Discussiones Mathematicae
General Algebra and Applications 43 (2023) 75–83
https://doi.org/10.7151/dmgaa.1402

ON RIGHT INVERSE ORDERED SEMIGROUPS

Amlan Jamadar and Kalyan Hansda

Department of Mathematics, Visva-Bharati University
Santiniketan, Bolpur-731235, West Bengal, India

e-mail: amlanjamadar@gmail.com
kalyanh4@gmail.com

Abstract

A regular ordered semigroup S is called right inverse if every principal left ideal of S is generated by an $\mathcal{R}$-unique positive element of it. We prove that a regular ordered semigroup is right inverse if and only if any two inverses of an element $a \in S$ are $\mathcal{R}$-related. Furthermore the class of right Clifford ordered semigroups have been characterized by the class of right inverse ordered semigroups.

Keywords: ordered regular, ordered inverse, positive element, completely regular, right inverse.

2020 Mathematics Subject Classification: 20M10, 06F05.

1. Introduction

Following Bailes [2], a right inverse semigroup is a regular semigroup with the property that each element has a unique left unit. Due to Bailes the class of right inverse semigroups are strictly between the class of orthodox semigroups and the class of inverse semigroups. Venkatesan [8] studied these semigroups under the name of right unipotent semigroups. He showed that a semigroup is right inverse if and only if every principal left ideal has a unique idempotent generator.

Bhuniya and Hansda [1] have deal with ordered semigroups in which any two inverses of an element are $\mathcal{H}$-related. These ordered semigroups are analogue to inverse semigroups. So it is a logical step to study ordered semigroups with the property that any two inverses of an element of it are $\mathcal{R}$-related. So the purpose of this paper is to characterise such ordered semigroups which we call right inverse ordered semigroups. We give a detailed exposition on the characterization of these ordered semigroups. Here we generalize such ordered semigroups into right inverse ordered semigroups. This paper is inspired by the works done by Venkatesan [8] and Bailes [2].

The presentation of this article is as follows: This section is followed by preliminaries. Definitions and basic properties of ordered semigroups are described in section 2. Section 3 is devoted to the right inverse ordered semigroups and its different characterizations.

2. Preliminaries

An ordered semigroup is a partiality ordered set $(S, \leq)$, and at the same time a semigroup $(S, \cdot)$ such that for all $a, b, x \in S$, $a \leq b$ implies $xa \leq xb$ and $ax \leq bx$. It is denoted by $(S, \cdot, \leq)$. For an ordered semigroup S and $H \subseteq S$, denote the download closure by

$$(H] := \{t \in S : t \leq h, \text{ for some } h \in H\}.$$

Let I be a non-empty subset of S. Then I is called a left(right) ideal [4] of S, if $SI \subseteq I (IS \subseteq I)$ and $(I] \subseteq I$. If I is both left and right ideal, then it is called an ideal of S. We call S a (left, right) simple ordered semigroup if it does not contain any proper (left, right) ideal. For $a \in S$, the smallest (left, right) ideal of S that contains a is denoted by $(L(a), R(a))I(a)$.

Following Kehayopulu [4] an ordered semigroup S is said to be regular (respectively, completely regular, right regular) if for every $a \in S$, $a \in (aSa]$ $(a \in (a^2Sa^2], a \in (a^2S])$. Due to Kehayopulu [4] Green's relations on a regular ordered semigroup given as follows:

$a\mathcal{L}b$ if $L(a) = L(b)$, $a\mathcal{R}b$ if $R(a) = R(b)$, $a\mathcal{J}b$ if $I(a) = I(b)$, $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$. These four relations $\mathcal{L}, \mathcal{R}, \mathcal{J}$, and $\mathcal{H}$ are equivalence relations.

