

THE AUTOMORPHISMS HAVING THE EXTENSION  
PROPERTY IN A CATEGORY OF A FINITE  
DIRECT SUM OF CYCLIC MODULES

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**Abstract**

It is well known that the problem of characterizing the automorphisms, in the category of abelian groups, with the extension property is resolved [1]. But in other categories, it is a very difficult problem. This paper extends the result in [1] to a category of modules. Let  $A$  be a unique factorization integral domain (UFD). Consider  $M$  a direct finite sum of cyclic modules over  $A$  where  $\text{Ann}_A(M) = \{0\}$  and  $\alpha$  an automorphism of  $M$ . We give a necessary and sufficient condition such that  $\alpha$  satisfies the extension property.

**Keywords:** integral domain, factorization, module, automorphism, torsion and torsion-free.

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1. INTRODUCTION

Let  $A$  be a unique factorization integral domain (UFD) and  $M$  be a module over  $A$ . We say that  $\alpha$ , an automorphism of  $M$ , satisfies the extension property if for

all monomorphisms  $\lambda : M \longrightarrow N$  there exists  $\tilde{\alpha}$  such that:

$$\begin{array}{ccc} M & \xrightarrow{\lambda} & N \\ \alpha \downarrow & & \downarrow \tilde{\alpha} \\ M & \xrightarrow{\lambda} & N \end{array}$$

is commutative, i.e.,  $\lambda \circ \alpha = \tilde{\alpha} \circ \lambda$ .

It is known that all automorphisms of a vector space satisfy the extension property. However, it is not always true for other categories. But there are some very important results in the category of groups. Schupp [7] proved that the automorphisms having the property of extension in the category of groups characterize the inner automorphisms. Pettet [6] provided a simpler proof of Schupp's result. Then, Ben Yacoub [4] proved that this result is not true in the algebra category. In order to generalize the result of Schupp, Abdelalim *et al.* [1] gave the characterization of the automorphisms having the property of extension in the category of abelian groups.

In another work [2], the authors considered the category that contains abelian groups. They proved that if a module  $M$  is a direct sum of cyclic torsion-free modules over a BFD, the automorphisms of  $M$  that satisfy the property of the extension are none other than the homotheties of invertible ratio.

Certainly, one of the generalizations of [1] other than [2] is the case where all or some of the co-generators of  $M$  are not torsion-free. To do this, consider the following example which illustrates where the problem of the extension's ownership lies. We know that  $\mathbb{Z}[i]$  is a UFD. Let  $n \in \mathbb{N} : n \geq 2$ . For all  $s \in \{2, 3, \dots, n\}$ , we will denote by  $p_s$  the  $(s-1)^e$  prime number. ( $p_2 = 2 < p_3 = 3 < p_4 = 5 < \dots < p_n$ ).

Consider, in the  $\mathbb{Z}[i]$ -module  $N = \mathbb{C}/\mathbb{Z}[i]$ , the sub-module  $M = \bigoplus_{s=1}^{s=n} \mathbb{Z}[i]\overline{x}_s$  where  $\overline{x}_1 \in \mathbb{C}/\mathbb{Z}[i] - \mathbb{Q}[i]/\mathbb{Z}[i]$  any torsion-free element and the torsion element  $\overline{x}_s = 1 + i \cdot \frac{1}{p_s}$  for all  $s \in \{2, \dots, n\}$ . We can prove that, for all  $s \in \{2, \dots, n\}$ , there exists  $a_s$  in  $\mathbb{Z}[i]$  such that  $\alpha(\overline{x}_s) = a_s \cdot \overline{x}_s$ . Without using that  $\alpha$  satisfies the extension property, we cannot conclude that  $a_2, \dots, a_n$  are units in  $\mathbb{Z}[i]$  neither (\*\*)  $\alpha(\overline{x}_1) \in \mathbb{Z}[i]\overline{x}_1$ . The key idea is to prove (\*\*).

In this paper, we are interested in the extension property of a special category of modules over  $A$  which is a unique factorization integral domain (not a field).

Let  $n$  be a non-zero natural integer and  $M = Ax_1 \oplus Ax_2 \oplus \dots \oplus Ax_n$  a direct finite sum of cyclic modules over  $A$ , such that  $\text{Ann}_A(M) = \{0_A\}$ .

