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SOME REMARKS ON THE COMPLEMENT OF THE ARMENDARIZ GRAPH OF A COMMUTATIVE RING

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Abstract

Let R be a commutative ring with identity which is not an integral domain. Let $Z(R)$ denote the set of all zero-divisors of R. Recall from [1] that the Armendariz graph of R denoted by $A(R)$ is an undirected graph whose vertex set is $Z(R[X])\backslash\{0\}$ and distinct vertices $f(X) = \sum_{i=0}^{n} a_iX^i$ and $g(X) = \sum_{j=0}^{m} b_jX^j$ are adjacent in $A(R)$ if and only if $a_ib_j = 0$ for all $i \in \{0, \ldots, n\}$ and $j \in \{0, \ldots, m\}$. The aim of this article is to study the interplay between the graph-theoretic properties of the complement of $A(R)$, that is, $(A(R))^c$ and the ring-theoretic properties of R.

Keywords: B-prime of (0), complement of the zero-divisor graph, diameter, domination number, maximal N-prime of (0), radius.

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1. Introduction

The rings considered in this article are commutative with identity which are not integral domains. Let R be a ring. Let us denote the set of all non-zero zero-divisors of R, that is, $Z(R)\backslash\{0\}$ by $Z(R)^*$. The study of interplay between ring theory and graph theory began with the research work of Beck [8]. Recall from [2] that the *zero-divisor graph of R*, denoted by $\Gamma(R)$ is an undirected graph whose vertex set is $Z(R)^*$ and distinct vertices x, y are adjacent in $\Gamma(R)$ if and only if $xy = 0$. For an inspiring and excellent survey on the zero-divisor graphs of commutative rings, the reader is referred to [3].

This article is motivated by the interesting results proved on the Armendariz graph of a commutative ring in [1]. For a ring R, we denote the polynomial ring in one variable X over R by $R[X]$. Recall from [1] that the *Armendariz graph of a ring R*, denoted by $A(R)$ is an undirected graph whose vertex set is $Z(R[X])^*$ and distinct vertices $f(X) = \sum_{i=0}^{n} a_iX^i$ and $g(X) = \sum_{j=0}^{m} b_jX^j$ are adjacent in $A(R)$ if and only if $a_ib_j = 0$ for all $i \in \{0, \ldots, n\}$ and $j \in \{0, \ldots, m\}$. Recall from [17] that a ring R is said to be *Armendariz* if $f(X) = \sum_{i=0}^{n} a_iX^i, g(X) = \sum_{j=0}^{m} b_jX^j \in R[X]$ are such that $f(X)g(X) = 0$, then $a_ib_j = 0$ for all $i \in \{0, \ldots, n\}$ and $j \in \{0, \ldots, m\}$. It was already observed in [1, Example 1] that if R is an Armendariz ring, then $A(R) = \Gamma(R[X])$. A ring R is said to be *reduced* if R has no non-zero nilpotent element. It is clear that any reduced ring is Armendariz and so, for a reduced ring R, $A(R) = \Gamma(R[X])$. In Section 2 of [1], several Examples of $A(R)$ were given and in [1, Theorem 1] necessary and sufficient conditions were determined for $A(R)$ to be complete. It was proved in [1, Theorem 2] that there exists $f(X) \in Z(R[X])^*$ such that $f(X)$ is adjacent in $A(R)$ to every other vertex of $A(R)$ if and only if $Z(R)$ is an annihilator ideal of R.

The graphs considered in this article are undirected and simple. Let $G = (V, E)$ be a simple graph. As in [7], we denote the complement of G by G^c. Let R be a ring such that $Z(R)^* \neq \emptyset$. For a graph G, we denote the vertex set of G by $V(G)$ and the edge set of G by $E(G)$. Notice that $V((A(R))^c) = V((\Gamma(R[X]))^c) = Z(R[X])^*$. Let $f(X) = \sum_{i=0}^{n} a_iX^i$ and $g(X) = \sum_{j=0}^{m} b_jX^j \in Z(R[X])^*$ be distinct. Observe that if $f(X)$ and $g(X)$ are adjacent in $(\Gamma(R[X]))^c$, then $f(X)g(X) \neq 0$ and so, $a_ib_j \neq 0$ for some $i \in \{0, \ldots, n\}$ and $j \in \{0, \ldots, m\}$. Hence, $f(X)$ and $g(X)$ are adjacent in $(A(R))^c$. The above observations imply that $(\Gamma(R[X]))^c$ is a spanning subgraph of $(A(R))^c$. In [18, 19], the graph-theoretic properties of $(\Gamma(R))^c$ were studied.

We denote the set of all prime ideals of a ring R by $Spec(R)$ and the set of all maximal ideals of R by $Max(R)$. Let I be an ideal of R with $I \neq R$. Recall from [13] that $\mathfrak{p} \in Spec(R)$ is said to be a *maximal N-prime of I* if $\mathfrak{p}$ is maximal with respect to the property of being contained in $Z_R(\frac{R}{I}) = \{r \in R \mid rx \in I$ for some $x \in R \backslash I\}$. Hence, $\mathfrak{p} \in Spec(R)$ is a maximal N-prime of (0) if $\mathfrak{p}$ is maximal with respect to the property of being contained in $Z(R)$. Let $x \in Z(R)$. Then the multiplicatively closed subset $S = R\backslash Z(R)$ of R is such that $Rx \cap S = \emptyset$. Hence, we obtain from Zorn's lemma and [14, Theorem 1] that there exists a maximal N-prime $\mathfrak{p}$ of (0) in R such that $x \in \mathfrak{p}$. It now follows that if $\{\mathfrak{p}_\alpha\}_{\alpha \in \Lambda}$ is the set of all maximal N-primes of (0) in R, then $Z(R) = \bigcup_{\alpha \in \Lambda} \mathfrak{p}_\alpha$. It is now clear that R has a unique maximal N-prime of (0) if and only if $Z(R)$ is an ideal of R. Let I be an ideal of R with $I \neq R$. Recall from [12] that $\mathfrak{p} \in Spec(R)$ is said to be an *associated prime of I in the sense of Bourbaki* if $\mathfrak{p} = (I :_R x)$ for some $x \in R$. In such a case, we say that $\mathfrak{p}$ is a B-prime of I. For basic definitions and concepts from graph theory that are used in this article, one can refer any standard textbook in Graph Theory (for example, see [7, 9]).

This article consists of three sections including the introduction. In Section 2 of this paper, for a ring R with $|Z(R)^*| \geq 1$, we discuss some results on the connectedness of $(A(R))^c$. In Propositions 2.3 and 2.5, necessary and sufficient conditions are determined in order that $(A(R))^c$ to be connected. If $(A(R))^c$ is connected, then the diameter and the radius of $(A(R))^c$ are determined (see Propositions 2.3, 2.6, 2.7, and 2.8). Let R be a ring such that $(A(R))^c$ is connected. It is proved in Theorem 2.12 that for any finite non-empty subset S of $(Z(R[X]))^*$, $(\Gamma(R[X]))^c - S$ is connected and so, $(A(R))^c - S$ is connected and it is deduced in Corollary 2.13 that $(\Gamma(R[X]))^c$ (respectively, $(A(R))^c$) does not admit any cut vertex.

In Section 3 of this paper, some more properties of $(A(R))^c$ are proved. For a graph G, we denote the girth of G by $gr(G)$. We set $gr(G) = \infty$ if G does not contain any cycle. It is proved in Proposition 3.5 that $gr((\Gamma(R[X]))^c) = gr((A(R))^c) \in \{3, \infty\}$ and moreover, necessary and sufficient conditions are determined such that $(A(R))^c$ does not contain any cycle. Some results on the domination number of $(\Gamma(R[X]))^c$ (respectively, $(A(R))^c$) are also proved in Section 3 (see Proposition 3.9 and Theorem 3.10). We denote the clique number of a graph G by $\omega(G)$. Section 3 also contains some results on $\omega((A(R))^c)$ (see Corollary 3.17 and Proposition 3.18). In Corollary 3.19, it is proved that $(A(R))^c$ is planar if and only if $(A(R))^c$ has no edges.

For any $n \geq 2$, we denote the ring of integers modulo n by $\mathbb{Z}_n$. The cardinality of a set A is denoted by $|A|$. For sets A, B, if A is a proper subset of B, then we denote it by $A \subset B$. The group of units of a ring R is denoted by $U(R)$. We use the abbreviation f.g. for finitely generated.

2. On the connectedness of $(A(R))^c$

For a connected graph G, we denote the diameter of G by $diam(G)$ and the radius of G by $r(G)$. Let R be a ring with $|Z(R)^*| \geq 1$. In this section, we discuss some results on the connectedness of $(A(R))^c$ and we determine $diam((A(R))^c)$ and $r((A(R))^c)$ in the case when $(A(R))^c$ is connected.

Lemma 2.1. *Let R be a ring such that $|Z(R)^*| \geq 1$. Then $Z(R[X])^*$ is infinite.*

Proof. As $R[X]$ is infinite and is not an integral domain, it follows from [10, Theorem 1] that $Z(R[X])^*$ is infinite. ∎

Let R be a ring. In Proposition 2.3 with the assumption that $Z(R)$ is an ideal of R, we determine necessary and sufficient conditions in order that $(A(R))^c$ to be connected. We use Lemma 2.2 in the proof of the moreover part of Proposition 2.3.

