

THE AUTOMORPHISMS HAVING THE EXTENSION
PROPERTY IN A CATEGORY OF A FINITE
DIRECT SUM OF CYCLIC MODULES

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Abstract

It is well known that the problem of characterizing the automorphisms, in the category of abelian groups, with the extension property is resolved [1]. But in other categories, it is a very difficult problem. This paper extends the result in [1] to a category of modules. Let A be a unique factorization integral domain (UFD). Consider M a direct finite sum of cyclic modules over A where $\text{Ann}_A(M) = \{0\}$ and α an automorphism of M . We give a necessary and sufficient condition such that α satisfies the extension property.

Keywords: integral domain, factorization, module, automorphism, torsion and torsion-free.

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1. INTRODUCTION

Let A be a unique factorization integral domain (UFD) and M be a module over A . We say that α , an automorphism of M , satisfies the extension property if for

all monomorphisms $\lambda : M \longrightarrow N$ there exists $\tilde{\alpha}$ such that:

$$\begin{array}{ccc} M & \xrightarrow{\lambda} & N \\ \alpha \downarrow & & \downarrow \tilde{\alpha} \\ M & \xrightarrow{\lambda} & N \end{array}$$

is commutative, i.e., $\lambda \circ \alpha = \tilde{\alpha} \circ \lambda$.

It is known that all automorphisms of a vector space satisfy the extension property. However, it is not always true for other categories. But there are some very important results in the category of groups. Schupp [7] proved that the automorphisms having the property of extension in the category of groups characterize the inner automorphisms. Pettet [6] provided a simpler proof of Schupp's result. Then, Ben Yacoub [4] proved that this result is not true in the algebra category. In order to generalize the result of Schupp, Abdelalim *et al.* [1] gave the characterization of the automorphisms having the property of extension in the category of abelian groups.

In another work [2], the authors considered the category that contains abelian groups. They proved that if a module M is a direct sum of cyclic torsion-free modules over a BFD, the automorphisms of M that satisfy the property of the extension are none other than the homotheties of invertible ratio.

Certainly, one of the generalizations of [1] other than [2] is the case where all or some of the co-generators of M are not torsion-free. To do this, consider the following example which illustrates where the problem of the extension's ownership lies. We know that $\mathbb{Z}[i]$ is a UFD. Let $n \in \mathbb{N} : n \geq 2$. For all $s \in \{2, 3, \dots, n\}$, we will denote by p_s the $(s-1)^e$ prime number. ($p_2 = 2 < p_3 = 3 < p_4 = 5 < \dots < p_n$).

Consider, in the $\mathbb{Z}[i]$ -module $N = \mathbb{C}/\mathbb{Z}[i]$, the sub-module $M = \bigoplus_{s=1}^{s=n} \mathbb{Z}[i]\overline{x}_s$ where $\overline{x}_1 \in \mathbb{C}/\mathbb{Z}[i] - \mathbb{Q}[i]/\mathbb{Z}[i]$ any torsion-free element and the torsion element $\overline{x}_s = 1 + i \cdot \frac{1}{p_s}$ for all $s \in \{2, \dots, n\}$. We can prove that, for all $s \in \{2, \dots, n\}$, there exists a_s in $\mathbb{Z}[i]$ such that $\alpha(\overline{x}_s) = a_s \cdot \overline{x}_s$. Without using that α satisfies the extension property, we cannot conclude that a_2, \dots, a_n are units in $\mathbb{Z}[i]$ neither $(**) \alpha(\overline{x}_1) \in \mathbb{Z}[i]\overline{x}_1$. The key idea is to prove $(**)$.

In this paper, we are interested in the extension property of a special category of modules over A which is a unique factorization integral domain (not a field).

Let n be a non-zero natural integer and $M = Ax_1 \oplus Ax_2 \oplus \dots \oplus Ax_n$ a direct finite sum of cyclic modules over A , such that $\text{Ann}_A(M) = \{0_A\}$.