An equivalence relation ρ is called left (right) congruence if for $a, b, c \in S$ $a\rho b$ implies $ca\rho cb$ $(ac\rho bc)$. By a congruence we mean both left and right congruence. A congruence ρ is called semilattice congruence if for all $a, b \in S$, $a\rho a^2$ and $ab\rho\, ba$. By a complete semilattice congruence we mean a semilattice congruence σ such that for $a, b \in S$, $a \leq b$ implies that $a\sigma ab$. An ordered semigroup S is called complete semilattice of subsemigroups of type τ if there exists a complete semilattice congruence ρ such that $(x)_\rho$ is a type τ subsemigroup of S.

A regular ordered semigroup S is said to be group-like (respectively, left group-like) [1] ordered semigroup if for every $a, b \in S$, $a \in (Sb]$ and $b \in (aS]$ (respectively, $a \in (Sb]$). Right group-like ordered semigroup can be defined dually. A regular ordered semigroup S is called a right (left) Clifford [1] if for all $a \in S$, $(Sa] \subseteq (aS]$, $((aS] \subseteq (Sa])$. Every right (left) group-like ordered semigroup is a right (left) Clifford ordered semigroup. An element $b \in S$ is said to be an inverse of $a \in S$ if $a \leq aba$ and $b \leq bab$. We denote the set of all inverses of an element a

by $V_{\leq}(a)$. In an ordered semigroup S, an element [7] $e \in S$ is said to be positive element if $e \leq e^2$. The set of all positive elements of S denoted by $E_{\leq}(S)$.

Following results are useful for the sake of this paper.

Theorem 1 [1]**.** *Let S be a regular ordered semigroup. Then the following statements are equivalent.*

(1) *S is right Clifford ordered semigroup;*

(2) *for all $e \in E_{\leq}(S), (Se] \subseteq (eS]$;*

(3) *for all $a \in S$, and $e \in E_{\leq}(S)$, there is $x \in S$ such that $ea \leq ax$;*

(4) *for all $a, b \in S$, there is $x \in S$ such that $ba \leq ax$;*

(5) *$\mathcal{L} \subseteq \mathcal{R}$ on S.*

Lemma 2 [1]**.** *Let S be a right Clifford ordered semigroup. Then the following conditions hold in S.*

(1) *$a \in (a^2Sa]$, for every $a \in S$;*

(2) *$ef \in (feSef]$, for every $e, f \in E_{\leq}(S)$.*

Theorem 3 [1]**.** *Let S be an ordered semigroup. Then S is right* (*left*) *Clifford ordered semigroup if and only if $\mathcal{R}(\mathcal{L})$ is the least complete semilattice congruence on S.*

Theorem 4 [1]**.** *Let S be a regular ordered semigroup. Then S is right* (*left*) *Clifford ordered semigroup if and only if it is a complete semilattice of right* (*left*) *group-like ordered semigroups.*

3. Right inverse ordered semigroup

Let S be an ordered semigroup and ρ be an equivalence relation on S. By the notion ρ-unique in S we mean the uniqueness in respect to the relation ρ.

Lemma 5. *Every principal left ideal in a regular ordered semigroup is generated by a positive element.*

Proof. Let S be a regular ordered semigroup. Consider a principal left ideal $J = (Sa]$ of S for $a \in S$. Since S is regular there is $x \in S$ such that $a \leq axa$. Then $xa \in E_{\leq}(S)$. Say $e = xa$. Clearly $a\mathcal{L}e$. Take $y \in J$. Then $y \leq za$ for some $z \in S$, so that $y \leq zaxa = zae$. Therefore $y \in (Se]$. Hence $J = (Se]$. ■

Lemma 6. *Let S be a regular ordered semigroup. Then S is a left group-like ordered semigroup if and only if any two positive elements of S are $\mathcal{L}$-related.*

Proof. Suppose that S is a left group-like ordered semigroup. Choose $e, f \in E_{\leq}(S)$. Then $e\mathcal{L}f$.

Conversely, assume that the given condition holds in S. Let $a, b \in S$. Since S is regular, there exists $s, t \in S$ such that $a \leq asa$ and $b \leq btb$. Clearly $as, sa, bt, tb \in E_{\leq}(S)$. Then by given condition, $sa \leq vtb$ for some $v \in S$. Then $a \leq asa \leq avtb$. Hence S is left group-like ordered semigroup. ■

Right group-like ordered semigroup can be characterized dually.