Our paper is organized as follows. In the first section, we will briefly give necessarily important results. In the second part, all the generators of  $M$  are torsion-free elements. We will prove that an automorphism  $\alpha : M \longrightarrow M$  satisfies the extension property, if and only if there exists a unit  $k$  in  $A$  such that  $\alpha = k \cdot 1_M$ . And in the last part,  $M$  is no longer considered as a torsion-free

module. Again with much more advanced techniques, we will prove that the homotheties of invertible ratio are the only automorphisms of  $M$  satisfying the extension property.

## 2. PRELIMINARIES AND NOTATIONS

**Definition.** Let  $A$  be an integral domain,  $M$  an  $A$ -module  $x$  an element of  $M$ . The annihilator of  $x$  is the ideal  $Ann_A(x) = \{a \in A / a \cdot x = 0_M\}$ . The annihilator of  $M$ , is the ideal  $Ann_A(M) = \{a \in A / \forall y \in M : a \cdot y = 0_M\}$ .

**Definition** [5].

- Let  $x$  be an element of  $M$ .  $x$  is a torsion-free element if  $Ann_A(x) = \{0_A\}$  and  $x$  is a torsion element if  $Ann_A(x) \neq \{0_A\}$ .
- We say that  $M$  is a torsion-free module if all its elements are torsion free.
- We say that  $M$  is a torsion module if all its elements are torsion.

**Lemma 1.** Let  $M = Ax_1 \oplus Ax_2 \oplus \dots \oplus Ax_n$  be a direct finite sum of cyclic modules over  $A$ .

$M$  is a torsion module if and only if  $Ann_A(M) \neq \{0_A\}$ .

**Proof.** The proof is easy. ■

**Lemma 2.** For an irreducible element  $p$  in  $A$ , the set  $I_p = \{p^n / n \in \mathbb{N}\}$  is infinite.

**Proof.** Suppose that  $I_p$  is finite, then necessarily  $\exists (n, m) \in \mathbb{N}^2$  such that  $n < m$  and  $p^n = p^m$ . So  $p^n(1 - p^{m-n}) = 1_A \implies p^{m-n} = 1_A$  (integral domain). Therefore  $p$  is a unit in  $A$ , which is not true. ■

**Notations.** Let  $E_p(M)$  denote the set of all  $A$ -automorphisms of  $M$  satisfying the extension property. For  $i \in \{1, 2, \dots, n\}$ ,  $E_i$  will denote an injective envelope of  $Ax_i$ ,  $\mu_i : Ax_i \longrightarrow E_i$  a monomorphism of  $A$ -modules and

$$M_i = E_i + \sum_{k=1, k \neq i}^{k=n} A \cdot x_k.$$

**Lemma 3.** If  $\alpha$  satisfies the extension property then  $\alpha^{-1}$  also satisfies the extension property.

**Proof.** Let  $M \xrightarrow{\lambda} N$  a monomorphism of  $A$ -modules, then there exists  $\widetilde{N} \xrightarrow{\widetilde{\alpha}} N$  an automorphism of  $A$ -modules such that  $\lambda \circ \alpha = \widetilde{\alpha} \circ \lambda$ . We define  $\widetilde{\alpha}^{-1} = \widetilde{\alpha}^{-1}$ , it's clear that  $\widetilde{\alpha}^{-1} \in Aut(N)$ . Let  $x \in M$ , so  $\exists y \in M : x = \alpha(y)$  then  $\lambda[\alpha^{-1}(x)] = \lambda(y)$ . On the other hand, if we put  $\widetilde{\alpha}^{-1}[\lambda(x)] = t$ . We will have  $\lambda(x) = \widetilde{\alpha}(t) \implies \widetilde{\alpha}(t) = \lambda[\alpha(y)]$ , then  $\widetilde{\alpha}(t) = \widetilde{\alpha}[\lambda(y)] \implies t = \lambda(y)$ . From where,  $\lambda \circ \alpha^{-1} = \alpha^{-1} \circ \lambda$  hence the lemma. ■

3. FINITE SUM OF TORSION FREE CYCLIC  $A$ -MODULES

In this section, we suppose that  $x_i$  is a torsion-free element for all  $i$  in  $\{1, 2, \dots, n\}$ . Let  $\mu_i : Ax_i \rightarrow E_i$  be a monomorphism of  $A$ -modules.