Lemma 2.2. *Let $G = (V, E)$ be a simple graph with $|V| \geq 2$. If both G and G^c are connected, then $r(G^c) \geq 2$ and $r(G) \geq 2$.*

Proof. Notice that $V(G) = V(G^c) = V$. As G is connected and $|V| \geq 2$ by hypothesis, we obtain from [19, Lemma 2.1] that $e(a) \geq 2$ in G^c for each $a \in V$. Hence, $r(G^c) \geq 2$. As G is the complement of G^c and G^c is connected by hypothesis, it follows that $r(G) \geq 2$. ■

Proposition 2.3. *Let R be a ring such that $|Z(R)^*| \geq 1$. Let $\mathfrak{p}$ be the unique maximal N-prime of (0) in R. The following statements are equivalent:*

(1) *$(A(R))^c$ is connected.*

(2) *$\mathfrak{p}$ is not a B-prime of (0) in R.*

(3) *$(\Gamma(R[X]))^c$ is connected.*

Moreover, if the statement (1) holds, then

$$diam((A(R))^c) = diam((\Gamma(R[X]))^c) = r((A(R))^c) = r((\Gamma(R[X]))^c) = 2.$$

Proof. (1) $\Rightarrow$ (2) Suppose that $\mathfrak{p}$ is a B-prime of (0) in R. Then there exists $r \in R\backslash\{0\}$ such that $\mathfrak{p} = ((0) :_R r)$. It is clear that $r \in \mathfrak{p}$ and $\mathfrak{p}[X] = ((0) :_{R[X]} r)$. We know from the proof of [18, Proposition 2.2(ii)] that $Z(R[X]) = \mathfrak{p}[X]$. Let $g(X) = r$. Let $h(X) = \sum_{i=0}^{n} a_i X^i \in Z(R[X])^*$ with $h(X) \neq g(X)$. As $a_i \in \mathfrak{p}$ for each $i \in \{0, \ldots, n\}$, it follows that $a_i g(X) = a_i r = 0$ for each $i \in \{0, \ldots, n\}$. This shows that $g(X)$ is an isolated vertex of $(A(R))^c$. As $Z(R[X])^*$ is infinite and $(A(R))^c$ admits an isolated vertex, we obtain that $(A(R))^c$ is not connected. This is in contradiction to the assumption that $(A(R))^c$ is connected. Therefore, $\mathfrak{p}$ is not a B-prime of (0) in R.

(2) $\Rightarrow$ (3) Let $a \in Z(R)^*$. As $Z(R) = \mathfrak{p}$ is not a B-prime of (0) in R and $((0) :_R a) \subseteq Z(R)$, we get that $\mathfrak{p} \not\subseteq ((0) :_R a)$. Let $b \in \mathfrak{p}$ such that $ab \neq 0$. If $a = ab$, then from $a(1 - b) = 0$, it follows that $1 - b \in \mathfrak{p}$. In such a case, $1 = b + 1 - b \in \mathfrak{p}$. This is a contradiction. Therefore, $a \neq ab$ and so, $|Z(R)^*| \geq 2$. Since $\mathfrak{p}$ is not a B-prime of (0) in R, it follows from [18, Lemma 1.5] that $(\Gamma(R[X]))^c$ is connected and $diam((\Gamma(R[X]))^c) \leq 2$.

(3) $\Rightarrow$ (1) As $(\Gamma(R[X]))^c$ is a spanning subgraph of $(A(R))^c$ and $(\Gamma(R[X]))^c$ is connected by assumption, we get that $(A(R))^c$ is connected.

Assume that the statement (1) holds. We know from the proof of (2) $\Rightarrow$ (3) of this proposition that $diam((\Gamma(R[X]))^c) \leq 2$. Hence, $diam((A(R))^c) \leq 2$. It follows from [1, Theorem 4] (respectively, [2, Theorem 2.3]) and Lemma 2.2 that $r((A(R))^c) \geq 2$ (respectively, $r((\Gamma(R[X]))^c) \geq 2$). Therefore, we obtain that $diam((A(R))^c) = diam((\Gamma(R[X]))^c) = r((A(R))^c) = r((\Gamma(R[X]))^c) = 2$. ■

A ring R is said to be *quasi-local* if $|Max(R)| = 1$. A Noetherian quasi-local ring is referred to as a *local ring*. The Krull dimension of a ring R is simply referred to as the dimension of R and is denoted by $dimR$. Example 2.4 is provided to illustrate Proposition 2.3.

Example 2.4. Let $(V, \mathfrak{m})$ be a rank one valuation domain which is not discrete. Let $m \in \mathfrak{m}\backslash\{0\}$. Let $R = \frac{V}{Vm}$. Let $\mathfrak{p} = \frac{\mathfrak{m}}{Vm}$. Let $T = R(+)R$ be the ring obtained by using Nagata's principle of idealization. Then the following statements hold:

(1) $\mathfrak{p}(+)R$ is the unique maximal N-prime of the zero ideal in T but it is not a B-prime of the zero ideal in T.

(2) $(A(T))^c$ is connected and $diam((A(T))^c) = r((A(T))^c) = 2$.

(3) $(A(T))^c \neq (\Gamma(T[X]))^c$.

Proof. (1) We know from the proof of [18, Example 3.1(ii)] that $\mathfrak{p}$ is the unique maximal N-prime of the zero ideal in R and $\mathfrak{p}$ is not a B-prime of the zero ideal in R. As R is quasi-local with $\mathfrak{p}$ as its unique maximal ideal, it follows that $T = R(+)R$ is quasi-local with $\mathfrak{p}(+)R$ as its unique maximal ideal. Hence, $Z(T) \subseteq \mathfrak{p}(+)R$. Let $(r, s) \in \mathfrak{p}(+)R$. Notice that $r \in \mathfrak{p} = Z(R)$ and so, $(r, 0+Vm) \in Z(T)$. Now, $(r, s) = (r, 0 + Vm) + (0 + Vm, s)$ and $(0 + Vm, s)^2 = (0 + Vm, 0 + Vm)$. From [15, Lemma 2.3], we get that $(r, s) \in Z(T)$. Therefore, $\mathfrak{p}(+)R \subseteq Z(T)$ and so, $Z(T) = \mathfrak{p}(+)R$. This shows that $\mathfrak{p}(+)R$ is the unique maximal N-prime of the zero ideal in T. From $\mathfrak{p}$ is not a B-prime of zero ideal in R, it follows that $\mathfrak{p}(+)R$ is not a B-prime of the zero ideal in T.

(2) It follows from (1) of this example and (2) $\Rightarrow$ (1) of Proposition 2.3 that $(A(T))^c$ is connected and from the moreover part of Proposition 2.3, we get that $diam((A(T))^c) = r((A(T))^c) = 2$.

(3) As $Spec(V) = \{(0), \mathfrak{m}\}$, it follows from [5, Proposition 1.14] that for each $a \in \mathfrak{m}\backslash\{0\}$, $\sqrt{Va} = \mathfrak{m}$. Since $\mathfrak{m}$ is not principal, it follows that $\mathfrak{m} \neq Vm$. Let $a \in \mathfrak{m}\backslash Vm$. Since the set of ideals of V is linearly ordered by inclusion, we get that $m \in Va$. Therefore, $m = av$ for some $v \in \mathfrak{m}$. Notice that $\sqrt{Va} = \sqrt{Vm} = \mathfrak{m}$. Let $n \geq 2$ be least with the property that $a^n \in Vm$. Then $a^{n-1} \notin Vm$ but $(a^{n-1})^2 \in Vm$. Let $f(X), g(X) \in T[X]$ be given by $f(X) = (a^{n-1} + Vm, v + Vm) + (a^{n-1} + Vm, 1 + Vm)X$ and $g(X) = (a^{n-1} + Vm, 0 + Vm) + (a^{n-1} + Vm, -1 + Vm)X$. Since $a \notin U(V)$, it follows that $v \notin Vm$ and so, $f(X) \neq g(X)$. Using the facts that $(a^{n-1})^2 \in Vm$ and $a^{n-1}v \in Vm$, it can be verified that $f(X)g(X) = (0 + Vm, 0 + Vm)$. Hence, $f(X)$ and $g(X)$ are not adjacent in $(\Gamma(T[X]))^c$. It can be verified that the product of the constant term of $f(X)$ and

the coefficient of X in $g(X)$ equals $(0+Vm, -a^{n-1}+Vm) \neq (0+Vm, 0+Vm)$ and so, $f(X)$ and $g(X)$ are adjacent in $(A(T))^c$. Therefore, we obtain that $(A(T))^c \neq (\Gamma(T[X]))^c$. ■

Let R be a ring such that R has exactly two maximal N-primes of (0). In Proposition 2.5, we determine necessary and sufficient conditions for $(A(R))^c$ to be connected.

Proposition 2.5. *Let R be a ring such that $\{\mathfrak{p}_i \mid i \in \{1,2\}\}$ is the set of all maximal N-primes of (0) in R. The following statements are equivalent:*

(1) $(A(R))^c$ *is connected.*

(2) $\bigcap_{i=1}^{2} \mathfrak{p}_i \neq (0)$.