Let S be a regular ordered semigroup and $a \in S$. Then there is $x \in S$ such that $a \leq axa \leq a(xax)a$. Also $xax \leq xaxaxax$. So there is $a' \in V_{\leq}(a)$ such that $a \leq aa'a$. Now we have the following theorem.

Theorem 7. *Let S be a regular ordered semigroup. Then any two inverses of an element of S are $\mathcal{L}$-related if and only if for every $e, f \in E_{\leq}(S), ef \in (eSfSe]$.*

Proof. First suppose that the given condition holds in S. Let $a \in S$ and $a',\ a'' \in V_{\leq}(a)$. Now $a' \leq a'aa' \leq a'aa''aa'$. Clearly aa' and $aa'' \in E_{\leq}(S)$ so that by the given condition we have $aa''aa' \leq aa''raa'taa''$ for some $r, t \in S$. Therefore $a \leq a'aa''raa'taa''$, that is, $a' \leq ma''$, where $m = a'aa''raa'ta$. Similarly $a'' \leq na'$ for some $n \in S$. Hence $a'\mathcal{L}a''$.

Conversely, let $a \in S$ and $a',\ a'' \in V_{\leq}(a)$ such that $a'\mathcal{L}a''$. Let $e, f \in E_{\leq}(S)$. Since S is regular, $V_{\leq}(ef) \neq \phi$. Let $x \in V_{\leq}(ef)$. Then $x \leq xefx$ and $ef \leq efxef$. Now $fxe \leq f(xefx)e \leq fxe(ef)fxe$ and $ef \leq ef(fxe)ef$. Then $ef \in V_{\leq}(fxe)$. Also $fxe \in E_{\leq}(S)$, and so $fxe \in V_{\leq}(fxe)$. This yields that $ef \in V_{\leq}(fxe)$, so $ef\mathcal{L}fxe$ by the given condition. Hence $ef \leq ufxe$ for some $u \in S$. Hence from above $ef \leq ef^2xeufxe$. Thus $ef \in (eSfSe]$. ■

Definition. A regular ordered semigroup S is called right inverse if every principal left ideal is generated by an $\mathcal{R}$-unique positive element of S.

Example 3.1. The ordered semigroup $S = \{a, e, f\}$ defined by multiplication and order below.

$\cdot$	a	e	f
a	a	e	f
e	a	e	f
f	a	e	f

$$' \leq' := \{(a, a), (a, e), (a, f), (e, e), (f, f)\}.$$

$(S, \cdot, \leq)$ is a right inverse ordered semigroup.

The following lemma is obvious so we omit its proof.

Lemma 8. *A regular ordered semigroup S is a right inverse if and only if for any two positive elements $e, f \in E_{\leq}(S)$, $e\mathcal{L}f$ implies $e\mathcal{H}f$.*

Left group-like ordered semigroups are generalizations of group-like ordered semigroups. Every right inverse and left group-like ordered semigroups are group-like ordered semigroups.

Lemma 9. *Every right inverse left group-like ordered semigroup is a group-like ordered semigroup.*

Proof. This follows from Lemma 6 and Lemma 8. ■

The characterization of right inverse ordered semigroups by the inverses of their elements has been given in the following theorem.

Theorem 10. *The following conditions are equivalent on a regular ordered semigroup S.*

(1) *S is right inverse;*

(2) *for every $a \in S$ and $a', a'' \in V_{\leq}(a)$, $a'\mathcal{R}a''$;*

(3) *for every $e, f \in E_{\leq}(S)$, $ef \in (fSeSf]$;*

(4) *for every $e, f \in E_{\leq}(S)$, $(eS] \cap (fS] = (efS]$;*

(5) *for $e \in E_{\leq}(S)$ and $x \in (Se]$ implies $x' \in (eS]$, where $x \in S$ and $x' \in V_{\leq}(x)$.*