Our paper is organized as follows. In the first section, we will briefly give necessarily important results. In the second part, all the generators of M are torsion-free elements. We will prove that an automorphism $\alpha : M \longrightarrow M$ satisfies the extension property, if and only if there exists a unit k in A such that $\alpha = k \cdot 1_M$. And in the last part, M is no longer considered as a torsion-free

module. Again with much more advanced techniques, we will prove that the homotheties of invertible ratio are the only automorphisms of M satisfying the extension property.

2. PRELIMINARIES AND NOTATIONS

Definition. Let A be an integral domain, M an A -module x an element of M . The annihilator of x is the ideal $Ann_A(x) = \{a \in A/a \cdot x = 0_M\}$. The annihilator of M , is the ideal $Ann_A(M) = \{a \in A/\forall y \in M : a \cdot y = 0_M\}$.

Definition [5].

- Let x be an element of M . x is a torsion-free element if $Ann_A(x) = \{0_A\}$ and x is a torsion element if $Ann_A(x) \neq \{0_A\}$.
- We say that M is a torsion-free module if all its elements are torsion free.
- We say that M is a torsion module if all its elements are torsion.

Lemma 1. Let $M = Ax_1 \oplus Ax_2 \oplus \dots \oplus Ax_n$ be a direct finite sum of cyclic modules over A .

M is a torsion module if and only if $Ann_A(M) \neq \{0_A\}$.

Proof. The proof is easy. ■

Lemma 2. For an irreducible element p in A , the set $I_p = \{p^n/n \in \mathbb{N}\}$ is infinite.

Proof. Suppose that I_p is finite, then necessarily $\exists (n, m) \in \mathbb{N}^2$ such that $n < m$ and $p^n = p^m$. So $p^n(1-p^{m-n}) = 1_A \implies p^{m-n} = 1_A$ (integral domain). Therefore p is a unit in A , which is not true. ■

Notations. Let $E_p(M)$ denote the set of all A -automorphisms of M satisfying the extension property. For $i \in \{1, 2, \dots, n\}$, E_i will denote an injective envelope of Ax_i , $\mu_i : Ax_i \rightarrow E_i$ a monomorphism of A -modules and

$$M_i = E_i + \sum_{k=1, k \neq i}^{k=n} A \cdot x_k.$$

Lemma 3. If α satisfies the extension property then α^{-1} also satisfies the extension property.

Proof. Let $M \xrightarrow{\lambda} N$ a monomorphism of A -modules, then there exists $\widetilde{N} \xrightarrow{\widetilde{\alpha}} N$ an automorphism of A -modules such that $\lambda \circ \alpha = \widetilde{\alpha} \circ \lambda$. We define $\widetilde{\alpha}^{-1} = \widetilde{\alpha}^{-1}$, it's clear that $\widetilde{\alpha}^{-1} \in Aut(N)$. Let $x \in M$, so $\exists y \in M : x = \alpha(y)$ then $\lambda[\alpha^{-1}(x)] = \lambda(y)$. On the other hand, if we put $\widetilde{\alpha}^{-1}[\lambda(x)] = t$. We will have $\lambda(x) = \widetilde{\alpha}(t) \implies \widetilde{\alpha}(t) = \lambda[\alpha(y)]$, then $\widetilde{\alpha}(t) = \widetilde{\alpha}[\lambda(y)] \implies t = \lambda(y)$. From where, $\lambda \circ \alpha^{-1} = \alpha^{-1} \circ \lambda$ hence the lemma. ■

3. FINITE SUM OF TORSION FREE CYCLIC A -MODULES

In this section, we suppose that x_i is a torsion-free element for all i in $\{1, 2, \dots, n\}$. Let $\mu_i : Ax_i \rightarrow E_i$ be a monomorphism of A -modules.