**Lemma 4.** *For all  $i \in \{1, 2, \dots, n\}$ , we have*

$$M_i = E_i \bigoplus \bigoplus_{k=1, k \neq i}^{k=n} A \cdot x_k$$

and

$$E_i = \bigcap_{a \in A^*} a \cdot M_i.$$

**Proof.** We know that  $E_1 \oplus E_2 \oplus \dots \oplus E_n$  is an injective envelope of  $Ax_1 + Ax_2 + \dots + Ax_n$  (see [3]). Then

$$M_i = E_i + \sum_{k=1, k \neq i}^{k=n} A \cdot x_k = E_i \bigoplus \bigoplus_{k=1, k \neq i}^{k=n} Ax_k.$$

As  $E_i$  is an injective envelope of  $Ax_i$  then a divisible module. For all  $a \in A^*$ ,

$$E_i = a \cdot E_i \subset a \cdot M_i \implies E_i \subset \bigcap_{a \in A^*} a \cdot M_i.$$

Reciprocally:

Let

$$t = e + \sum_{j=1, j \neq i}^{j=n} m_j \cdot x_j \in \bigcap_{a \in A^*} a \cdot M_i \subset \bigcap_{k \in \mathbb{N}} p^k \cdot M_i \text{ where } e \in E_i \text{ and } m_j \in A.$$

Then  $(\forall k \in \mathbb{N}) (\exists e_k \in E_i) (\exists (m_{j,k})_{j=1, j \neq i, j=n} \in A^{n-1})$  such that

$$t = p^k \left( e_k + \sum_{j=1, j \neq i}^{j=n} m_{j,k} \cdot x_j \right).$$

Moreover, as  $M_i$  is a direct sum of  $A$ -modules, by identification, we have

$$e = p^k e_k \text{ and } \sum_{j=1, j \neq i}^{j=n} m_j \cdot x_j = p^k \cdot \left( \sum_{j=1, j \neq i}^{j=n} m_{j,k} \cdot x_j \right).$$

If  $\sum_{j=1, j \neq i}^{j=n} m_j \cdot x_j \neq 0_M$  then there exists  $j \neq i \in \{1, 2, \dots, n\}$  such that  $m_j \neq 0_A$  and  $m_j = p^k \cdot m_{j,k}$ . So,  $p^k \mid m_j$  for all integer  $k$ , which is false in a factorial integral domain by Lemma 2.

Hence  $t = p^k e_k \in E_i$ , that concludes the proof. ■

Under the previous notations, we have the following lemma.

**Lemma 5.** *If  $\alpha \in E_p(M)$ , then  $\tilde{\alpha}(E_i) = E_i$ .*

**Proof.** We have

$$E_i = \bigcap_{a \in A^*} aM_i \text{ then } \tilde{\alpha}(E_i) = \bigcap_{a \in A^*} a\tilde{\alpha}(M_i) \subset E_i.$$

Let now  $y$  in the divisible module  $E_i$ , then  $\forall a \in A^*, \exists y_a \in M_i$  such that  $y = a \cdot y_a$ . As  $\tilde{\alpha} \in \text{Aut}(M_i)$  then there exists  $x_a \in M_i$ :  $y_a = \tilde{\alpha}(x_a)$ , so

$$y = \tilde{\alpha}(a \cdot x_a) \implies y \in \tilde{\alpha}\left(\bigcap_{a \in A^*} a \cdot M_i\right) = \tilde{\alpha}(E_i)$$

Therefore  $\tilde{\alpha}(E_i) = E_i$ . ■

**Lemma 6.** *For all  $i$  and  $j$  in  $\{1, 2, \dots, n\}$ . If  $i \neq j$ , then*

$$M = A(x_i + x_j) \oplus \bigoplus_{k=1, k \neq i}^{k=n} A \cdot x_k.$$

**Proof.** Let  $i$  and  $j$  in  $\{1, 2, \dots, n\}$  such that  $i \neq j$ . Consider  $(a_1, a_2, \dots, a_n) \in A^n$  and suppose that  $a_i(x_i + x_j) + \sum_{k=1, k \neq i}^{k=n} a_k x_k = 0_M$ .