Proof. (1) $\Rightarrow$ (2): Let S be a right inverse semigroup and $a \in S$. Consider $a', a'' \in V_{\leq}(a)$. Then $a'a, a''a \in E_{\leq}(S)$. Say $L = (Sa]$ and let $x \in (Sa]$. Then there is $s \in S$ such that $x \leq sa$. Thus $x \leq saa'a$, which implies that $x \in (Sa'a]$ and so $(Sa] \subseteq (Sa'a]$. Also $(Sa'a] \subseteq (Sa]$. And thus $(Sa] = (Sa'a]$. Similarly $(Sa] = (Sa''a]$. Since S is right inverse, we have $a'a\mathcal{R}a''a$. Now $a'' \leq a''aa'' \leq a'az_1a''$ for some $z_1 \in S$. Hence $a'' \leq a't_1$, where $t_1 = az_1a''$. Also $a' \leq a'aa'' \leq a''az_2a'$ for some $z_2 \in S$. Hence $a' \leq a''t_2$, where $t_2 = az_2a'$. Hence $a'\mathcal{R}a''$.

(2) $\Rightarrow$ (3): Suppose that for every $a \in S$ and $a', a'' \in V_{\leq}(a)$, we have $a'\mathcal{R}a''$. Since S is regular, $V_{\leq}(ef) \neq \phi$. Let $x \in V_{\leq}(ef)$. Then $x \leq xefx$ and $ef \leq efxef$ and so $fxe \leq fxefxe \leq fxe(ef)fxe$ and $ef \leq ef^2xe^2f$. Thus

$$ef \leq ef(fxe)ef \tag{1}$$

and thus $ef \in V_{\leq}(fxe)$. Also $fxe \leq (fxe)^2$, so that is $fxe \in V_{\leq}(fxe)$. Hence $ef, fxe \in V_{\leq}(fxe)$. Then by the condition (2), we have $ef\mathcal{R}fxe$, so $ef \leq fxeu$ for some $u \in S$. Also $ef \leq fxeufxe^2f$ from inequality (1). Hence $ef \in (fSeSf]$.

(3) $\Rightarrow$ (4): Let $x \in (eS] \cap (fS]$. Then there are $s_1, s_2 \in S$ such that $x \leq es_1$ and $x \leq fs_2$. Since S is regular there is $z \in V_{\leq}(x)$ such that $x \leq xzx$, so $x \leq es_1zfs_2$. Also $es_1z \leq es_1zxz \leq es_1zes_1z$, thus $es_1z \in E_{\leq}(S)$. Now by condition (3) $es_1zf \in (fSes_1zSf]$, and thus $es_1zf \leq fs_3es_1zs_4f$ for some $s_3, s_4 \in S$. Therefore $es_1zf \leq fm$, where $m = s_3es_1zs_4f$. Finally $x \leq es_1zfs_2 \leq e(es_1zf)s_2 \leq efms_2$. Thus $x \in (efS]$ and hence $(eS] \cap (fS] \subseteq (efS]$.

Next let $y \in (efS]$ then $y \in (eS]$. Also $ef \in (fSeSf]$, by given condition. Now $y \in (efS]$ implies that $y \leq efq$ for some $q \in S$. Then there are $s_6, s_7 \in S$ such that $y \leq fs_6es_7fq$. Thus $y \in (fS]$, and so $y \in (eS] \cap (fS]$. Therefore $(efS] \subseteq (eS] \cap (fS]$ and hence $(eS] \cap (fS] = (efS]$.

(4) $\Rightarrow$ (5): Given $x \in (Se]$. Then there is $s \in S$ such that $x \leq se$. Also $x \leq xx'x$ and $x' \leq x'xx'$ for some $x' \in V_{\leq}(x)$. Now $x'se \leq x'sex'se$, so $x'se \in E_{\leq}(S)$. Then for $x'se$, $e \in E_{\leq}(S)$ we have that $(x'seeS] = (eS] \cap (x'seS]$, by condition (4). Also from $x' \leq x'xx'$ we have $x' \leq x'sex'$. Therefore $x' \in (x'SeeS]$ and hence $x' \in (eS]$.