Lemma 4. *For all $i \in \{1, 2, \dots, n\}$, we have*

$$M_i = E_i \bigoplus \bigoplus_{k=1, k \neq i}^{k=n} A \cdot x_k$$

and

$$E_i = \bigcap_{a \in A^*} a \cdot M_i.$$

Proof. We know that $E_1 \oplus E_2 \oplus \dots \oplus E_n$ is an injective envelope of $Ax_1 + Ax_2 + \dots + Ax_n$ (see [3]). Then

$$M_i = E_i + \sum_{k=1, k \neq i}^{k=n} A \cdot x_k = E_i \bigoplus \bigoplus_{k=1, k \neq i}^{k=n} Ax_k.$$

As E_i is an injective envelope of Ax_i then a divisible module. For all $a \in A^*$,

$$E_i = a \cdot E_i \subset a \cdot M_i \implies E_i \subset \bigcap_{a \in A^*} a \cdot M_i.$$

Reciprocally:

Let

$$t = e + \sum_{j=1, j \neq i}^{j=n} m_j \cdot x_j \in \bigcap_{a \in A^*} a \cdot M_i \subset \bigcap_{k \in \mathbb{N}} p^k \cdot M_i \text{ where } e \in E_i \text{ and } m_j \in A.$$

Then $(\forall k \in \mathbb{N}) (\exists e_k \in E_i) (\exists (m_{j,k})_{j=1, j \neq i, j=n} \in A^{n-1})$ such that

$$t = p^k \left(e_k + \sum_{j=1, j \neq i}^{j=n} m_{j,k} \cdot x_j \right).$$

Moreover, as M_i is a direct sum of A -modules, by identification, we have

$$e = p^k e_k \text{ and } \sum_{j=1, j \neq i}^{j=n} m_j \cdot x_j = p^k \cdot \left(\sum_{j=1, j \neq i}^{j=n} m_{j,k} \cdot x_j \right).$$

If $\sum_{j=1, j \neq i}^{j=n} m_j \cdot x_j \neq 0_M$ then there exists $j \neq i \in \{1, 2, \dots, n\}$ such that $m_j \neq 0_A$ and $m_j = p^k \cdot m_{j,k}$. So, $p^k \mid m_j$ for all integer k , which is false in a factorial integral domain by Lemma 2.

Hence $t = p^k e_k \in E_i$, that concludes the proof. \blacksquare

Under the previous notations, we have the following lemma.

Lemma 5. *If $\alpha \in E_p(M)$, then $\tilde{\alpha}(E_i) = E_i$.*

Proof. We have

$$E_i = \bigcap_{a \in A^*} aM_i \text{ then } \tilde{\alpha}(E_i) = \bigcap_{a \in A^*} a\tilde{\alpha}(M_i) \subset E_i.$$

Let now y in the divisible module E_i , then $\forall a \in A^*, \exists y_a \in M_i$ such that $y = a \cdot y_a$. As $\tilde{\alpha} \in \text{Aut}(M_i)$ then there exists $x_a \in M_i$: $y_a = \tilde{\alpha}(x_a)$, so

$$y = \tilde{\alpha}(a \cdot x_a) \implies y \in \tilde{\alpha}\left(\bigcap_{a \in A^*} a \cdot M_i\right) = \tilde{\alpha}(E_i)$$

Therefore $\tilde{\alpha}(E_i) = E_i$. ■

Lemma 6. *For all i and j in $\{1, 2, \dots, n\}$. If $i \neq j$, then*

$$M = A(x_i + x_j) \oplus \bigoplus_{k=1, k \neq i}^{k=n} A \cdot x_k.$$

Proof. Let i and j in $\{1, 2, \dots, n\}$ such that $i \neq j$. Consider $(a_1, a_2, \dots, a_n) \in A^n$ and suppose that $a_i(x_i + x_j) + \sum_{k=1, k \neq i}^{k=n} a_k x_k = 0_M$.

