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STRONGLY REGULAR MODULES

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Abstract

The notion of strongly regular modules over a ring which is not necessarily commutative is introduced. The relation between F-regular, GF-regular and vn-regular modules that are defined over commutative rings and strongly regular module is obtained. We have shown that a remark that if R is a reduced ring, then the R-module M is F-regular if and only if M is GF-regular is false. We have obtained the necessary and sufficient condition under which the remark is true. We have shown that if R is a commutative ring and if M is finitely generated multiplication module then the notion of F-regular, GF-regular, vn-regular and strongly regular are equivalent.

Keywords: strong M-vn-regular element, strongly regular module, F-regular module, GF-reguar module, vn-regular module, weak commutative module.

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1. Introduction

In this paper we introduce the notion of strongly regular modules over rings which are not necessarily commutative. Following [4], a module M is a Fieldhouse regular module, called F-regular if each submodule of M is pure [5]. Majid Ali [10] have demonstrated about pure submodules. Anderson and Fuller [1], Fieldhouse [6] described the submodule K a pure submodule of M if $AK = K \cap AM$ for every ideal A of R. Ribenboim [12] described K to be pure in M if $aM \cap K = aK$ for each a in R. If M is a module over a commutative ring R, then the first condition implies the second and these descriptions are not equivalent in general [9, p.158], also in [7] they have followed the second definition. In this paper, we imitate the definition of purity as in Ribenboim [12]. Recall that an R-module M is called a multiplication module if for every submodule K of M there exists an ideal A of R such that $K = AM$. For an R-module M, the annihilator of $m \in M$ in R is $(0 : m) = \{a \in R : am = 0\}$ and thus $(0 : M)$ is the annihilator of M. A torsion free R-module M is expressed as, for any $r \in R$ and $m \in M$, if $rm = 0$, then either $r = 0$ or $m = 0$. A submodule K of M is called complimented submodule if there exists a submodule L of M such that $K + L = M$ and $K \cap L = 0$.

Following [2], a module M is called GF-regular(generalised F-regular) if for each $m \in M$ and $r \in R$, there exists $t \in R$ and a positive integer n such that $r^n tr^n m = r^n m$. Jayaram and Tekir [7] introduced Von Neumann regular module (vn-regular module for short). For a module M over a ring R, an element a of R is called M-vn-regular if $aM = a^2M$. An R-module M is said to be vn-regular module if for any m in M, $Rm = aM$ for some a in R. All these three regularities namely, F-regular, GF-regular, vn-regular modules are defined over commutative rings. In [14], we introduced the notion of VN-regular module M over a ring R which is not necessarily commutative. A module M over a ring R is communicated as a strongly regular module if given $a \in R$ and $m \in M$, there exists $x \in R$ such that $am = xa^2m$. This is infact a generalization of strongly regular rings to strongly regular modules. We know that a ring R is strongly regular if for every $r \in R$, there exists some $r' \in R$ such that $r = r'r^2$ and a ring R is strongly regular iff R is a reduced regular ring, [3, 8].

In this paper we find necessary and sufficient condition for a module M to be strongly regular. We have shown that if M is a module over a commutative ring R, then the notions of strongly regular module and F-regular module coincide. We have given an example of a F-regular module which is not strongly regular. We have obtained necessary and sufficient condition for a GF-regular module to be strongly regular. We have also shown that if M is a finitely generated multiplication module over a commutative ring then all the four notions of regularities namely, F-regular, GF-regular, strongly regular and vn-regular coincide.

Abduldaim [2] made a remark (Remark 5(1)) that if R is a reduced ring, then the R-module M is F-regular iff M is a GF-regular R-module. We show by an example that the remark is not true. We have given an example of a GF-regular module over a reduced ring R which is not F-regular. We have obtained condition under which the remark holds.

Throughout this paper, unless stated R stands for a ring with nonzero identity and all modules are nonzero unital left R-modules. If and only if is described as iff.

2. Characterizations of strongly regular modules

The upcoming section is a study about strongly regular modules. Initiated with the succeeding definition.

Definition 2.1. An element a of R is called strong M-vn-regular if for any given $m \in M$, there exists $x \in R$ such that $am = xa^2m$. An R-module M is called strongly regular module if every element of R is strong M-vn-regular.

We now give an example of strongly regular module.

Example 2.2. Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \Big/ a, b, c \in Z_2 \right\}$ be the ring with usual matrices addition and multiplication. Then the R-module $M = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$ is a strongly regular module.