(3) $(\Gamma(R[X]))^c$ *is connected.*

Proof. (1) $\Rightarrow$ (2) Suppose that $\bigcap_{i=1}^{2} \mathfrak{p}_i = (0)$. Then R is reduced. Hence, $A(R) = \Gamma(R[X])$ and so, $(A(R))^c = (\Gamma(R[X]))^c$. From $(\Gamma(R[X]))^c$ is connected, we obtain from [18, Proposition 2.6(i)] that $(\Gamma(R))^c$ is connected. It now follows from [18, Proposition 1.7(i)] that $\bigcap_{i=1}^{2} \mathfrak{p}_i \neq (0)$. This is a contradiction and so, $\bigcap_{i=1}^{2} \mathfrak{p}_i \neq (0)$.

(2) $\Rightarrow$ (3) As $\bigcap_{i=1}^{2} \mathfrak{p}_i \neq (0)$ by assumption, it follows from [18, Proposition 1.7(i)] that $(\Gamma(R))^c$ is connected and we know from [18, Proposition 2.6(i)] that $(\Gamma(R[X]))^c$ is connected.

(3) $\Rightarrow$ (1) We are assuming that $(\Gamma(R[X]))^c$ is connected. As $(\Gamma(R[X]))^c$ is a spanning subgraph of $(A(R))^c$, we obtain that $(A(R))^c$ is connected. ■

Proposition 2.6. *Let $R, \mathfrak{p}_1, \mathfrak{p}_2$ be as in the statement of Proposition 2.5. If $(A(R))^c$ is connected, then the following statements hold:*

(1) $2 \leq diam((A(R))^c) \leq diam((\Gamma(R[X]))^c) \leq 3$. *If $diam((\Gamma(R[X]))^c) = 2$, then $diam((A(R))^c) = 2$.*

(2) $diam((A(R))^c) = 3$ *if and only if $\mathfrak{p}_i$ is a B-prime of (0) in R for each $i \in \{1,2\}$.*

Proof. We are assuming that $(A(R))^c$ is connected.

(1) From the proof of (2) $\Rightarrow$ (3) of Proposition 2.5, we get that $(\Gamma(R))^c$ is connected and $(\Gamma(R[X]))^c$ is connected. We know from [18, Proposition 1.7(ii)] that $2 \leq diam((\Gamma(R))^c) \leq 3$ and $diam((\Gamma(R))^c) = 3$ if and only if $\mathfrak{p}_i$ is a B-prime of (0) in R for each $i \in \{1,2\}$. Moreover, we obtain from [18, Proposition 2.6(ii)] that $diam((\Gamma(R[X]))^c) = diam((\Gamma(R))^c) \in \{2,3\}$. It follows from [1, Theorem 4] and Lemma 2.2 that $r((A(R))^c) \geq 2$ and so, $2 \leq diam((A(R))^c)$. As $(\Gamma(R[X]))^c$ is a spanning subgraph of $(A(R))^c$, it follows that $diam((A(R))^c) \leq diam((\Gamma(R[X]))^c)$. Therefore, we get that

$2 \leq diam((A(R))^c) \leq diam((\Gamma(R[X]))^c) \leq 3$. If $diam((\Gamma(R[X]))^c) = 2$, then it is clear that $diam((A(R))^c) = 2$.

(2) If $diam((A(R))^c) = 3$, then $diam((\Gamma(R[X]))^c) = 3$. Hence, it follows from the proof of (1) that $\mathfrak{p}_i$ is a B-prime of (0) in R for each $i \in \{1, 2\}$. Conversely, assume that $\mathfrak{p}_i$ is a B-prime of (0) in R for each $i \in \{1, 2\}$. Let $u, v \in R \backslash \{0\}$ be such that $\mathfrak{p}_1 = ((0) :_R u)$ and $\mathfrak{p}_2 = ((0) :_R v)$. It is clear that $\mathfrak{p}_1[X] = ((0) :_{R[X]} u)$ and $\mathfrak{p}_2[X] = ((0) :_{R[X]} v)$. We know from the proof of [18, Proposition 2.6 $(ii)(b)$] that $Z(R[X]) = \bigcup_{i=1}^{2} \mathfrak{p}_i[X]$. We claim that $d(u, v) \geq 3$ in $(A(R))^c$. From [8, Lemma 3.6], we get that $uv = 0$. Hence, u and v are not adjacent in $(A(R))^c$. Let $h(X) \in Z(R[X])^* \backslash \{u, v\}$. Either $h(X) \in \mathfrak{p}_1[X]$ or $h(X) \in \mathfrak{p}_2[X]$. If $h(X) \in \mathfrak{p}_1[X]$, then $h(X)u = 0$ and so, u and $h(X)$ are not adjacent in $(A(R))^c$. If $h(X) \in \mathfrak{p}_2[X]$, then $h(X)v = 0$ and so, $h(X)$ and v are not adjacent in $(A(R))^c$. This shows that there exists no path of length two between u and v in $(A(R))^c$. Therefore, $d(u, v) \geq 3$ in $(A(R))^c$ and hence, $diam((A(R))^c) \geq 3$. From $diam((A(R))^c) \leq 3$, we obtain that $diam((A(R))^c) = 3$. ■

Proposition 2.7. *Let R be a ring such that R admits at least three maximal N-primes of* (0). *Then both $(\Gamma(R[X]))^c$ and $(A(R))^c$ are connected and $diam((A(R))^c) = diam((\Gamma(R[X]))^c) = 2$.*

Proof. By hypothesis, R has at least three maximal N-primes of (0). It follows from [18, Proposition 2.8] that $(\Gamma(R[X]))^c$ is connected with $diam((\Gamma(R[X]))^c) = 2$. Since $(\Gamma(R[X]))^c$ is a spanning subgraph of $(A(R))^c$, we obtain that $(A(R))^c$ is connected and $diam((A(R))^c) \leq 2$. It follows from [1, Theorem 4] and Lemma 2.2 that $r((A(R))^c) \geq 2$ and so, $2 \leq diam((A(R))^c)$. Therefore, both $(\Gamma(R[X]))^c$ and $(A(R))^c$ are connected with $diam((A(R))^c) = diam((\Gamma(R[X]))^c) = 2$. ■

Proposition 2.8. *Let R be a ring such that R has at least two maximal N-primes of* (0). *If $(A(R))^c$ is connected, then $r((A(R))^c) = r((\Gamma(R[X]))^c) = 2$.*

Proof. Suppose that $(A(R))^c$ is connected. It is already noted in the proof of Proposition 2.6(1) and Proposition 2.7 that $(\Gamma(R[X]))^c$ is connected and $r((A(R))^c) \geq 2$. We know from [19, Theorem 2.5] that $r((\Gamma(R[X]))^c) = 2$. Since $(\Gamma(R[X]))^c$ is a spanning subgraph of $(A(R))^c$, it follows that $r((A(R))^c) \leq 2$ and so, $r((A(R))^c) = r((\Gamma(R[X]))^c) = 2$. ■

We provide Examples 2.9, 2.10, and 2.11 to illustrate Propositions 2.5, 2.6, 2.7, and 2.8.

Example 2.9. Let T be as in Example 2.4 and let $S = T \times \mathbb{Z}_8$ be the direct product of rings T and $\mathbb{Z}_8$. Then the following statements hold:

(1) S has exactly two maximal N-primes of its zero ideal.

(2) $(A(S))^c$ is connected with $diam((A(S))^c) = r((A(S))^c) = 2$.

(3) $(A(S))^c \neq (\Gamma(S[X]))^c$.

Proof. In the notation of Example 2.4, $T = R(+)R$ is quasi-local with $Z(T) = \mathfrak{p}(+)R$ as its unique maximal ideal.

(1) Notice that $Z(S) = (Z(T)\times\mathbb{Z}_8)\cup(T\times Z(\mathbb{Z}_8)) = ((\mathfrak{p}(+)R)\times\mathbb{Z}_8)\cup(T\times 2\mathbb{Z}_8)$. Let $\mathfrak{p}_1 = (\mathfrak{p}(+)R)\times\mathbb{Z}_8$ and let $\mathfrak{p}_2 = T\times 2\mathbb{Z}_8$. Observe that $\mathfrak{p}_i \in Max(S)$ for each $i \in \{1,2\}$, $\mathfrak{p}_1 \neq \mathfrak{p}_2$, and $Z(S) = \bigcup_{i=1}^{2}\mathfrak{p}_i$. Therefore, we get that $\{\mathfrak{p}_i \mid i \in \{1,2\}\}$ is the set of all maximal N-primes of the zero ideal in S.

(2) As $\bigcap_{i=1}^{2}\mathfrak{p}_i = (\mathfrak{p}(+)R)\times 2\mathbb{Z}_8$ is not the zero ideal of S, we obtain from (2) $\Rightarrow$ (1) of Proposition 2.5 that $(A(S))^c$ is connected. It is already observed in the proof of Example 2.4 that $\mathfrak{p}(+)R$ is not a B-prime of the zero ideal in T and hence, we obtain that $\mathfrak{p}_1$ is not a B-prime of the zero ideal in S. Therefore, it follows from Proposition 2.6(1) and (2) that $diam((A(S))^c) = 2$ and from Proposition 2.8, we obtain that $r((A(S))^c) = 2$.