Then  $a_i x_i + (a_i + a_j)x_j + \sum_{k=1, k \neq i, k \neq j}^{k=n} a_k x_k = 0_M$ . As  $\{x_1, x_2, \dots, x_n\}$  is  $A$  free, then  $a_i = a_i + a_j = a_k = 0_A$  for all  $k \in \{1, 2, \dots, n\} \setminus \{i, j\}$ . So,  $a_1 = a_2 = \dots = a_n = 0_A$ . It's clear that  $A(x_i + x_j) \oplus Ax_j \subset Ax_i \oplus Ax_j$ . Let  $x \in Ax_i \oplus Ax_j$  then  $x = ax_i + bx_j$  where  $a$  and  $b$  are in  $A$ . So,  $x = b(x_i + x_j) + (a - b)x_i \in A(x_i + x_j) \oplus Ax_i$ . Then the lemma follows. ■

**Theorem 7.** *If  $\alpha \in E_p(M)$  then  $(\forall i \in \{1, 2, \dots, n\}) (\exists k_i \in A^*) (\alpha(x_i) = k_i \cdot x_i)$ .*

**Proof.** Let  $\lambda : M \rightarrow M_i$  defined by If  $x = \sum_{j=1}^{j=n} m_j \cdot x_j$ , where  $m_j \in A$  for all  $j \in \{1, 2, \dots, n\}$ ,  $\lambda(x) = m_i \cdot \mu_i(x_i) + \sum_{j=1, j \neq i}^{j=n} m_j \cdot x_j$ .

As defined  $\lambda$  is a morphism of  $A$ -modules and we have

$$\lambda(x) = m_i \cdot \mu_i(x_i) + \sum_{j=1, j \neq i}^{j=n} m_j \cdot x_j = 0_M \implies m_i \cdot \mu_i(x_i) = 0_M \text{ and } \sum_{j=1, j \neq i}^{j=n} m_j \cdot x_j = 0_M.$$

However  $\mu_i$  is injective then  $m_i = 0_A$ . As  $\{x_1, x_2, \dots, x_n\} \setminus \{x_i\}$  is  $A$ -free, then  $m_j = 0_A$  for all  $j \in \{1, 2, \dots, n\} \setminus \{i\}$ . Then,  $x = 0_M$ . Therefore  $\lambda$  is a monomorphism of  $A$ -modules. We know that

$$\alpha(x_i) = k_i \cdot x_i + \sum_{j=1, j \neq i}^{j=n} k_j \cdot x_j \text{ where } k_1, k_2, \dots, k_{n-1} \text{ and } k_n \text{ are in } A.$$

As  $\lambda\alpha = \tilde{\alpha}\lambda$  and by Lemma 5 we have

$$\lambda[\alpha(x_i)] = k_i \cdot \mu(x_i) + \sum_{j=1, j \neq i}^{j=n} k_j \cdot \lambda(x_j) = k_i \cdot \mu_i(x_i) + \sum_{j=1, j \neq i}^{j=n} k_j \cdot x_j = \tilde{\alpha}[\lambda(x_i)] \in E_i.$$

And since  $E_i$  is a direct factor in  $M_i$ ,  $\lambda(\alpha(x_i)) = k_i x_i \implies \sum_{j=1, j \neq i}^{j=n} k_j x_j = 0_M$ . So  $\alpha(x_i) = k_i \cdot x_i$ . ■

**Corollary 8.** *We have  $E_p(M) = \{k id_M / k \in A^\times\}$ , where  $A^\times$  is the group of units in  $A$ .*

**Proof.** Applying the Theorem 7 for all  $i \in \{1, 2, \dots, n\}$  there exists  $k_i \in A^*$  such that  $\alpha(x_i) = k_i \cdot x_i$ . We must find  $k_i = k_j$ , for all  $i$  and  $j$  in  $\{1, 2, \dots, n\}$  such that  $i \neq j$ . Applying now the Theorem 7, to  $M$  as asserted in Lemma 6, there exists  $k_{i,j} \in A^*$  such that  $\alpha(x_i + x_j) = k_{i,j} \cdot (x_i + x_j) = k_{i,j} \cdot x_i + k_{i,j} \cdot x_j = k_i \cdot x_i + k_j \cdot x_j \implies k_{i,j} = k_i = k_j$  since  $\{x_i, x_j\}$  is  $A$ -free. Then, there is  $k \in A^*$  such that for all  $i \in \{1, 2, \dots, n\}$ ,  $\alpha(x_i) = k \cdot x_i$ . Therefore  $\alpha = k \cdot id_M$ .