Then $a_i x_i + (a_i + a_j)x_j + \sum_{k=1, k \neq i, k \neq j}^{k=n} a_k x_k = 0_M$. As $\{x_1, x_2, \dots, x_n\}$ is A free, then $a_i = a_i + a_j = a_k = 0_A$ for all $k \in \{1, 2, \dots, n\} \setminus \{i, j\}$. So, $a_1 = a_2 = \dots = a_n = 0_A$. It's clear that $A(x_i + x_j) \oplus Ax_j \subset Ax_i \oplus Ax_j$. Let $x \in Ax_i \oplus Ax_j$ then $x = ax_i + bx_j$ where a and b are in A . So, $x = b(x_i + x_j) + (a - b)x_i \in A(x_i + x_j) \oplus Ax_i$. Then the lemma follows. ■

Theorem 7. *If $\alpha \in E_p(M)$ then $(\forall i \in \{1, 2, \dots, n\}) (\exists k_i \in A^*) (\alpha(x_i) = k_i \cdot x_i)$.*

Proof. Let $\lambda : M \rightarrow M_i$ defined by If $x = \sum_{j=1}^{j=n} m_j \cdot x_j$, where $m_j \in A$ for all $j \in \{1, 2, \dots, n\}$, $\lambda(x) = m_i \cdot \mu_i(x_i) + \sum_{j=1, j \neq i}^{j=n} m_j \cdot x_j$.

As defined λ is a morphism of A -modules and we have

$$\lambda(x) = m_i \cdot \mu_i(x_i) + \sum_{j=1, j \neq i}^{j=n} m_j \cdot x_j = 0_M \implies m_i \cdot \mu_i(x_i) = 0_M \text{ and } \sum_{j=1, j \neq i}^{j=n} m_j \cdot x_j = 0_M.$$

However μ_i is injective then $m_i = 0_A$. As $\{x_1, x_2, \dots, x_n\} \setminus \{x_i\}$ is A -free, then $m_j = 0_A$ for all $j \in \{1, 2, \dots, n\} \setminus \{i\}$. Then, $x = 0_M$. Therefore λ is a monomorphism of A -modules. We know that

$$\alpha(x_i) = k_i \cdot x_i + \sum_{j=1, j \neq i}^{j=n} k_j \cdot x_j \text{ where } k_1, k_2, \dots, k_{n-1} \text{ and } k_n \text{ are in } A.$$

As $\lambda\alpha = \tilde{\alpha}\lambda$ and by Lemma 5 we have

$$\lambda[\alpha(x_i)] = k_i \cdot \mu(x_i) + \sum_{j=1, j \neq i}^{j=n} k_j \cdot \lambda(x_j) = k_i \cdot \mu_i(x_i) + \sum_{j=1, j \neq i}^{j=n} k_j \cdot x_j = \tilde{\alpha}[\lambda(x_i)] \in E_i.$$

And since E_i is a direct factor in M_i , $\lambda(\alpha(x_i)) = k_i x_i \implies \sum_{j=1, j \neq i}^{j=n} k_j x_j = 0_M$. So $\alpha(x_i) = k_i \cdot x_i$. ■

Corollary 8. *We have $E_p(M) = \{k id_M / k \in A^\times\}$, where A^\times is the group of units in A .*

Proof. Applying the Theorem 7 for all $i \in \{1, 2, \dots, n\}$ there exists $k_i \in A^*$ such that $\alpha(x_i) = k_i \cdot x_i$. We must find $k_i = k_j$, for all i and j in $\{1, 2, \dots, n\}$ such that $i \neq j$. Applying now the Theorem 7, to M as asserted in Lemma 6, there exists $k_{i,j} \in A^*$ such that $\alpha(x_i + x_j) = k_{i,j} \cdot (x_i + x_j) = k_{i,j} \cdot x_i + k_{i,j} \cdot x_j = k_i \cdot x_i + k_j \cdot x_j \implies k_{i,j} = k_i = k_j$ since $\{x_i, x_j\}$ is A -free. Then, there is $k \in A^*$ such that for all $i \in \{1, 2, \dots, n\}$, $\alpha(x_i) = k \cdot x_i$. Therefore $\alpha = k \cdot id_M$.