(3) In the notation of Example 2.4, recall that $f(X), g(X) \in T[X]$ are such that $f(X) = (a^{n-1}+Vm, v+Vm)+(a^{n-1}+Vm, 1+Vm)X$ and $g(X) = (a^{n-1}+Vm, 0+Vm)+(a^{n-1}+Vm, -1+Vm)X$. Let $f_1(X), g_1(X) \in S[X]$ be given by $f_1(X) = ((a^{n-1}+Vm, v+Vm), 0)+((a^{n-1}+Vm, 1+Vm), 0)X$ and $g_1(X) = ((a^{n-1}+Vm, 0+Vm), 0)+((a^{n-1}+Vm, -1+Vm), 0)X$. From the choice of a and v, it follows as in the proof of Example 2.4 that $f_1(X) \neq g_1(X)$, $f_1(X)g_1(X)$ is the zero polynomial, and the product of the constant term of $f_1(X)$ and the coefficient of X in $g_1(X)$ is not the zero element of S. Therefore, $f_1(X)$ and $g_1(X)$ are not adjacent in $(\Gamma(S[X]))^c$ but they are adjacent in $(A(S))^c$. Hence, $(A(S))^c \neq (\Gamma(S[X]))^c$. ■

Example 2.10. Let $R = \mathbb{Z}_8(+)\mathbb{Z}_8$ be the ring obtained by using Nagata's principle of idealization. Let $T = R\times R$ be the direct product of rings R and R. Then the following statements hold.

(1) T has exactly two maximal N-primes of its zero ideal and both are B-primes of the zero ideal in T.
(2) $(A(T))^c$ is connected with $diam((A(T))^c) = 3$ and $r((A(T))^c) = 2$.
(3) $(A(T))^c \neq (\Gamma(T[X]))^c$.

Proof. Notice that $R = \mathbb{Z}_8(+)\mathbb{Z}_8$ is local with $\mathfrak{p} = 2\mathbb{Z}_8(+)\mathbb{Z}_8$ as its unique maximal ideal. Observe that $Z(R) = \mathfrak{p} = ((0,0) :_R (0,4))$ is a B-prime of the zero ideal in R.

(1) As $T = R\times R$, we get that $Z(T) = (Z(R)\times R)\cup(R\times Z(R)) = (\mathfrak{p}\times R)\cup(R\times\mathfrak{p})$. Let $\mathfrak{p}_1 = \mathfrak{p}\times R$ and let $\mathfrak{p}_2 = R\times\mathfrak{p}$. Notice that $\mathfrak{p}_i \in Max(T)$ for each $i \in \{1,2\}$, $\mathfrak{p}_1 \neq \mathfrak{p}_2$, and $Z(T) = \bigcup_{i=1}^{2}\mathfrak{p}_i$. Hence, it follows that $\{\mathfrak{p}_i \mid i \in \{1,2\}\}$ is the set of all maximal N-primes of the zero ideal in T. Let $u = ((0,4),(0,0))$ and let $v = ((0,0),(0,4))$. It is clear that $\mathfrak{p}_1 = ((0_R, 0_R) :_T u)$ and $\mathfrak{p}_2 = ((0_R, 0_R) :_T v)$,

where $0_R = (0, 0)$ is the zero element of R. Therefore, $\mathfrak{p}_i$ is a B-prime of the zero ideal in T for each $i \in \{1, 2\}$.

(2) As $\bigcap_{i=1}^{2} \mathfrak{p}_i = \mathfrak{p} \times \mathfrak{p}$ is not the zero ideal of T, we obtain from (2) $\Rightarrow$ (1) of Proposition 2.5 that $(A(T))^c$ is connected. Since $\mathfrak{p}_i$ is a B-prime of the zero ideal in T for each $i \in \{1, 2\}$, it follows from Proposition 2.6(2) that $diam((A(T))^c) = 3$ and from Proposition 2.8, we obtain that $r((A(T))^c) = 2$.

(3) Let $f(X), g(X) \in Z(R[X])^*$ be given by $f(X) = (4, 2) + (4, 1)X$ and $g(X) = (4, 0) + (4, 1)X$. It was already noted in the proof of [1, Example 2] that $f(X)g(X)$ is the zero polynomial but $f(X)$ and $g(X)$ are not adjacent in $A(R)$. Hence, $f(X)$ and $g(X)$ are not adjacent in $(\Gamma(R[X]))^c$ but they are adjacent in $(A(R))^c$. Let $f_1(X) = ((4, 2), (0, 0)) + ((4, 1), (0, 0))X$ and let $g_1(X) = ((4, 0), (0, 0)) + ((4, 1), (0, 0))X$. It can be shown as in the proof of Example 2.9(3) that $f_1(X)$ and $g_1(X)$ are not adjacent in $(\Gamma(T[X]))^c$ but they are adjacent in $(A(T))^c$. Therefore, $(A(T))^c \neq (\Gamma(T[X]))^c$. ■

Example 2.11. Let T be as in Example 2.4 and let $S = T \times \mathbb{Z}_8 \times \mathbb{Z}_8$ be the direct product of rings $T, \mathbb{Z}_8$, and $\mathbb{Z}_8$. Then the following statements hold:

(1) S has exactly three maximal N-primes of its zero ideal.

(2) $(A(S))^c$ is connected with $diam((A(S))^c) = r((A(S))^c) = 2$.

(3) $(A(S))^c \neq (\Gamma(S[X]))^c$.

Proof. In the notation of Example 2.4, T is quasi-local with $Z(T) = \mathfrak{p}(+)R$ as its unique maximal ideal.

(1) It follows as in the proof of (1) of Example 2.10 that $Z(S) = (Z(T) \times \mathbb{Z}_8 \times \mathbb{Z}_8) \cup (T \times 2\mathbb{Z}_8 \times \mathbb{Z}_8) \cup (T \times \mathbb{Z}_8 \times 2\mathbb{Z}_8)$. Let $\mathfrak{p}_1 = (\mathfrak{p}(+)R) \times \mathbb{Z}_8 \times \mathbb{Z}_8$, $\mathfrak{p}_2 = T \times 2\mathbb{Z}_8 \times \mathbb{Z}_8$, and $\mathfrak{p}_3 = T \times \mathbb{Z}_8 \times 2\mathbb{Z}_8$. It is clear that $\mathfrak{p}_i \in Max(S)$ for each $i \in \{1, 2, 3\}$, $\mathfrak{p}_i \neq \mathfrak{p}_j$ for all distinct $i, j \in \{1, 2, 3\}$, and $Z(S) = \bigcup_{i=1}^{3} \mathfrak{p}_i$. Hence, it follows that $\{\mathfrak{p}_i \mid i \in \{1, 2, 3\}\}$ is the set of all maximal N-primes of the zero ideal in S.

(2) As S has more than two maximal N-primes of its zero ideal, it follows from Proposition 2.7 that $(A(S))^c$ is connected and $diam((A(S))^c) = 2$ and we know from Proposition 2.8 that $r((A(S))^c) = 2$.

(3) Using the fact that $(A(T))^c \neq (\Gamma(T[X]))^c$ (see Example 2.4(3)), it can be shown as in the proof of Example 2.9(3) that $(A(S))^c \neq (\Gamma(S[X]))^c$. ■

In [20, Theorem 5.1], rings R with $|Z(R)^*| \geq 1$ and $(\Gamma(R))^c$ is connected were characterized in order that $(\Gamma(R))^c$ to admit a cut vertex. If $(A(R))^c$ is connected, then we prove in Theorem 2.12 that $(A(R))^c$ does not admit any finite vertex cut.

Theorem 2.12. *Let R be a ring such that $(A(R))^c$ is connected. Let S be any finite non-empty subset of $Z(R[X])^*$. Then $(\Gamma(R[X]))^c - S$ is connected and so, $(A(R))^c - S$ is connected.*

Proof. We are assuming that $(A(R))^c$ is connected. Hence, it follows from (1) $\Rightarrow$ (3) of Proposition 2.3 (respectively, Proposition 2.5) and Proposition 2.7 that $(\Gamma(R[X]))^c$ is connected. Moreover, we obtain from [18, Propositions 2.2, 2.6, and 2.8] that $diam((\Gamma(R[X]))^c) \in \{2, 3\}$. Let S be a finite non-empty subset of $Z(R[X])^*$. Let $f(X), g(X) \in Z(R[X])^* \backslash S$ be such that $f(X) \neq g(X)$. Let $S = \{f_i(X) \mid i \in \{1, \ldots, k\}\}$. Let $deg(f_i(X)) = n_i$ for each $i \in \{1, \ldots, k\}$. If $f(X)g(X) \neq 0$, then $f(X) - g(X)$ is a path in $(\Gamma(R[X]))^c - S$. Suppose that $f(X)g(X) = 0$. Notice that $d(f(X), g(X)) = 2$ or 3 in $(\Gamma(R[X]))^c$. Let $f(X) - h_1(X) - \cdots - h_m(X) - g(X)$ be a path of shortest length between $f(X)$ and $g(X)$ in $(\Gamma(R[X]))^c$. It is clear that $m \in \{1, 2\}$. Let $n \in \mathbb{N}$ be such that $n > n_i$ for each $i \in \{1, \ldots, k\}$. Let $i \in \{1, \ldots, m\}$. Observe that $X^n h_i(X) \notin S$ and from the fact that $X^n \notin Z(R[X])^*$, it follows that $X^n h_i(X) \in Z(R[X])^*$. It is clear that $X^n h_i(X) \neq X^n h_j(X)$ for all distinct $i, j \in \{1, \ldots, m\}$ and $f(X) - X^n h_1(X) - \cdots - X^n h_m(X) - g(X)$ is a path in $(\Gamma(R[X]))^c - S$.