The succeeding theorem offers a depiction of strongly regular modules in connection with F-regular modules. In advance we recite the definitions of F-regular module as in [5] and a pure submodule as in [12, 7]. Also an R-module M is professed to be an IFP-module if for any $r \in R$ and $m \in M$, if $rm = 0$ then $rRm = 0$ [13]. If M is a module over a commutative ring, then M is clearly an IFP-module.

Theorem 2.3. *Presuming R to be a commutative ring. Then an R-module M is strongly regular iff M is a F-regular R-module.*

Proof. Grant M to be a strongly regular module. Let K be a submodule of M and let $a \in R$. Clearly $aK \subseteq aM \cap K$. Let $y \in aM \cap K$. Then $y = k = am$ for some $k \in K$ and $m \in M$. As M is strongly regular, there exists $x \in R$ such that $am = xa^2m$. Then $y = xa^2m = xa(am) = axk \in aK$. Thus $aM \cap K \subseteq aK$. Hence M is F-regular.

Conversly, grant M to be a F-regular module. Let $a \in R$ and $m \in M$. Clearly $< a > m$, a submodule of M. Then $aM \cap < a > m = a(< a > m)$. Clearly $am \in aM \cap < a > m$. Then $am \in a < a > m$. This implies $am = a\big(\sum_i r_i a\big)m$ for some $r_i \in R$, where the sum is finite. Subsequently $am = \big(\sum_i r_i\big)a^2m = xa^2m$ for some $x = \sum_i r_i \in R$. Consequently M is a strongly regular module.

The following example will show that the above statement need not hold for a ring R, which is not necessarily commutative. ∎

Example 2.4. Let $R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Big/ a, b, c, d \in Z_2 \right\}$ be the ring with usual matrices addition and multiplication. In both sense the R-module R_R is a F-regular module as the only ideal J in R is either $\{0\}$ or R and hence if $J = \{0\}$, for any submodule K of M we have $\{0\} = JK = K \cap JM$. If $J = R$ then $K = JK = K \cap JM$.

But, the element $a = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ of R is not a strong M-vn-regular element, as given $m = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ there is no $x \in R$ such that $am = xa^2m$.

Proposition 2.5. *Suppose M is an IFP-module and a F-regular module. Then the Prime radical of $R/(0:m)$ is zero for each $0 \neq m \in M$.*

Proof. Let $0 \neq m \in M$. Let $\bar{a} = a + (0:m) \in R/(0:m)$. Suppose that $\bar{a}^2 = \bar{0}$. Since M is F-regular, we hope $< a > m \cap aM = a(< a > m)$. Clearly $am \in < a > m \cap aM$. Then $am \in a(< a > m)$. It follows that $am = a\big(\sum_{i,j} r_i a r_j\big)m$ for some $r_i, r_j \in R$, where the sum is finite. Since $\bar{a}^2 = \bar{0}$, we have $a^2m = 0$ and as M is an IFP-module, accordingly $am = 0$. Hence $\bar{a} = \bar{0}$.

Another portrayal of a strongly regular module in connection with GF-regular modules is given in the next result. Before that we recall the definition of GF-regular modules as in [2]. ∎

Theorem 2.6. *Presuming R to be a commutative ring and the Prime radical of $R/(0:m)$ is zero for each $0 \neq m \in M$, then an R-module M is a strongly regular module iff M is a GF-regular module.*

Proof. Grant M to be a strongly regular module. Followed by Theorem 2.3, M is a F-regular module. Since every F-regular module implies GF-regular. Accordingly M is a GF-regular module.

Conversly, grant M to be a GF-regular module. Let $a \in R$ and let K be any submodule of M. It is clear that $aK \subseteq aM \cap K$. Let $x \in aM \cap K$. Then $x = am$, where $m \in M$.

As M is GF-regular. For $a \in R$ and $m \in M$, there exist $t \in R$ and a positive integer n such that $a^n t a^n m = a^n m$. This implies that $a(a^{n-1}ta^n - a^{n-1}) \in (0:m)$. Since $(0:m)$ is an ideal, we have

$$\big(a^{n-1}ta^{n-1}\big)a\big(a^{n-1}ta^n - a^{n-1}\big) = a^{n-1}ta^n\big(a^{n-1}ta^n - a^{n-1}\big) \in (0:m).$$

Now,

$$a^{n-2}\big(a\big(a^{n-1}ta^n - a^{n-1}\big)\big) = a^{n-1}\big(a^{n-1}ta^n - a^{n-1}\big) \in (0:m).$$

Then, $(a^{n-1}ta^n - a^{n-1})^2 \in (0:m)$. It follows that $\overline{(a^{n-1}ta^n - a^{n-1})}^2 = \bar{0}$.