(5) $\Rightarrow$ (1): Since S is regular, by Lemma 5 we have that every principal left ideal is generated by a positive element. Let $e, f \in E_{\leq}(S)$ such that $(Se] = (Sf]$. Also $e \leq ee$, so $e \in (Se] = (Sf]$. Now $e \in V_{\leq}(e)$. So from the given condition $e \in (fS]$. Similarly $f \in (eS]$ and thus $e\mathcal{R}f$. Hence S is a right inverse ordered semigroup. ∎

Corollary 3.2. *Let S be a right inverse ordered semigroup. Then any two positive elements $e, f \in E_{\leq}(S)$ are $\mathcal{H}$-commutative if and only if $(Se] \cap (Sf] = (Sef]$.*

Proof. This follows from Theorem 10 and ([1], Theorem 3.10). ∎

Let S be a regular ordered semigroup. For $A \subseteq S$, denote $A' = \{x \in S : x \in V_{\leq}(y)$ where $y \in A\}$, the set of all ordered inverses of the elements of A.

In the following lemma we characterize these subsets A' in an ordered semigroup.

Lemma 11. *Let S be a regular ordered semigroup. Then following are true in S.*

(1) *For any subset R of S, $(R')' \subseteq R$.*

(2) *For any subset A, B of S, $A \subseteq B$ implies $A' \subseteq B'$.*

Proof. (1) Let $x \in (R')'$. So there $x' \in R'$ such that $x \in V_{\leq}(x')$. Also $x' \in V_{\leq}(x)$ which implies that $x \in R$. So $(R')' \subseteq R$.

(2) Let $x \in A'$. So there exists $y \in A$ such that $x \in V_{\leq}(y)$. Now $y \in A$ implies $y \in B$. So $x \in B'$. So $A' \subseteq B'$. ∎

In the following theorem we characterize a right inverse ordered semigroup S by the set of all inverses of elements of $\mathcal{R}$-class of a positive element of S.

Theorem 12. *Let S be a regular ordered semigroup. Then S is a right inverse ordered semigroup if and only if $L_e \subseteq (R_e)'$ for any $e \in E_{\leq}(S)$.*

Proof. Let $e \in E_{\leq}(S)$. Say $R = R_e$ and $L = L_e$. Suppose that S is a right inverse ordered semigroup. Let $x \in L_e$ and $x' \in V_{\leq}(x)$. Then $x' \in (L_e)'$ and $x'x \in E_{\leq}(S)$. Now $x'x \leq (x'xx')x$ and $x \leq x(x'x)$, which gives that $x\mathcal{L}x'x$.

Hence $x'x$ is a positive element in L_e. Therefore $L = (Se] = (Sx'x]$. Since S is a right inverse ordered semigroup, so $e\mathcal{R}x'x$, by definition. Also $x'\mathcal{R}x'x$, then $x' \in R_e$ which implies that $x \in (R_e)'$ and hence $L \subseteq R'$.

Conversely assume that the given condition holds in S. Since S is regular, by Lemma 5 we have every principal left ideal is generated by a positive element. Let $e, f \in E_{\leq}(S)$ such that $(Se] = (Sf]$. Clearly $e, f \in L_e$. Also $L_e \subseteq (R_e)'$ implies $(L_e)' \subseteq ((R_e)')' \subseteq R_e$ by Lemma 11. Now $e, f \in L_e$ implies $e, f \in (L_e)'$. So $e, f \in R_e$. Hence S is right inverse ordered semigroup. ∎

Theorem 13. *An ordered semigroup S is right Clifford if and only if S is right inverse and for every $a \in S$, $a \in (a^2Sa]$.*

Proof. First suppose that S is a right inverse ordered semigroup and for every $a \in S$, $a \in (a^2Sa]$. Then $a \leq a^2xa$ for some $x \in S$. So we have $a^2x \leq a^2xa^2x$, which implies that $a^2x \in E_{\leq}(S)$. Let $e \in E_{\leq}(S)$. Since S is right inverse, there are $x_1, x_2 \in S$ such that $ea^2x \leq a^2xx_1ex_2a^2x$ by Theorem 2. Now $ea \leq ea^2xa \leq a^2xx_1ex_2a^2xa = a(axx_1ex_2a^2xa) = ax_3$, where $x_3 = axx_1ex_2a^2xa$. Thus $ea \in (aS]$. So S is a right Clifford ordered semigroup.