From the above discussion, it is clear that $(\Gamma(R[X]))^c - S$ is connected. Since $(\Gamma(R[X]))^c - S$ is a spanning subgraph of $(A(R))^c - S$, we obtain that $(A(R))^c - S$ is connected. ∎

Corollary 2.13. *Let R be a ring such that $|Z(R)^*| \geq 1$ and $(A(R))^c$ is connected. Then $(\Gamma(R[X]))^c$ and $(A(R))^c$ do not admit any cut vertex.*

Proof. Let $f(X) \in Z(R[X])^*$. We know from Theorem 2.12 that both $(\Gamma(R[X]))^c - f(X)$ and $(A(R))^c - f(X)$ are connected. This proves that both the graphs $(\Gamma(R[X]))^c$ and $(A(R))^c$ do not admit any cut vertex. ∎

3. Some more results on $(A(R))^c$

Let R be a ring such that $|Z(R)^*| \geq 1$. The aim of this section is to discuss some more properties of $(A(R))^c$. First, we prove some results on $gr((A(R))^c)$.

Lemma 3.1. *Let R be a ring such that $|Z(R)^*| \geq 1$. The following statements are equivalent:*

(1) *$(A(R))^c$ has no edges.*

(2) *$(\Gamma(R[X]))^c$ has no edges.*

(3) *$Z(R)$ is an ideal of R with $Z(R)^2 = (0)$.*

Proof. (1) $\Rightarrow$ (2) This is clear, since $(\Gamma(R[X]))^c$ is a spanning subgraph of $(A(R))^c$.

(2) $\Rightarrow$ (3) Suppose that $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ as rings. Let us denote $\mathbb{Z}_2 \times \mathbb{Z}_2$ by T. It was already noted in [6, page 2045] that $f(X) = (1, 0) + (1, 0)X, g(X) = (1, 0) + (1, 0)X^2 \in Z(T[X])^*$ are such that $f(X) - g(X)$ is an edge of $(\Gamma(T[X]))^c$. So, we

get that $(\Gamma(R[X]))^c$ has at least one edge. Therefore, if (2) holds, then $R \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$ as rings. As (2) holds, it follows that $(\Gamma(R))^c$ has no edges (equivalently, $\Gamma(R)$ is complete). In such a case, we obtain from [2, Theorem 2.8] that $Z(R)$ is an ideal of R with $Z(R)^2 = (0)$.

(3) $\Rightarrow$ (1) Let $f(X), g(X) \in Z(R[X])^*$ be distinct. Let $f(X) = \sum_{i=0}^{n} a_i X^i$ and let $g(X) = \sum_{j=0}^{m} b_j X^j$. It follows from McCoy's Theorem [16, Theorem 2] that $a_i, b_j \in Z(R)$ for all $i \in \{0, \ldots, n\}$ and $j \in \{0, \ldots, m\}$. By (3), $Z(R)$ is an ideal of R with $Z(R)^2 = (0)$ and so, $a_i b_j = 0$ for all $i \in \{0, \ldots, n\}$ and $j \in \{0, \ldots, m\}$. Hence, $f(X)$ and $g(X)$ are not adjacent in $(A(R))^c$. Therefore, $(A(R))^c$ has no edges. ■

Proposition 3.2. *Let R be a ring and let $f(X), g(X) \in Z(R[X])^*$ be such that $f(X) - g(X)$ is an edge of $(A(R))^c$. Then there exists $h(X) \in Z(R[X])^*$ such that $f(X) - h(X) - g(X) - f(X)$ is a cycle of length three in $(A(R))^c$ with $f(X)h(X) \neq 0$ and $h(X)g(X) \neq 0$.*

Proof. Let $f(X), g(X) \in Z(R[X])^*$ be such that $f(X) - g(X)$ is an edge of $(A(R))^c$. Let $deg(f(X)) = m$ and let $deg(g(X)) = k$. Let $f(X) = \sum_{i=0}^{m} a_i X^i$ and let $g(X) = \sum_{j=0}^{k} b_j X^j$. It follows from $f(X) - g(X)$ is an edge of $(A(R))^c$ that $a_s b_t \neq 0$ for some $s \in \{0, \ldots, m\}$ and $t \in \{0, \ldots, k\}$. Let $n \in \mathbb{N}$ be such that $n > max(m, k)$. We consider the following cases.

Case (1). $a_s^2 = 0 = b_t^2$. Notice that $a_s + b_t$ is nilpotent and from $a_s^2 = 0$ and $a_s b_t \neq 0$, it follows that $a_s + b_t \neq 0$. It is clear that $f(X)(a_s + b_t) \neq 0$ and $g(X)(a_s + b_t) \neq 0$. Let $h(X) = (a_s + b_t)X^n$. As $X^n \notin Z(R[X])$ and $a_s + b_t \in Z(R)^*$, it follows that $X^n(a_s + b_t) \in Z(R[X])^*$, $f(X)h(X) \neq 0, h(X)g(X) \neq 0$, and by the choice of n, we get that $h(X) \notin \{f(X), g(X)\}$. Therefore, $h(X)$ is adjacent to both $f(X)$ and $g(X)$ in $(\Gamma(R[X]))^c$ and so, in $(A(R))^c$. Hence, we obtain that $f(X) - h(X) - g(X) - f(X)$ is a cycle of length three in $(A(R))^c$ with $f(X)h(X) \neq 0$ and $h(X)g(X) \neq 0$.

Case (2). At least one between a_s^2 and b_t^2 is not equal to 0. Without loss of generality, we can assume that $a_s^2 \neq 0$. Let $h(X) = a_s X^n$. It follows from [16, Theorem 2] that $a_s \in Z(R)^*$. From $X^n \notin Z(R[X])$, it follows that $h(X) \in Z(R[X])^*$. As $a_s^2 \neq 0$, $a_s b_t \neq 0$, we obtain that $f(X)h(X) \neq 0$ and $h(X)g(X) \neq 0$. By the choice of n, it is clear that $h(X) \notin \{f(X), g(X)\}$. Thus $f(X) - h(X) - g(X) - f(X)$ is a cycle of length three in $(A(R))^c$ with $f(X)h(X) \neq 0$ and $h(X)g(X) \neq 0$.

This completes the proof. ■

Corollary 3.3. *Let R be a ring such that $(\Gamma(R[X]))^c$ admits at least one edge. Then any edge of $(\Gamma(R[X]))^c$ is an edge of a triangle in $(\Gamma(R[X]))^c$.*

Proof. Let $f(X), g(X) \in Z(R[X])^*$ be such that $f(X) - g(X)$ is an edge of $(\Gamma(R[X]))^c$. Then $f(X) - g(X)$ is also an edge of $(A(R))^c$. Hence, we obtain from Proposition 3.2 that there exists $h(X) \in Z(R[X])^*$ such that $f(X) - h(X) - g(X) - f(X)$ is a cycle of length three in $(\Gamma(R[X]))^c$. This proves that any edge of $(\Gamma(R[X]))^c$ is an edge of a triangle in $(\Gamma(R[X]))^c$. ∎

Corollary 3.4. *Let R be a ring such that $(\Gamma(R[X]))^c$ admits at least one edge. Then $gr((\Gamma(R[X]))^c) = gr((A(R))^c) = 3$.*

Proof. We know from Corollary 3.3 that any edge of $(\Gamma(R[X]))^c$ is an edge of a triangle in $(\Gamma(R[X]))^c$. Therefore, we get that $gr((\Gamma(R[X]))^c) = 3$. Since $(\Gamma(R[X]))^c$ is a spanning subgraph of $(A(R))^c$, it follows that $gr((A(R))^c) = 3$. ∎

Proposition 3.5. *Let R be a ring such that $|Z(R)^*| \geq 1$. Then the following statements hold:*

(1) $gr((\Gamma(R[X]))^c) = gr((A(R))^c) \in \{3, \infty\}$.

(2) $gr((\Gamma(R[X]))^c) = gr((A(R))^c) = \infty$ *if and only if $Z(R)$ is an ideal of R with $Z(R)^2 = (0)$.*

Proof. (1) If $(\Gamma(R[X]))^c$ admits at least one edge, then we know from Corollary 3.4 that $gr((\Gamma(R[X]))^c) = gr((A(R))^c) = 3$. Suppose that $(\Gamma(R[X]))^c$ contains no cycle. Then $(\Gamma(R[X]))^c$ has no edges and hence, we obtain from (2) $\Rightarrow$ (1) of Lemma 3.1 that $(A(R))^c$ has no edges. Therefore, $gr((A(R))^c) = \infty$. If $(A(R))^c$ does not contain any cycle, then as $(\Gamma(R[X]))^c$ being a spanning subgraph of $(A(R))^c$, it follows that $gr((\Gamma(R[X]))^c) = \infty$. This proves that $gr((\Gamma(R[X]))^c) = gr((A(R))^c) \in \{3, \infty\}$.