Then by assumption, we have $\overline{(a^{n-1}ta^n - a^{n-1})} = \bar{0}$. This implies that $(a^{n-1}ta^n - a^{n-1})m = 0$. Similarly proceeding, we have $(ata^n - a)m = 0$. Then $x = at'(am)$ where $t' = ta^{n-1} \in R$. Thus $x \in aK$. Hence $aM \cap K = aK$. Followed by Theorem 2.3, M is a strongly regular module.

Now, recall that an element $e' \in R$ is claimed to be weak idempotent if $e' - e'^2 \in (0 : M)$ [7] and $< a >$ denotes the principal ideal generated by $a \in R$. Also an R-module M is a colon distributive module if $(K_1 : M) + (K_2 : M) = (K_1 + K_2 : M)$ for all submodules K_1, K_2 of M [7].

Hence we have the next result on colon distributive module. However, first we require Lemma 2.7 which is shown in [14]. ■

Lemma 2.7. *Assuming M an R-module. Taking the ideals A_1, A_2 of R in such a way that $A_1 + A_2 = R$ and $A_1 A_2 \subseteq (0 : M)$. Then*

(i) $A_1 + (0 : M) =< e' > +(0 : M)$ *for some weak idempotent* $e' \in A_1$

(ii) $A_2 + (0 : M) =< 1 - e' > +(0 : M)$ *for some weak idempotent* $(1 - e') \in A_2$

(iii) $A_1 M =< e' > M$ *and* $A_2 M =< 1 - e' > M$ *for some weak idempotent elements* e' *and* $(1 - e')$ *such that* $e' \in A_1$ *and* $(1 - e') \in A_2$.

Proposition 2.8. *Assuming M a colon distributive module. If K is a complemented submodule of M, then $K =< e' > M$ for some weak idempotent element $e' \in R$.*

Proof. Suppose K has a complement. Surely there exist a submodule K' in M such that $K + K' = M$ and $K \cap K' = 0$. Now, $R = (M : M) = (K + K' : M) = (K : M) + (K' : M)$. Also $(K : M) \cap (K' : M) = (K \cap K' : M) = (0 : M)$, and hence $(K : M)(K' : M) \subseteq (0 : M)$. Followed by Lemma 2.7(iii), $(K : M)M =< e' > M$ for some weak idempotent element $e' \in R$. Again $K = K \cap M = K \cap ((K : M)M + K')$ as $(K' : M)M \subseteq K'$ and $(K : M)M + (K' : M)M = RM = M$. By the modular law, we have $K = (K : M)M + (K \cap K') = (K : M)M + 0 =< e' > M$ for some weak idempotent element $e' \in R$.

The succeeding Lemma gives a condition for an element of R to be M-vn-regular. In advance we recite the definitions of M-vn-regular element and vn-regular module as in[7]. ■

Lemma 2.9. *Suppose R is a commutative ring and an element $a \in R$ is strong M-vn-regular then $a \in R$ is a M-vn-regular element.*

Proof. Let $a \in R$ be a strong M-vn-regular element. Because of this, for any $m \in M$ there exist $x \in R$ in such a way that $am = xa^2m = a^2xm \in a^2M$. This implies that $aM \subseteq a^2M$. Then clearly $aM = a^2M$. That being the case.

Now we can compile the characterizations of strongly regular modules with those of F-regular modules, GF-regular modules, vn-regular modules in the following. ■

Theorem 2.10. *If R is a commutative ring and M is a finitely generated multiplication R-module. Then the axioms that follows are parallel to each other.*

(i) *M is a strongly regular module.*

(ii) *Every element of R is M-vn-regular.*

(iii) *M is a F-regular module.*

(iv) *M is a GF-regular module and the prime radical of $R/(0:m)$ is zero foreach $0 \neq m \in M$.*

Proof. (i)$\Longrightarrow$(ii). Emulates from Lemma 2.9.

(ii)$\Longrightarrow$(iii). Emulates from [7, Theorem 1].