We must prove that  $k$  is a unit in  $A$ . As  $\alpha$  satisfies the extension property then  $\alpha^{-1}$  also satisfies the extension property by Lemma 3. Then there exists  $r \in A^*$  such that  $\alpha^{-1} = r \cdot id_M$ . As  $\alpha \circ \alpha^{-1} = \alpha^{-1} \circ \alpha = k \cdot r \cdot id_M$  then  $r \cdot k = 1_A$  which proved that  $k$  is a unit in  $A$ . ■

#### 4. DIRECT FINITE SUM OF TORSION FREE AND TORSION CYCLICS $A$ -MODULES

Let  $n' \in \mathbb{N}^*$  such that  $n' \leq n - 1$ . In this section, we assume that  $x_1, x_2, \dots, x_{n'}$  are the torsion-free co-generators and  $x_{n'+1}, x_{n'+2}, \dots, x_n$  are the torsions co-generators.

For all  $i \in \{1, 2, \dots, n'\}$ ,  $E_i$  will denote an injective envelope of  $A \cdot x_i$  and  $\mu_i : A \cdot x_i \rightarrow E_i$  a monomorphism of  $A$ -modules. Same as the Theorem 7 and corollary 8, we have the same result, but with a demonstration using the notion of a torsion element.

**Lemma 9.** *For all  $i \in \{1, \dots, n'\}$ , we have,  $E_i = \bigcap_{a \in A^*} a \cdot M_i$ .*

**Proof.** It's clear that  $E_1$  is a divisible  $A$ -module, so  $E_1 \subset a \cdot E_1 \subset a \cdot M_1$  for all  $a$  in  $A^* \implies E_1 \subset \bigcap_{a \in A^*} a \cdot M_1$ . Reciprocally, let  $x \in M_1$  then  $(\exists (f, c_2, c_3, \dots, c_n) \in E_1 \times (A^*)^{n-1}) (x = f + c_2 \cdot x_2 + c_3 \cdot x_3 + \dots + c_n \cdot x_n)$ .

We know that  $(\forall i \in \{n'+1, n'+2, \dots, n\}) (\exists b_i \in A^* : b_i \cdot x_i = 0_M)$ . Let  $b = b_{n'+1} b_{n'+2} \dots b_n \in A^*$ . Therefore, for all  $i \in \{n'+1, n'+2, \dots, n\}$ ,  $b \cdot x_i = 0$ . Suppose that for all  $a \in A^*$ ,  $x \in a \cdot M_1$ , then for all  $m$  a nonzero naturel integer  $x \in b^m \cdot M_1$ , consequently.

If  $n' = 1$ , then  $x \in E_1$ . If  $n' \neq 1$ , then

$$\begin{aligned} \exists (e_m, a_{2,m}, a_{3,m}, \dots, a_{n,m}) \in E_1 \times (A^*)^{n-1} : \\ x = b^m e_m + \sum_{j=2}^{j=n} a_{j,m} b^m x_j = f + \sum_{j=2}^{j=n'} c_j x_j. \end{aligned}$$

Let us suppose that  $c_2 x_2 + c_3 x_3 + \dots + c_{n'} x_{n'} \neq 0$ , then there exists  $j \in \{2, 3, \dots, n'\}$  such that  $c_j \neq 0_A$  and  $c_j = a_{j,m} b^m$ . So, there is an infinity divisors of  $c_j$ , which is false (because  $A$  is UFD). Therefore,  $c_2 \cdot x_2 + c_3 \cdot x_3 + \dots + c_{n'} \cdot x_{n'} = 0 \implies x \in E_1$ . For any  $i \in \{2, 3, \dots, n'\}$  with the simple transposition  $(i, 1)$ , we find the first case. Then the lemma follows. ■

**Lemma 10.** *If  $\alpha \in E_p(M)$ , then  $(\forall i \in \{1, 2, \dots, n'\}) (\exists k_i \in A^* : \alpha(x_i) = k_i x_i)$ .*

**Proof.** Let  $\lambda : M \rightarrow M_1$  be the morphism defined by  $\lambda(a_1 x_1 + a_2 x_2 + \dots + a_n x_n) = a_1 \mu_1(x_1) + a_2 x_2 + \dots + a_n x_n$ . It's clear that  $\lambda$  is a monomorphism of  $A$  modules.