We must prove that k is a unit in A . As α satisfies the extension property then α^{-1} also satisfies the extension property by Lemma 3. Then there exists $r \in A^*$ such that $\alpha^{-1} = r \cdot id_M$. As $\alpha \circ \alpha^{-1} = \alpha^{-1} \circ \alpha = k \cdot r \cdot id_M$ then $r \cdot k = 1_A$ which proved that k is a unit in A . ■

4. DIRECT FINITE SUM OF TORSION FREE AND TORSION CYCLICS A -MODULES

Let $n' \in \mathbb{N}^*$ such that $n' \leq n - 1$. In this section, we assume that $x_1, x_2, \dots, x_{n'}$ are the torsion-free co-generators and $x_{n'+1}, x_{n'+2}, \dots, x_n$ are the torsion co-generators.

For all $i \in \{1, 2, \dots, n'\}$, E_i will denote an injective envelope of $A \cdot x_i$ and $\mu_i : A \cdot x_i \rightarrow E_i$ a monomorphism of A -modules. Same as the Theorem 7 and corollary 8, we have the same result, but with a demonstration using the notion of a torsion element.

Lemma 9. *For all $i \in \{1, \dots, n'\}$, we have, $E_i = \bigcap_{a \in A^*} a \cdot M_i$.*

Proof. It's clear that E_1 is a divisible A -module, so $E_1 \subset a \cdot E_1 \subset a \cdot M_1$ for all a in $A^* \implies E_1 \subset \bigcap_{a \in A^*} a \cdot M_1$. Reciprocally, let $x \in M_1$ then $(\exists (f, c_2, c_3, \dots, c_n) \in E_1 \times (A^*)^{n-1}) (x = f + c_2 \cdot x_2 + c_3 \cdot x_3 + \dots + c_n \cdot x_n)$.

We know that $(\forall i \in \{n'+1, n'+2, \dots, n\}) (\exists b_i \in A^* : b_i \cdot x_i = 0_M)$. Let $b = b_{n'+1} b_{n'+2} \dots b_n \in A^*$. Therefore, for all $i \in \{n'+1, n'+2, \dots, n\}$, $b \cdot x_i = 0$. Suppose that for all $a \in A^*$, $x \in a \cdot M_1$, then for all m a nonzero natural integer $x \in b^m \cdot M_1$, consequently.

If $n' = 1$, then $x \in E_1$. If $n' \neq 1$, then

$$\begin{aligned} \exists (e_m, a_{2,m}, a_{3,m}, \dots, a_{n,m}) \in E_1 \times (A^*)^{n-1} : \\ x = b^m e_m + \sum_{j=2}^{j=n} a_{j,m} b^m x_j = f + \sum_{j=2}^{j=n'} c_j x_j. \end{aligned}$$

Let us suppose that $c_2 x_2 + c_3 x_3 + \dots + c_{n'} x_{n'} \neq 0$, then there exists $j \in \{2, 3, \dots, n'\}$ such that $c_j \neq 0_A$ and $c_j = a_{j,m} b^m$. So, there is an infinity divisor of c_j , which is false (because A is UFD). Therefore, $c_2 \cdot x_2 + c_3 \cdot x_3 + \dots + c_{n'} \cdot x_{n'} = 0 \implies x \in E_1$. For any $i \in \{2, 3, \dots, n'\}$ with the simple transposition $(i, 1)$, we find the first case. Then the lemma follows. ■

Lemma 10. *If $\alpha \in E_p(M)$, then $(\forall i \in \{1, 2, \dots, n'\}) (\exists k_i \in A^* : \alpha(x_i) = k_i x_i)$.*

Proof. Let $\lambda : M \rightarrow M_1$ be the morphism defined by $\lambda(a_1 x_1 + a_2 x_2 + \dots + a_n x_n) = a_1 \mu_1(x_1) + a_2 x_2 + \dots + a_n x_n$. It's clear that λ is a monomorphism of A modules.