(i)$\Longleftrightarrow$(iii). Emulates from Theorem 2.3

(i)$\Longrightarrow$(iv). Let (i) holds. Then clearly M, a GF-regular module. Now let $0 \neq m \in M$. Let $\bar{a} = a+(0:m) \in R/(0:m)$ such that $\bar{a}^2 = \bar{0}$. As M is F-regular, $aM \cap am = aM\cap < a > m = a(< a > m)$ is clear. Since $am \in aM\cap < a > m$, it follows that $am = a\big(\sum_i r_i a\big)m$ for some $r_i \in R$, where the sum is finite. Since $\bar{a}^2 = \bar{0}$, it follows that $a^2m = 0$ and then $am = 0$. Hence $\bar{a} = \bar{0}$.

(iv)$\Longrightarrow$(i). Emulates from Theorem 2.6. Hence concluded. ■

Theorem 2.11. *If R is a commutative ring and M is a finitely generated R-module. Then the axioms that follows are parallel to each other.*

(i) *M is a strongly regular module and a multiplication R-module.*

(ii) *M is a vn-regular module.*

Proof. (i)$\Longrightarrow$(ii) Theorem 2.10 induces every element of R is M-vn-regular. Then we conclude that M is a vn-regular module by [7, Theorem 2].

(ii)$\Longrightarrow$(i) [7, Theorem 2] induces M is a multiplication module and a F-regular module. Then (i) holds by Theorem 2.10.

Abduldaim [2] had a Remark 5 that if R is a reduced ring, then an R-module M is F-regular iff M is a GF-regular module. We show that the remark is not true. As Z, a reduced ring, the Z-module Z_4 is a GF-regular module, however Z_4 is not an F-regular Z-module. For the submodule $K = \{0, 2\}$ and for $2 \in Z$, $2Z_4 \cap K = \{0, 2\}$, where as $2K = \{0\}$ hence $2Z_4 \cap K \neq 2K$. Also $K\cap < 2 > Z_4 = \{0, 2\}$ and $< 2 > K = \{0\}$. Hence $K \cap IZ_4 \neq IK$. Thus K is not a pure submodule in the sense of [7, 12] and in the sense of [1, 6]. Now we find condition under which the remark is true. ■

Proposition 2.12. *Presuming R as a commutative and a reduced ring and M as a torsion free R-module. Then M is a F-regular module iff M is a GF-regular module.*

Proof. Consider M to be a F-regular module, then clearly M is a GF-regular module. Conversly, let M be a GF-regular module. Let $a \in R$ and $m \in M$. Then there exists t in R such that $a^n t a^n m = a^n m$ for some integer n. If $m = 0$, then $xa^2m = am$ for any $x \in R$. Suppose $m \neq 0$. Now $(a^n t a^n - a^n)m = 0$.

Since $m \neq 0$, we have $(a^n t a^n - a^n) = 0$. Then $(a^n t a^{n-1} - a^{n-1})a = 0$. Hence $(a^n t a^{n-1} - a^{n-1})a^{n-1} = 0$ and $(a^n t a^{n-1} - a^{n-1})a^{n-1} a t a^{n-1} = 0$. Thus $(a^n t a^{n-1} - a^{n-1})a^n t a^{n-1} = 0$. Hence $(a^n t a^{n-1} - a^{n-1})^2 = 0$. Since R is reduced, we have $(a^n t a^{n-1} - a^{n-1}) = 0$. Similarly proceeding $(a^n t a - a) = 0$. Thus $a(a^{n-1}t)a - a = 0$. Let $a^{n-1}t = x$. Hence $axa - a = 0$ and this implies that $axa = a$ and thus $a = xa^2$. Thus $am = xa^2 m$. Because of this $a \in R$ is a strong M-vn-regular element and we conclude M is a strongly regular module. Thereby M is a F-regular module by Theorem 2.3. ■

Proposition 2.13. *Presuming R as a reduced ring and M as a reduced R-module. Then an R-module M is F-regular iff M is a GF-regular module.*

Proof. Grant M as a F-regular module. Because of this, M is a GF-regular module. Conversely, assume M, a GF-regular module. Let K be any submodule of M and let $a \in R$. Clearly $aK \subseteq aM \cap K$. Let $y \in aM \cap K$. Then $y = am$ for some $m \in M$. Because of M, a GF-regular module, there exists $t \in R$ and a positive integer n such that $a^n t a^n m = a^n m$.

