Therefore, the condition (18) is valid. ■

Claims for (right) quasi-interior ideals of a quasi-ordered semigroup can be designed without major difficulties analogously to previous claims. Thus, for example, we transform Theorem 13 into the following theorem:

Theorem 14. *Suppose that a quasi-ordered semigroup S satisfies one additional condition.*

(D) *For every elements $a, b \in S$ the following holds $a \preceq ba$.*

Then the interior ideals and the right quasi-interior ideals in S coincide.

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