(2) It follows from the proof of (1) that $gr((\Gamma(R[X]))^c) = gr((A(R))^c) = \infty$ if and only if $(\Gamma(R[X]))^c$ has no edges and we obtain from (2) $\Leftrightarrow$ (3) of Lemma 3.1 that $(\Gamma(R[X]))^c$ has no edges if and only if $Z(R)$ is an ideal of R with $Z(R)^2 = (0)$. ∎

Let $G = (V, E)$ be a graph. Recall from [4] that two distinct vertices u, v of G are said to be *orthogonal*, written $u \perp v$ if u and v are adjacent in G and there is no vertex w of G which is adjacent to both u and v in G. A vertex v of G is said to be a *complement of* u if $u \perp v$ [4]. Moreover, recall from [4] that G is *complemented* if each vertex of G admits a complement in G. In Section 3 of [4] Anderson et al. determined rings R for which the zero-divisor graphs $\Gamma(R)$ are complemented. For a ring R with $|Z(R)^*| \geq 1$, we verify in Corollary 3.6 that no vertex of $(A(R))^c$ (respectively, $(\Gamma(R[X]))^c$) admits a complement in $(A(R))^c$ (respectively, $(\Gamma(R[X]))^c$).

Corollary 3.6. *Let R be a ring such that $|Z(R)^*| \geq 1$. Then no vertex of $(A(R))^c$ (respectively, $(\Gamma(R[X]))^c$) admits a complement in $(A(R))^c$ (respectively, $(\Gamma(R[X]))^c$).*

Proof. Let $f(X) \in Z(R[X])^* = V((\Gamma(R[X]))^c) = V((A(R))^c)$. Since any edge of $(A(R))^c$ (respectively, $(\Gamma(R[X]))^c$) is an edge of a triangle in $(A(R))^c$ (respectively, $(\Gamma(R[X]))^c$) by Proposition 3.2 (respectively, Corollary 3.3), it follows that $f(X)$ does not admit any complement in $(A(R))^c$ (respectively, $(\Gamma(R[X]))^c$). ∎

Let R be a ring such that $|Z(R)^*| \geq 1$. We next discuss some results on the dominating sets and the domination number of $(A(R))^c$ (respectively, $(\Gamma(R[X]))^c$). For a graph G, we denote the domination number of G by $\gamma(G)$.

Lemma 3.7. *Let $G = (V, E)$ be a simple graph such that $|V| \geq 2$. If G is connected, then $\gamma(G^c) \geq 2$.*

Proof. Let $v \in V$. As $|V| \geq 2$ and G is connected, we can find $u \in V$ such that v and u are adjacent in G. Therefore, u is not adjacent to v in G^c. This implies that $\{v\}$ is not a dominating set of G^c for any $v \in V$. Therefore, $\gamma(G^c) \geq 2$. ∎

Corollary 3.8. *Let R be a ring such that $|Z(R)^*| \geq 1$. Then $\gamma((A(R))^c) \geq 2$ (respectively, $\gamma((\Gamma(R[X]))^c) \geq 2$).*

Proof. We know from Lemma 2.1 that $Z(R[X])^*$ is infinite. It follows from [1, Theorem 4] (respectively, [2, Theorem 2.3]) and Lemma 3.7 that $\gamma((A(R))^c) \geq 2$ (respectively, $\gamma((\Gamma(R[X]))^c) \geq 2$). ∎

Proposition 3.9. *Let R be a ring such that $Z(R)$ is not an ideal of R. Then $\gamma((A(R))^c) = \gamma((\Gamma(R[X]))^c) = 2$.*

Proof. Notice that $Z(R[X]) \cap R = Z(R)$. As $Z(R)$ is not an ideal of R by hypothesis, it follows that $Z(R[X])$ is not an ideal of $R[X]$. Hence, we obtain from [21, Lemma 2.3] that $\gamma((\Gamma(R[X]))^c) = 2$. Since $(\Gamma(R[X]))^c$ is a spanning subgraph of $(A(R))^c$, we get that $\gamma((A(R))^c) \leq 2$. It now follows from Corollary 3.8 that $\gamma((A(R))^c) = \gamma((\Gamma(R[X]))^c) = 2$. ∎

Let R be a ring such that $|Z(R)^*| \geq 1$ and $Z(R)$ is an ideal of R. We next discuss some results on the dominating sets and the domination number of $(A(R))^c$ (respectively, $(\Gamma(R[X]))^c$). We prove in Theorem 3.10 that $(A(R))^c$ admits a finite dominating set if and only if $Z(R[X])$ is not an ideal of $R[X]$.

Theorem 3.10. *Let R be a ring such that $|Z(R)^*| \geq 1$ and $Z(R)$ is an ideal of R. The following statements are equivalent:*

(1) *$(A(R))^c$ admits a finite dominating set.*

(2) *$Z(R[X])$ is not an ideal of $R[X]$.*

(3) *$(\Gamma(R[X]))^c$ admits a finite dominating set.*

Moreover, if the statement (1) holds, then $\gamma((A(R))^c) = \gamma((\Gamma(R[X]))^c) = 2$.

Proof. (1) $\Rightarrow$ (2) Let D be a finite dominating set of $(A(R))^c$. Notice that $D \subset Z(R[X])^*$. Let $D = \{f_i(X) \mid i \in \{1, \ldots, n\}\}$. It follows from Corollary 3.8 that $n \geq 2$. Observe that $\{Xf_i(X) \mid i \in \{1, 2, \ldots, n\}\}$ is also a dominating set of $(A(R))^c$. Hence, on replacing $f_i(X)$ by $Xf_i(X)$ (if necessary) for each $i \in \{1, 2, \ldots, n\}$, we can assume without loss of generality that $deg(f_i(X)) > 0$ for each $i \in \{1, 2, \ldots, n\}$. For each $i \in \{1, 2, \ldots, n\}$, let A_{f_i} denote the ideal of R generated by the coefficients of $f_i(X)$ and it follows from [16, Theorem 2] that $A_{f_i} \subseteq Z(R)$. We assert that $Z(R[X])$ is not an ideal of $R[X]$. Suppose that $Z(R[X])$ is an ideal of $R[X]$. Notice that $A = \sum_{i=1}^{n} A_{f_i}$ is a f.g. ideal of R and $A \subseteq Z(R)$. Hence, we obtain from [15, Theorem 3.3] that there exists $r \in R \backslash \{0\}$ such that $rA_{f_i} = (0)$ for each $i \in \{1, 2, \ldots, n\}$. Observe that $r \in Z(R)^* \subset Z(R[X])^*$. It is clear that $r \notin D$. Since D is a dominating set of $(A(R))^c$, we obtain that there exists $t \in \{1, 2, \ldots, n\}$ such that r and $f_t(X)$ are adjacent in $(A(R))^c$. This implies that $rf_t(X) \neq 0$. This is a contradiction and so, $Z(R[X])$ is not an ideal of $R[X]$.

(2) $\Rightarrow$ (3) As $Z(R[X])$ is not an ideal of $R[X]$ by assumption, we obtain from [21, Lemma 2.3] that $\gamma((\Gamma(R[X]))^c) = 2$. If $f(X), g(X) \in Z(R[X])^*$ are such that $f(X) + g(X) \notin Z(R[X])$, then we know from the proof of [21, Lemma 2.3] that $\{f(X), g(X)\}$ is a dominating set of $(\Gamma(R[X]))^c$.

(3) $\Rightarrow$ (1) Since $(\Gamma(R[X]))^c$ is a spanning subgraph of $(A(R))^c$, any dominating set of $(\Gamma(R[X]))^c$ is a dominating set of $(A(R))^c$. As $(\Gamma(R[X]))^c$ admits a finite dominating set by assumption, we obtain that $(A(R))^c$ admits a finite dominating set.

Assume that (1) holds. It is noted in the proof of (2) $\Rightarrow$ (3) of this theorem that $\gamma((\Gamma(R[X]))^c) = 2$. Hence, we obtain that $\gamma((A(R))^c) \leq 2$. It now follows from Corollary 3.8 that $\gamma((A(R))^c) = 2$. Therefore, $\gamma((A(R))^c) = \gamma((\Gamma(R[X]))^c) = 2$. ∎

Let R be a ring such that $|Z(R)^*| \geq 1$. If $Z(R)$ is a f.g. ideal of R and is not a B-prime of (0) in R, then we verify in Corollary 3.11 that $\gamma((A(R))^c) = \gamma((\Gamma(R[X]))^c) = 2$.

Corollary 3.11. *Let R be a ring such that $|Z(R)^*| \geq 1$ and suppose that R admits $\mathfrak{p}$ as its unique maximal N-prime of (0). If there exists a f.g. ideal I of R with $I \subseteq \mathfrak{p}$ such that I is not annihilated by any non-zero element of R, then $\gamma((A(R))^c) = \gamma((\Gamma(R[X]))^c) = 2$. In particular, if $\mathfrak{p}$ is a f.g. ideal of R and is not a B-prime of (0) in R, then $\gamma((A(R))^c) = \gamma((\Gamma(R[X]))^c) = 2$.*

Proof. By hypothesis, $Z(R)$ is an ideal of R and there exists a f.g. ideal I of R with $I \subseteq Z(R)$ such that $Ir \neq (0)$ for any non-zero $r \in R$. Hence, we obtain from [15, Theorem 3.3] that $Z(R[X])$ is not an ideal of $R[X]$. Therefore, we obtain from the moreover part of Theorem 3.10 that $\gamma((A(R))^c) = \gamma((\Gamma(R[X]))^c) = 2$.