As α satisfies the extension property, there exists $\tilde{\alpha} \in \text{Aut}(M_1)$ such that $\lambda \circ \alpha = \tilde{\alpha} \circ \lambda$. As $\alpha(x_1) \in M$ then $\alpha(x_1) = d_1 \cdot x_1 + d_2 \cdot x_2 + \dots + d_n \cdot x_n$, where $(d_1, d_2, \dots, d_n) \in A^n$. So $\lambda[\alpha(x_1)] = d_1 \cdot \mu_1(x_1) + \sum_{j=2}^{j=n} d_j \cdot x_j = \tilde{\alpha}[\lambda(x_1)]$. And as we know that $\tilde{\alpha}[\lambda(x_1)] \in E_1$, which is a direct factor in M_1 , then $\sum_{j=2}^{j=n} d_j \cdot x_j = 0_M \implies \alpha(x_1) = d_1 \cdot x_1$. Same proof for all i in $\{2, 3, \dots, n'\}$, hence $\alpha(x_i) = d_i \cdot x_i$ ($k_i = d_i$). ■

Lemma 11. *If $n' \neq 1$, let $i_0 \in \{2, 3, \dots, n'\}$. We have*

- $x_1 + x_{i_0}$ is a torsion-free element.
- $M = A(x_1 + x_{i_0}) \oplus \bigoplus_{i=1, i \neq i_0}^{i=n'} A x_i \oplus \bigoplus_{j=n'+1}^{i=n} A x_j$ (*).

Proof.

- We have $x_{i_0} = x_1 + x_{i_0} - x_1 \in A(x_1 + x_{i_0}) \oplus \bigoplus_{i=1, i \neq i_0}^{i=n'} A x_i \oplus \bigoplus_{j=n'+1}^{i=n} A x_j$. Let $a \in A$ such that $a(x_1 + x_{i_0}) = 0_M$ then $a x_1 + a x_{i_0} = 0_M \implies a = 0_A$. ($\{x_1, x_{i_0}\}$ is A free). Therefore, $\{x_1 + x_{i_0}\}$ is a A -free.
- Let $(a_1, a_2, \dots, a_n) \in A^n$ and suppose that

$$a_{i_0}(x_1 + x_{i_0}) + \sum_{i=1, i \neq i_0}^{i=n} a_i x_i = 0_M.$$

Then

$$(a_{i_0} + a_1)x_1 + \sum_{i=2}^{i=n} a_i x_i = 0_M \implies a_1 = a_2 = \dots = a_{n'} = 0_A$$

and $a_{n'+1}x_{n'+1} = a_{n'+2}x_{n'+2} = \cdots = a_n x_n = 0_M$ ($M = \bigoplus_{i=1}^{i=n} Ax_i$ and $\{x_1, \dots, x_{n'}\}$ is A free). So $M = A(x_1 + x_{i_0}) \oplus \bigoplus_{i=1, i \neq i_0}^{i=n'} Ax_i \oplus \bigoplus_{j=n'+1}^{j=n} Ax_j$ (*).

Then the lemma follows. \blacksquare

Corollary 12. For all $i \in \{1, 2, \dots, n'\}$, $k_i = k_1$.

Proof. By Lemma 10 we have for all i in $\{1, 2, \dots, n'\}$, there exists an $k_i \in A^*$ such that $\alpha(x_i) = k_i \cdot x_i$.

Let M be defined as in Lemma 11, then by Lemma 10, there exists an $r_{i_0} \in A^*$ such that $\alpha(x_1 + x_{i_0}) = r_{i_0}(x_1 + x_{i_0})$. So $r_{i_0} = k_{i_0} = k_1$, and this for all i_0 a fixed element in $\{2, \dots, n'\}$. Hence the corollary. \blacksquare

Lemma 13. For all $i \in \{1, 2, \dots, n'\}$, $x_i + x_{n'+1}$ is a torsion-free element and

$$M = \bigoplus_{i=1}^{i=n'} A(x_i + x_{n'+1}) \oplus \bigoplus_{j=n'+1}^{j=n} Ax_j.$$