Then $0 = (a^n t a^n - a^n)m = a^2((a^{n-2} t a^n - a^{n-2})m)$. Because of M, a reduced module, we get $a((a^{n-2} t a^n - a^{n-2})m = 0$. Similarly proceeding we have $a(ta^n - 1)m = 0$. Thus $y = ata^n m = a(ta^{n-1})(am) \in aK$. Hence $aM \cap K = aK$. Hence the proof. ■

3. MAIN RESULTS ON STRONGLY REGULAR MODULES

Lemma 3.1. *Let M be an IFP-module. Then the axioms that follows are equivalent.*

(i) *M is a strongly regular module.*

(ii) *$R/(0 : m)$ is a strongly regular ring for each $0 \neq m \in M$.*

Proof. (i)$\Longrightarrow$(ii) Let $0 \neq m \in M$. Let $\bar{a} = a + (0 : m) \in R/(0 : m)$. Since a is a strong M-vn-regular element, there exists x in R such that $am = xa^2 m$. Then $a - xa^2 \in (0 : m)$. It follows that $\bar{a} = \bar{x}\bar{a}^2$. Hence (ii) holds.

(ii)$\Longrightarrow$(i) Let $a \in R$ and let $m \in M$. Then for $\bar{a} = a + (0 : m) \in R/(0 : m)$, there exist $\bar{x} \in R/(0 : m)$ in such a way that $\bar{a} = \bar{x}\bar{a}^2$. Then $a - xa^2 \in (0 : m)$. Thus $am = xa^2 m$ and therefore $a \in R$ is strong M-vn-regular. Hence (i) holds. ■

Lemma 3.2. *Let M be an R-module. If $R/(0 : M)$ is a strongly regular ring, then M is a strongly regular module.*

Proof. Let $R/(0 : M)$ be a strongly regular ring. Let $a \in R$. For $\bar{a} = a + (0 : M) \in R/(0 : M)$, there exists $\bar{x} \in R/(0 : M)$ such that $\bar{a} = \bar{x}\bar{a}^2$. It follows that $a - xa^2 \in (0 : M)$. This implies that $a - xa^2 = x'$ for some $x' \in (0 : M)$. Let m be an arbitrary element in M. Then $am = xa^2 m$. ■

The upcoming Theorem offers parallel condition for M to be strongly regular.

Theorem 3.3. *Take M, a finitely generated IFP-module. Here we get the following equivalent statements.*

(i) *$R/(0:M)$ is strongly regular.*

(ii) *M is a strongly regular module.*

Proof. (i)$\Longrightarrow$(ii). Emulates from Lemma 3.2.

(ii)$\Longrightarrow$(i). As M is finitely generated, let $\{m_1, m_2, \ldots, m_n\}$ be a finite set of generators of M. Then $(0:M) = \bigcap_i (0:m_i)$, $1 \le i \le n$.

Let $N' = \{a + (0:m_1), a + (0:m_2), \ldots, a + (0:m_n) : a \in R\}$. Clearly N' is a subring of the ring $\sum_{i=1}^n R/(0:m_i)$. Now we define a mapping ϕ: $R/(0:M) \to N'$ by $\phi(a + (0:M)) = (a + (0:m_1), a + (0:m_2), \ldots, a + (0:m_n))$ for each $a + (0:M) \in R/(0:M)$.

Clearly ϕ is an isomorphism. Now we claim that N' is strongly regular. By Lemma 3.1, $R/(0:m_i)$ is a strongly regular ring. Thus for each $a \in R$ and $1 \le i \le n$, there exist $x_i \in R$ such that $a + (0:m_i) = x_i a^2 + (0:m_i)$. Then $a - x_i a^2 \in (0:m_i)$. This implies $am_i = x_i a^2 m_i$ and hence $(1 - x_i a)am_i = 0$.

Define x by the relation $1 - xa = \prod_{i=1}^n (1 - x_i a)$. Then $(1 - xa)am_i = \big(\prod_{i=1}^n (1 - x_i a)\big)am_i$. Now for $i = 1$, we have $(1 - xa)am_1 = \big(\prod_{i=1}^n (1 - x_i a)\big)am_1$. Since $(1 - x_1 a)am_1 = 0$, we have $(1 - x_1 a)m' = 0$ for some $m' = am \in M$.

As M is an IFP-module, we have $(1 - x_1 a)Rm' = 0$. It follows that $(1 - x_1 a)[(1 - x_2 a)(1 - x_3 a) \cdots (1 - x_n a)]m' = 0$. Hence $(1 - x_1 a)(1 - x_2 a)(1 - x_3 a) \cdots (1 - x_n a)am_1 = 0$. Thus $(1 - xa)am_1 = 0$.