Suppose that $\mathfrak{p} = Z(R)$ is a f.g. ideal of R with $\mathfrak{p}$ is not a B-prime of (0) in R. Hence, $\mathfrak{p}r \neq (0)$ for any $r \in R\backslash\{0\}$. Therefore, we obtain using the arguments given in the previous paragraph of this proof that $\gamma((A(R))^c) = \gamma((\Gamma(R[X]))^c) = 2$. ■

We provide Example 3.12 to illustrate Corollary 3.11.

Example 3.12. Let $S = K[X,Y]$ be the polynomial ring in two variables X, Y over a field K. Let $\mathfrak{m} = SX + SY$. Let $T = S_{\mathfrak{m}}$. Let M be the T-module given by $M = \frac{K(X,Y)}{T}$, where $K(X,Y)$ is the field of rational functions in two variables X, Y over K. Let $R = T(+)M$ be the ring obtained by using Nagata's principle of idealization. Let $\mathfrak{p} = \mathfrak{m}T(+)M$. Then R has $\mathfrak{p}$ as its unique maximal N-prime of its zero ideal, $\mathfrak{p}$ is not a B-prime of the zero ideal in R, $\gamma((A(R))^c) = \gamma((\Gamma(R[Z]))^c) = 2$, where $R[Z]$ is the polynomial ring in one variable Z over R.

Proof. It is clear that $\mathfrak{m} \in Max(S)$. It follows from [5, Example 1, page 38] that T has $\mathfrak{m}T$ as its unique maximal ideal. We know from [5, Corollary 7.6 and Proposition 7.3] that T is Noetherian. Notice that $K(X,Y)$ is the quotient field of T. As $\mathfrak{m} = SX + SY$, it follows that $\mathfrak{m}T = TX + TY$. We claim that $Z(R) = \mathfrak{m}T(+)M$. Since $\mathfrak{m}T$ is the unique maximal ideal of T, we obtain that $\mathfrak{m}T(+)M$ is the unique maximal ideal of R. Hence, $Z(R) \subseteq \mathfrak{m}T(+)M$. Let $(t,m) \in \mathfrak{m}T(+)M$. It is clear that $(t,m) = (t, 0+T)+(0,m)$. From $(0,m)^2 = (0, 0+T)$, in view of [15, Lemma 2.3] to prove $(t,m) \in Z(R)$, it is enough to show that $(t, 0+T) \in Z(R)$. If $t = 0$, then it is clear that $(0, 0+T) \in Z(R)$. Suppose that $t \neq 0$. From $t \in \mathfrak{m}T$, it follows that $\frac{1}{t} \in K(X,Y)\backslash T$. Notice that $\frac{1}{t} + T$ is a non-zero element of M and $(t, 0+T)(0, \frac{1}{t} + T) = (0, 0+T)$ is the zero element of R. This shows that $(t,m) \in Z(R)$ for any $(t,m) \in \mathfrak{m}T(+)M$. Therefore, $\mathfrak{m}T(+)M \subseteq Z(R)$ and so, $Z(R) = \mathfrak{m}T(+)M$. This proves that $\mathfrak{p} = \mathfrak{m}T(+)M$ is the unique maximal N-prime of the zero ideal in R. We verify that $\mathfrak{p} = R(X, 0+T)+R(Y, 0+T)$. It is clear that $R(X, 0+T) + R(Y, 0+T) \subseteq \mathfrak{p}$. Let $(t,m) \in \mathfrak{p}$. Notice that $t \in \mathfrak{m}T$ and $m \in M$ and $(t,m) = (t, 0+T) + (0,m)$. Now, $t = t_1X + t_2Y$ for some $t_1, t_2 \in T$. Hence, $(t, 0+T) = (t_1X + t_2Y, 0+T) = (t_1, 0+T)(X, 0+T) + (t_2, 0+T)(Y, 0+T) \in R(X, 0+T) + R(Y, 0+T)$. Since $M = \frac{K(X,Y)}{T}$, $m = \frac{f(X,Y)}{g(X,Y)} + T$ for some $f(X,Y), g(X,Y) \in S = K[X,Y]$. Therefore, $(0,m) = (X, 0+T)(0, \frac{f(X,Y)}{g(X,Y)X} + T) \in R(X, 0+T)$. Hence, $(t,m) \in R(X, 0+T) + R(Y, 0+T)$. This proves that $\mathfrak{p} \subseteq R(X, 0+T) + R(Y, 0+T)$ and so, $\mathfrak{p} = R(X, 0+T) + R(Y, 0+T)$. Thus $\mathfrak{p}$ is a f.g. ideal of R. Suppose that $\mathfrak{p}$ is a B-prime of the zero ideal in R. Then there exists $(t,m) \in \mathfrak{p}\backslash\{(0, 0+T)\}$ such that $\mathfrak{p} = ((0, 0+T) :_R (t,m))$. This implies that $(X, 0+T)(t,m) = (0, 0+T)$ and so, $tX = 0$. Hence, $t = 0$. Therefore, $m \neq 0+T$. Since $K[X,Y]$ is a unique factorization domain (UFD), it follows from [5, Proposition 3.11(*iv*)] and [14, Theorem 5] that T is a UFD. Notice that $K(X,Y)$ is the quotient field of T. It is possible to find $t_1, t_2 \in T$

such that t_1 and t_2 are relatively prime in T and $m = \frac{t_1}{t_2} + T$. From $m \neq 0 + T$, it follows that $\frac{t_1}{t_2} \notin T$. Now, $(X, 0)(0, m) = (Y, 0)(0, m) = (0, 0 + T)$. Hence, $Xt_1 = t_2t_3$ and $Yt_1 = t_2t_4$ for some $t_3, t_4 \in T$. Notice that $TX \in Spec(T)$ and from $\frac{t_1}{t_2} \notin T$, $Xt_1 = t_2t_3$, we get that $t_2 \in TX$. From $Yt_1 = t_2t_4$, it follows that $Yt_1 \in TX$. As $Y \notin TX$, we obtain that $t_1 \in TX$. This is impossible, as t_1 and t_2 are relatively prime in T. This shows that there exists no non-zero $r \in R$ such that $\mathfrak{p} = ((0, 0 + T) :_R r)$ and so, $\mathfrak{p}$ is not a B-prime of the zero ideal in R. Thus the ring R satisfies the hypotheses of Corollary 3.11 and hence, it follows from Corollary 3.11 that $\gamma((A(R))^c) = \gamma((\Gamma(R[Z]))^c) = 2$. ■

We provide Example 3.13 to illustrate that the in particular part of Corollary 3.11 can fail to hold if the ideal $Z(R)$ is f.g. is omitted. The example of the reduced ring R given in Example 3.13 is due to Gilmer and Heinzer [11, Example, page 16].

Example 3.13. Let $\{X_i\}_{i=1}^{\infty}$ be a set of indeterminates over a field K. Let $D = \bigcup_{n=1}^{\infty} K[[X_1, \ldots, X_n]]$, where for each $n \in \mathbb{N}$, $K[[X_1, \ldots, X_n]]$ is the power series ring in $X_1, \ldots, X_n$ over K. Let I be the ideal of D generated by $\{X_iX_j \mid i, j \in \mathbb{N}, i \neq j\}$. Let $R = \frac{D}{I}$. Then $(A(R))^c = (\Gamma(R[X]))^c$ and moreover, $(\Gamma(R[X]))^c$ does not admit any finite dominating set.

Proof. For each $i \in \mathbb{N}$, let us denote $X_i + I$ by x_i. It was already noted in [11, Example, page 16] that R is reduced and it is quasi-local with $\mathfrak{m} = \sum_{n=1}^{\infty} Rx_n$ as its unique maximal ideal. It was observed in [21, Example 2.4] that $Z(R) = \mathfrak{m}$ and so, $\mathfrak{m}$ is the unique maximal N-prime of the zero ideal in R. As R is reduced, we get that $\mathfrak{m}$ is not a B-prime of the zero ideal in R. By [16, Theorem 2], we obtain that $Z(R[X]) \subseteq Z(R)[X] = \mathfrak{m}[X]$. It was verified in the proof of [21, Example 2.4] that any f.g. proper ideal of R is annihilated by a non-zero element of R. Let $f(X) \in \mathfrak{m}[X]$ with $f(X) \neq 0$. Let C be the ideal of R generated by the coefficients of $f(X)$. Then C is a non-zero f.g. proper ideal of R. If $r \in R\backslash\{0 + I\}$ is such that $Cr = (0 + I)$, then $f(X)r = 0 + I$. Hence, $f(X) \in Z(R[X])$. This shows that $\mathfrak{m}[X] \subseteq Z(R[X])$. Therefore, $Z(R[X]) = \mathfrak{m}[X]$ is an ideal of $R[X]$. Hence, we obtain from (1) $\Rightarrow$ (2) of Theorem 3.10 that $(A(R))^c$ does not admit any finite dominating set. Since R is reduced, it follows that $(A(R))^c = (\Gamma(R[X]))^c$. Therefore, we obtain that $(A(R))^c = (\Gamma(R[X]))^c$ does not admit any finite dominating set. ■