As  $\alpha$  satisfies the extension property, there exists  $\tilde{\alpha} \in \text{Aut}(M_1)$  such that  $\lambda \circ \alpha = \tilde{\alpha} \circ \lambda$ . As  $\alpha(x_1) \in M$  then  $\alpha(x_1) = d_1 \cdot x_1 + d_2 \cdot x_2 + \dots + d_n \cdot x_n$ , where  $(d_1, d_2, \dots, d_n) \in A^n$ . So  $\lambda[\alpha(x_1)] = d_1 \cdot \mu_1(x_1) + \sum_{j=2}^{j=n} d_j \cdot x_j = \tilde{\alpha}[\lambda(x_1)]$ . And as we know that  $\tilde{\alpha}[\lambda(x_1)] \in E_1$ , which is a direct factor in  $M_1$ , then  $\sum_{j=2}^{j=n} d_j \cdot x_j = 0_M \implies \alpha(x_1) = d_1 \cdot x_1$ . Same proof for all  $i$  in  $\{2, 3, \dots, n'\}$ , hence  $\alpha(x_i) = d_i \cdot x_i$  ( $k_i = d_i$ ). ■

**Lemma 11.** *If  $n' \neq 1$ , let  $i_0 \in \{2, 3, \dots, n'\}$ . We have*

- $x_1 + x_{i_0}$  is a torsion-free element.
- $M = A(x_1 + x_{i_0}) \oplus \bigoplus_{i=1, i \neq i_0}^{i=n'} A x_i \oplus \bigoplus_{j=n'+1}^{i=n} A x_j$  (\*).

**Proof.**

- We have  $x_{i_0} = x_1 + x_{i_0} - x_1 \in A(x_1 + x_{i_0}) \oplus \bigoplus_{i=1, i \neq i_0}^{i=n'} A x_i \oplus \bigoplus_{j=n'+1}^{i=n} A x_j$ . Let  $a \in A$  such that  $a(x_1 + x_{i_0}) = 0_M$  then  $a x_1 + a x_{i_0} = 0_M \implies a = 0_A$ . ( $\{x_1, x_{i_0}\}$  is  $A$  free). Therefore,  $\{x_1 + x_{i_0}\}$  is a  $A$ -free.
- Let  $(a_1, a_2, \dots, a_n) \in A^n$  and suppose that

$$a_{i_0}(x_1 + x_{i_0}) + \sum_{i=1, i \neq i_0}^{i=n} a_i x_i = 0_M.$$

Then

$$(a_{i_0} + a_1)x_1 + \sum_{i=2}^{i=n} a_i x_i = 0_M \implies a_1 = a_2 = \dots = a_{n'} = 0_A$$

and  $a_{n'+1}x_{n'+1} = a_{n'+2}x_{n'+2} = \cdots = a_n x_n = 0_M$  ( $M = \bigoplus_{i=1}^{i=n} Ax_i$  and  $\{x_1, \dots, x_{n'}\}$  is  $A$  free). So  $M = A(x_1 + x_{i_0}) \oplus \bigoplus_{i=1, i \neq i_0}^{i=n'} Ax_i \oplus \bigoplus_{j=n'+1}^{j=n} Ax_j$  (\*).

Then the lemma follows.  $\blacksquare$

**Corollary 12.** For all  $i \in \{1, 2, \dots, n'\}$ ,  $k_i = k_1$ .

**Proof.** By Lemma 10 we have for all  $i$  in  $\{1, 2, \dots, n'\}$ , there exists an  $k_i \in A^*$  such that  $\alpha(x_i) = k_i \cdot x_i$ .

Let  $M$  be defined as in Lemma 11, then by Lemma 10, there exists an  $r_{i_0} \in A^*$  such that  $\alpha(x_1 + x_{i_0}) = r_{i_0}(x_1 + x_{i_0})$ . So  $r_{i_0} = k_{i_0} = k_1$ , and this for all  $i_0$  a fixed element in  $\{2, \dots, n'\}$ . Hence the corollary.  $\blacksquare$

**Lemma 13.** For all  $i \in \{1, 2, \dots, n'\}$ ,  $x_i + x_{n'+1}$  is a torsion-free element and

$$M = \bigoplus_{i=1}^{i=n'} A(x_i + x_{n'+1}) \oplus \bigoplus_{j=n'+1}^{j=n} Ax_j.$$