Conversely, assume that S is a right Clifford ordered semigroup. Then from Lemma 2, $a \in (a^2Sa]$ and $ef \in (feSef]$. So there is $x_4 \in S$ such that $ef \leq fex_4ef \leq f(e)e(x_4e)f = f(e)e(x_5)f$, where $x_5 = x_4e$ and thus $ef \in (fSeSf]$. Hence S is right inverse ordered semigroup, by Theorem 10. ∎

Bhuniya and Hansda [1] showed that every Clifford ordered semigroup is a union of group-like ordered semigroups. Here we have shown that a left Clifford ordered semigroup is a union of group-like ordered semigroups in right inverse ordered semigroup.

Theorem 14. *Let S be a right inverse ordered semigroup. If S is left Clifford then S is a union of group-like ordered semigroups.*

Proof. Suppose that S is a left Clifford ordered semigroup. Then $\mathcal{L}$ is a semilattice congruence on S by Theorem 4. Let $a \in S$ and $a' \in V_{\leq}(a)$. Then $a \leq aa'a$ and $a' \leq a'aa'$. Let us denote $aa' = e$ and $a'a = f$. Clearly $e, f \in E_{\leq}(S)$. Also $a \leq aa'a$ and $a'a \leq a'aa'a$, which implies that $a\mathcal{L}f$. Since $\mathcal{L}$ is a congruence on S, $ea\mathcal{L}ef$. Also $a \leq aa'a = ea \leq eea$ and $ea \leq eea$, so $a\mathcal{L}ea$. Thus $a\mathcal{L}ef$ implies that $f\mathcal{L}ef$. Similarly it can be shown that $a'\mathcal{L}e$ and $e\mathcal{L}fe$. Since $\mathcal{L}$ is a semilattice congrence, so $ef\mathcal{L}fe$. Hence $ef\mathcal{L}fe\mathcal{L}a\mathcal{L}a'\mathcal{L}f\mathcal{L}e$.

Also $a\mathcal{R}e$ and $a'\mathcal{R}f$. Hence $a\mathcal{H}e$ and $a'\mathcal{H}f$. Now let $x \in (Se]$. Hence there are $s_1, s_2 \in S$ such that $x \leq s_1e \leq s_1s_2f$, since $e\mathcal{L}f$.

So $x \in (Sf]$. Hence $(Se] \subseteq (Sf]$. Similarly it can be shown that $(Sf] \subseteq (Se]$. Hence $(Se] = (Sf]$. Since S is right inverse ordered semigroup, we have $e\mathcal{R}f$. Hence $e\mathcal{H}f$. So $a\mathcal{H}f\mathcal{H}e\mathcal{H}a'$.

Since $a\mathcal{H}f$ and $a\mathcal{H}a'$, so there is $x_1, x_2 \in S$ such that $a \leq aa'a \leq aa'aa'a = afa'a \leq aax_1x_2aa = a^2x_1x_2a^2$.

So, $a \in (a^2Sa^2]$. Thus S is a completely regular ordered semigroup and hence S is a union of group-like ordered semigroups by Theorem 4.8 of [1]. ■

In the following we show that in a right inverse ordered semigroup $\mathcal{R}$ is a congruence if and only if $\mathcal{L} = \mathcal{H}$.