Proposition 3.14. *Let R be a ring such that $|Z(R)^*| \geq 1$. If $(\Gamma(R[X]))^c$ (respectively, $(A(R))^c$) admits a finite dominating set, then so does $(\Gamma(R))^c$.*

Proof. Suppose that $(A(R))^c$ (respectively, $(\Gamma(R[X]))^c$) admits a finite dominating set. In such a case, we know from (1) $\Rightarrow$ (2) (respectively, (3) $\Rightarrow$ (2)) of Theorem 3.10 that $Z(R[X])$ is not an ideal of $R[X]$. If $f_1(X), f_2(X) \in Z(R[X])^*$

are such that $f_1(X) + f_2(X) \notin Z(R[X])$, then $A = \{f_1(X), f_2(X)\}$ is a dominating set of $(\Gamma(R[X]))^c$ (respectively, $(A(R))^c$). Notice that $\{Xf_i(X) \mid i \in \{1, 2\}\}$ is a dominating set of $(\Gamma(R[X]))^c$ (respectively, $(A(R))^c$). Hence on replacing $f_i(X)$ by $Xf_i(X)$ (if necessary) for each $i \in \{1, 2\}$, we can assume without loss of generality that $deg(f_i(X)) > 0$ for each $i \in \{1, 2\}$. It is clear that $A \subset Z(R[X])^*$. Let $i \in \{1, 2\}$ and let C_i be the set consisting of distinct non-zero coefficients of $f_i(X)$. As $f_i(X) \in Z(R[X])^*$, it follows from [16, Theorem 2] that $C_i \subseteq Z(R)^*$. Let $C = \bigcup_{i=1}^{2} C_i$. Then C is a finite non-empty subset of $Z(R)^*$. Let $a \in Z(R)^* \backslash C$. Notice that $a \in Z(R[X])^* \backslash A$. Since A is a dominating set of $(\Gamma(R[X]))^c$ (respectively, $(A(R))^c$), there exists $i \in \{1, 2\}$ such that a and $f_i(X)$ are adjacent in $(\Gamma(R[X]))^c$ (respectively, $(A(R))^c$). Hence, there exists $c \in C_i$ such that $ac \neq 0$ and so, a and c are adjacent in $(\Gamma(R))^c$. This shows that C is a dominating set of $(\Gamma(R))^c$. Hence, we obtain that $(\Gamma(R))^c$ admits a finite dominating set. ■

In Example 3.15, we mention an example of a ring R such that $\gamma((\Gamma(R))^c) = 1$ but $(A(R))^c$ (respectively, $(\Gamma(R[X]))^c$) does not admit any finite dominating set.

Example 3.15. Let $R = \mathbb{Z}_4$. Then $\gamma((\Gamma(R))^c) = 1$ but $(A(R))^c$ (respectively, $(\Gamma(R[X]))^c$) does not admit any finite dominating set.

Proof. Notice that R is local with $2R$ as its unique maximal ideal and $Z(R) = 2R$. As $Z(R)^* = \{2\}$, it follows that $\gamma((\Gamma(R))^c) = 1$. Observe that $Z(R[X]) = 2R[X]$ is an ideal of $R[X]$. From $Z(R[X])^2 = (0)$, it follows from $(3) \Rightarrow (1)$ (respectively, $(3) \Rightarrow (2)$) of Lemma 3.1 that $(A(R))^c$ (respectively, $(\Gamma(R[X]))^c$) has no edges. Hence, $(A(R))^c = (\Gamma(R[X]))^c$ and $Z(R[X])^*$ is the only dominating set of $(A(R))^c$. As $Z(R[X])^*$ is infinite, we get that $(A(R))^c = (\Gamma(R[X]))^c$ does not admit any finite dominating set. ■

Let R be a ring such that $|Z(R)^*| \geq 1$. We next discuss some results on $\omega((A(R))^c)$ (respectively, $\omega((\Gamma(R[X]))^c)$).

Lemma 3.16. *Let R be a ring such that $|Z(R)^*| \geq 1$. If there exists $a \in Z(R)^*$ such that $a^2 \neq 0$, then $(\Gamma(R[X]))^c$ admits an infinite clique.*

Proof. Notice that for any $n \in \mathbb{N}$, $aX^n \in Z(R[X])^*$ and for any distinct $m, n \in \mathbb{N}$, $aX^m \neq aX^n$. From $a^2 \neq 0$ and $X \notin Z(R[X])^*$, it follows that $(aX^m)(aX^n) = a^2X^{m+n} \neq 0$. Therefore, the subgraph of $(\Gamma(R[X]))^c$ induced by $\{aX^n \mid n \in \mathbb{N}\}$ is an infinite clique. ■

Corollary 3.17. *Let R be a ring such that R has at least two maximal N-primes of (0). Then $(\Gamma(R[X]))^c$ admits an infinite clique.*

Proof. By hypothesis, R has at least two maximal N-primes of (0). Let $\mathfrak{p}_1, \mathfrak{p}_2$ be distinct maximal N-primes of (0) in R. Then $\mathfrak{p}_1 \not\subseteq \mathfrak{p}_2$. Let $a \in \mathfrak{p}_1 \backslash \mathfrak{p}_2$. Then $a \in Z(R)^*$ and $a^2 \neq 0$. Hence, we obtain from Lemma 3.16 that $(\Gamma(R[X]))^c$ admits an infinite clique. ■

Proposition 3.18. *Let R be a ring such that $|Z(R)^*| \geq 1$. Suppose that R has $\mathfrak{p}$ as its unique maximal N-prime of (0). Then the following statements are equivalent:*

(1) $\omega((A(R))^c) < \infty$.

(2) $\omega((\Gamma(R[X]))^c) < \infty$.

(3) $(\Gamma(R[X]))^c$ *does not admit any infinite clique.*

(4) $\mathfrak{p}^2 = (0)$.

Proof. (1) $\Rightarrow$ (2) This is clear, since $(\Gamma((R[X]))^c$ is a spanning subgraph of $(A(R))^c$.

(2) $\Rightarrow$ (3) This is clear.

(3) $\Rightarrow$ (4) As $(\Gamma(R[X]))^c$ does not admit any infinite clique by assumption, it follows from Lemma 3.16 that $a^2 = 0$ for each $a \in Z(R) = \mathfrak{p}$. Suppose that $\mathfrak{p}^2 \neq (0)$. Then there exist $a, b \in Z(R)^* = \mathfrak{p}\backslash\{0\}$ such that $ab \neq 0$. Let $n \in \mathbb{N}$. From $a^2 = b^2 = 0$, it follows that $a + bX^n$ is a nilpotent element of $R[X]$ and hence, $a + bX^n \in Z(R[X])^*$. Let us denote $a + bX^n$ by $f_n(X)$. It is clear that $f_m(X) \neq f_n(X)$ for all distinct $m, n \in \mathbb{N}$. Let $m, n \in \mathbb{N}$ with $m \neq n$. Observe that ab is the coefficient of X^m in $f_m(X)f_n(X)$. From $ab \neq 0$, we get that $f_m(X)f_n(X) \neq 0$ and so, the subgraph of $(\Gamma(R[X]))^c$ induced by $\{f_n(X) \mid n \in \mathbb{N}\}$ is an infinite clique. This is a contradiction and so, we obtain that $\mathfrak{p}^2 = (0)$.

(4) $\Rightarrow$ (1) By hypothesis, $Z(R) = \mathfrak{p}$ is an ideal of R. We are assuming that $\mathfrak{p}^2 = (0)$. Hence, we obtain from (3) $\Rightarrow$ (1) of Lemma 3.1 that $(A(R))^c$ has no edges. Therefore, $\omega((A(R))^c) = 1 < \infty$. ■

Corollary 3.19. *Let R be a ring such that $|Z(R)^*| \geq 1$. Then the following statements are equivalent:*

(1) $(A(R))^c$ *is planar.*

(2) $(\Gamma(R[X]))^c$ *is planar.*

(3) $Z(R)$ *is an ideal of* R *with* $Z(R)^2 = (0)$.

(4) $(A(R))^c$ *has no edges.*

Proof. (1) $\Rightarrow$ (2) As $(\Gamma(R[X]))^c$ is a spanning subgraph of $(A(R))^c$ and $(A(R))^c$ is planar, we obtain that $(\Gamma(R[X]))^c$ is planar.

(2) $\Rightarrow$ (3) We are assuming that $(\Gamma(R[X]))^c$ is planar. As K_5 is non-planar by Kuratowski's Theorem [9, Theorem 5.9] we obtain that $\omega((\Gamma(R[X]))^c) \leq 4$.

Therefore, it follows from Corollary 3.17 that $Z(R)$ is necessarily an ideal of R. It now follows from (2) $\Rightarrow$ (4) of Proposition 3.18 that $Z(R)^2 = (0)$.

(3) $\Rightarrow$ (4) We are assuming that $Z(R)$ is an ideal of R with $Z(R)^2 = (0)$. Hence, we obtain from (3) $\Rightarrow$ (1) of Lemma 3.1 that $(A(R))^c$ has no edges.

(4) $\Rightarrow$ (1) This is clear. ■

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