**Proof.** We have  $x_i = x_i + x_{n'+1} - x_{n'+1}$  for all  $i \in \{1, 2, \dots, n'\}$ , then  $M = A(x_1 + x_{n'+1}) + A(x_2 + x_{n'+1}) + \cdots + A(x_{n'} + x_{n'+1}) + Ax_{n'+1} + Ax_{n'+2} + \cdots + Ax_n$ . Let  $(a_1, a_2, \dots, a_n) \in A^n$  such that:

$$x = \sum_{i=1}^{i=n'} a_i(x_i + x_{n'+1}) + \sum_{j=n'+1}^{j=n} a_j x_j = 0_M.$$

Then

$$\sum_{i=1}^{i=n'} a_i \cdot x_i + \left( \left( \sum_{i=1}^{i=n'} a_i \right) + a_{n'+1} \right) \cdot x_{n'+1} + \sum_{j=n'+2}^{j=n} a_j \cdot x_j = 0_M.$$

So,

$$\sum_{i=1}^{i=n'} a_i \cdot x_i = 0_M, \left( \left( \sum_{i=1}^{i=n'} a_i \right) + a_{n'+1} \right) \cdot x_{n'+1} = 0_M \text{ and } \sum_{j=n'+2}^{j=n} a_j \cdot x_j = 0_M.$$

And as  $x_1, x_2, \dots, x_{n'}$  are a torsion-free elements, then  $a_i = 0_A$  for all  $i \in \{1, 2, \dots, n'\}$  and  $a_j \cdot x_j = 0_M$  for all  $j \in \{n'+1, n'+2, \dots, n\}$ . Let  $i \in \{1, 2, \dots, n'\}$  and suppose that  $a_i \cdot (x_i + x_{n'+1}) = 0_M$ . Then  $a_i \cdot x_i + a_i \cdot x_{n'+1} = 0_M \implies a_i = 0_A \implies x_i + x_{n'+1}$  is a torsion-free element.  $\blacksquare$

**Theorem 14.** We have  $E_p(M) = \{kid_M / k \in A^\times\}$ .

**Proof.** Let  $b \in \text{Ann}_A(x_{n'+1}) \setminus \{0_A\}$ , so  $b \cdot x_{n'+1} = 0_M$ . As  $\alpha(x_{n'+1}) \in M$ , then  $\alpha(x_{n'+1}) = \sum_{j=1}^{j=n} a_j x_j$ , where  $(a_1, a_2, \dots, a_n) \in A^n$ . Then

$$\begin{aligned} \alpha(bx_{n'+1}) &= ba_1x_1 + ba_2x_2 + \dots + ba_nx_n = 0_M \\ \implies ba_1x_1 &= ba_2x_2 = \dots = ba_{n'}x_{n'} = 0_M \implies a_1 = a_2 = \dots = a_{n'} = 0_A \\ \implies \alpha(x_{n'+1}) &= a_{n'+1}x_{n'+1} + a_{n'+2}x_{n'+2} + \dots + a_nx_n. \end{aligned}$$

By applying the result of Lemma 10 to  $M = \bigoplus_{i=1}^{i=n'} A(x_i + x_{n'+1}) \oplus \bigoplus_{j=n'+1}^{j=n} Ax_j$  there exists  $k_{n'+1} \in A^*$  such that  $\alpha(x_i + x_{n'+1}) = k_{n'+1}x_i + k_{n'+1}x_{n'+1} = k_1x_i + \alpha(x_{n'+1})$ . Then  $(k_{n'+1} - k_1)x_i + k_{n'+1}x_{n'+1} = a_{n'+1}x_{n'+1} + a_{n'+2}x_{n'+2} + \dots + a_nx_n$ .

Also,  $M = \bigoplus_{k=1}^{k=n} Ax_k$  and  $x_i$  is a torsion free element, then  $k_{n'+1} = k_1$  and  $\sum_{j=n'+2}^{j=n} a_j x_j = 0_M \implies \alpha(x_{n'+1}) = k_1x_{n'+1}$ . The same proof for  $\alpha(x_{n'+j}) = k_1x_{n'+j}$  for  $j \in \{2, 3, \dots, n - n'\}$ , so  $\alpha = k_1 \text{id}_M$ . As we know  $\alpha^{-1}$  satisfies the extension property, then there exists  $r \in A^*$  such that  $\alpha^{-1} = r \text{id}_M$ . Consequently,  $r \cdot k_1 = 1_A$ . Which completes the proof of the theorem. ■

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