Similarly $(1 - xa)am_i = \big(\prod_{i=1}^n (1 - x_i a)\big)am_i = 0$ for each i. Thus for any $(a + (0:m_1), a + (0:m_2), \ldots, a + (0:m_n)) \in N'$ we have, $(a + (0:m_1), a + (0:m_2), \ldots, a + (0:m_n)) = (x + (0:m_1), x + (0:m_2), \ldots, x + (0:m_n))(a^2 + (0:m_1), a^2 + (0:m_2), \ldots, a^2 + (0:m_n))$ where $x \in R$ is defined by the relation $(1 - xa) = \prod_{i=1}^n (1 - x_i a)$. Hence N' is a strongly regular ring and hence $R/(0:M)$ is strongly regular. ■

Proposition 3.4. *Every homomorphic image of a strongly regular module is a strongly regular module.*

Proof. Suppose M_1 is a strongly regular module and ϕ: $M_1 \to M_2$ is an epimorphism. Let $a \in R$ and let $m_2 \in M_2$. Then $m_2 = \phi(m_1)$ for some $m_1 \in M_1$.

Thus clearly $am_1 = xa^2 m_1$ for some $x \in R$ since M_1 is a strongly regular module. Now $am_2 = a\phi(m_1) = \phi(am_1) = \phi(xa^2 m_1) = xa^2\phi(m_1) = xa^2 m_2$. Hence M_2 is strongly regular. ■

The succeeding corollary is an instant outcome of Proposition 3.4.

Corollary 3.5. *Suppose M is a strongly regular module and K is a submodule of M. Then M/K is a strongly regular module.*

Definition 3.6. An R-module M is defined to be a weak commutative module if for any a, $b \in R$, $m \in M$ there exists $b' \in R$ such that $abm = b'am$.

Proposition 3.7. *Take M, a finitely generated IFP R-module. Hereby we get the equivalent axioms.*

(i) *M is a strongly regular module.*

(ii) *For every left ideals L_1,L_2 and every submodule K of M, $(L_1 \cap L_2)K \subseteq L_1L_2K$ and M is weak commutative.*

(iii) *For every left ideal L, every ideal I and every submodule K of M, $(I \cap L)K = ILK$ and M is weak commutative.*

(iv) *For every ideals I_1,I_2 and every submodule K of M, $(I_1 \cap I_2)K = I_1I_2K$ and M is weak commutative.*

Proof. (i)$\Longrightarrow$(ii) Let L_1,L_2 be the left ideals of R and let K be a submodule of M. Now let $x \in (L_1 \cap L_2)K$. Then $x = \sum_i l_ik_i$ where the sum is finite and for some $l_i \in L_1 \cap L_2$ and $k_i \in K$. For any i, $l_ik_i = y_il_i^2k_i$ for some $y_i \in R$. Then $x = \sum_i y_il_i^2k_i = \sum_i(y_il_i)(l_i)k_i \in L_1L_2K$. Hence $(L_1 \cap L_2)K \subseteq L_1L_2K$.

Let a, $b \in R$ and $m \in M$. By Theorem 3.4, $R/(0 : M)$ is strongly regular. Then for $a \in R$ there exists $\bar{x} \in R/(0 : M)$ such that $\bar{a} = \bar{x}\bar{a}^2$. It follows that $\bar{a} = \bar{a}\bar{x}\bar{a}$. Since $\bar{x}\bar{a}$ is central, we have $\bar{a}\bar{b} = (\bar{a}\bar{x}\bar{a})\bar{b} = \bar{a}\bar{b}(\bar{x}\bar{a}) = \bar{b}'\bar{a}$ for some $\bar{b}' = \bar{a}\bar{b}\bar{x} \in R/(0 : M)$. Then $abm = b'am$ for all $m \in M$. Hence M is weak commutative.

(ii)$\Longrightarrow$(iii)$\Longrightarrow$(iv) Are all obvious.

(iv)$\Longrightarrow$(i) Let $a \in R$ and $m \in M$. Since $am \in (< a > \cap < a >)(Rm)$, we have $am \in < a >< a > (Rm)$ by our assumption. Then $am = r'a^2m$ for some $r' \in R$ since M is a weak commutative module. This completed the proof. ∎

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