Proof. We have $x_i = x_i + x_{n'+1} - x_{n'+1}$ for all $i \in \{1, 2, \dots, n'\}$, then $M = A(x_1 + x_{n'+1}) + A(x_2 + x_{n'+1}) + \cdots + A(x_{n'} + x_{n'+1}) + Ax_{n'+1} + Ax_{n'+2} + \cdots + Ax_n$. Let $(a_1, a_2, \dots, a_n) \in A^n$ such that:

$$x = \sum_{i=1}^{i=n'} a_i(x_i + x_{n'+1}) + \sum_{j=n'+1}^{j=n} a_j x_j = 0_M.$$

Then

$$\sum_{i=1}^{i=n'} a_i \cdot x_i + \left(\left(\sum_{i=1}^{i=n'} a_i \right) + a_{n'+1} \right) \cdot x_{n'+1} + \sum_{j=n'+2}^{j=n} a_j \cdot x_j = 0_M.$$

So,

$$\sum_{i=1}^{i=n'} a_i \cdot x_i = 0_M, \left(\left(\sum_{i=1}^{i=n'} a_i \right) + a_{n'+1} \right) \cdot x_{n'+1} = 0_M \text{ and } \sum_{j=n'+2}^{j=n} a_j \cdot x_j = 0_M.$$

And as $x_1, x_2, \dots, x_{n'}$ are a torsion-free elements, then $a_i = 0_A$ for all $i \in \{1, 2, \dots, n'\}$ and $a_j \cdot x_j = 0_M$ for all $j \in \{n'+1, n'+2, \dots, n\}$. Let $i \in \{1, 2, \dots, n'\}$ and suppose that $a_i \cdot (x_i + x_{n'+1}) = 0_M$. Then $a_i \cdot x_i + a_i \cdot x_{n'+1} = 0_M \implies a_i = 0_A \implies x_i + x_{n'+1}$ is a torsion-free element. \blacksquare

Theorem 14. We have $E_p(M) = \{kid_M / k \in A^\times\}$.

Proof. Let $b \in \text{Ann}_A(x_{n'+1}) \setminus \{0_A\}$, so $b \cdot x_{n'+1} = 0_M$. As $\alpha(x_{n'+1}) \in M$, then $\alpha(x_{n'+1}) = \sum_{j=1}^{j=n} a_j x_j$, where $(a_1, a_2, \dots, a_n) \in A^n$. Then

$$\begin{aligned} \alpha(bx_{n'+1}) &= ba_1x_1 + ba_2x_2 + \dots + ba_nx_n = 0_M \\ \implies ba_1x_1 &= ba_2x_2 = \dots = ba_{n'}x_{n'} = 0_M \implies a_1 = a_2 = \dots = a_{n'} = 0_A \\ \implies \alpha(x_{n'+1}) &= a_{n'+1}x_{n'+1} + a_{n'+2}x_{n'+2} + \dots + a_nx_n. \end{aligned}$$

By applying the result of Lemma 10 to $M = \bigoplus_{i=1}^{i=n'} A(x_i + x_{n'+1}) \oplus \bigoplus_{j=n'+1}^{j=n} Ax_j$ there exists $k_{n'+1} \in A^*$ such that $\alpha(x_i + x_{n'+1}) = k_{n'+1}x_i + k_{n'+1}x_{n'+1} = k_1x_i + \alpha(x_{n'+1})$. Then $(k_{n'+1} - k_1)x_i + k_{n'+1}x_{n'+1} = a_{n'+1}x_{n'+1} + a_{n'+2}x_{n'+2} + \dots + a_nx_n$.

Also, $M = \bigoplus_{k=1}^{k=n} Ax_k$ and x_i is a torsion free element, then $k_{n'+1} = k_1$ and $\sum_{j=n'+2}^{j=n} a_j x_j = 0_M \implies \alpha(x_{n'+1}) = k_1x_{n'+1}$. The same proof for $\alpha(x_{n'+j}) = k_1x_{n'+j}$ for $j \in \{2, 3, \dots, n - n'\}$, so $\alpha = k_1 \text{id}_M$. As we know α^{-1} satisfies the extension property, then there exists $r \in A^*$ such that $\alpha^{-1} = r \text{id}_M$. Consequently, $r \cdot k_1 = 1_A$. Which completes the proof of the theorem. ■

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