Theorem 15. *Let S be a right inverse ordered semigroup. Then following conditions are equivalent.*

(1) *$\mathcal{R}$ is a congruence on S;*

(2) *$\mathcal{L} = \mathcal{H}$;*

(3) *S is a complete semilattice of right group-like ordered semigroups.*

Proof. (1) $\Rightarrow$ (2). Let $\mathcal{R}$ be a congruence on S. Since S is right inverse, S is regular. So for every $a \in S$ there is $a' \in V_{\leq}(a)$. Now $a \leq aa'a$ and $a' \leq a'aa'$. Denote $aa' = e$ and $a'a = f$. Therefore $a\mathcal{R}e$ and $a'\mathcal{R}f$. Since $\mathcal{R}$ is a congruence on S, $aa'\mathcal{R}ef$. That is $e\mathcal{R}ef$.

Again $a' \leq (a'e)e$ and $a'e \leq a'(ee)$. Therefore $a'\mathcal{R}a'e$ which implies that $a'\mathcal{R}fe$. Thus $f\mathcal{R}fe$. So there is $z \in S$ such that $f \leq fez$. Since S is right inverse, there are $x, y \in S$ such that $ef \leq fxeyf \leq fezxeyf$. Hence $ef \leq fet_1$, where $t_1 = zxeyf$. Similarly $fe \leq eft_2$, for some $t_2 \in S$. Therefore $ef\mathcal{R}fe$. Hence $a\mathcal{R}e\mathcal{R}ef\mathcal{R}fe\mathcal{R}f\mathcal{R}a'$. Now $a \leq aa'a \leq aawa$, for some $w \in S$. Hence $a \in (a^2Sa]$. So S is a right clifford ordered semigroup. Hence $\mathcal{L} \subseteq \mathcal{R}$. Thus $\mathcal{L} = \mathcal{H}$.

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1). These implications follows from Theorem 4 and Theorem 3. ■

Our paper ends up with the following corollary which follows from Theorem 15 and Theorem 14. This gives a condition for a right inverse ordered semigroup to become a completely regular ordered semigroup.

Corollary 3.3. *Let S be a right inverse and left regular ordered semigroup. Then following conditions are equivalent.*

(1) *$\mathcal{R}$ is a congruence on S;*

(2) *$\mathcal{L} = \mathcal{H}$;*

(3) *S is a complete semilattice of right group-like ordered semigroups;*

(4) *S is completely regular.*

Acknowledgements

We express our deepest gratitude to the editor of the journal Professor Joanna Skowronek-Kaziów for communicating the paper and communicating the paper and to the referee of the paper for their important valuable comments and suggestions to enrich the quality of the paper both in value and content.

References

[1] A.K. Bhuniya and K. Hansda, *Completely regular and Clifford ordered semigroups*, Afrika Matematika **31** (2020) 1029–1045.
https://doi.org/10.1007/s13370-020-00778-1

[2] G.L. Bailes, *Right inverse semigroups*, J. Algebra **26** (1973) 492–507.
https://doi.org/10.1016/0021-8693(73)90010-0

[3] N. Kehayopulu, *Remark in ordered semigroups*, Math. Japonica **35** (1990) 1061–1063.

[4] N. Kehayopulu, *Ideals and Green's relations in ordered semigroups*, Int. J. Math. and Math. Sci. (2006) 1–8.
https://doi.org/10.1155/IJMMS/2006/61286

[5] S.K. Lee and Y.I. Kwon, *On completely regular and quasi-completely regular ordered semigroups*, Sci. Math. **2** (1998) 247–251.

[6] T. Saito, *Ordered idempotent semigroups*, J. Math. Soc. Japan **14 (2)** (1962) 150–169.

[7] T. Saito, *Ordered inverse semigroups*, Trans. Amer. Math. Soc **153** (1971) 99–138.
https://doi.org/10.2307/1995550

[8] P.S. Venkatesan, *Right (left) inverse semigroups*, J. Algebra **31** (1974) 209–217.
https://doi.org/10.1016/0021-8693(74)90064-7

[9] P.S. Venkatesan, *On right unipotent semigroups*, Pacific J. Math. **63** (1976) 555–561.
https://doi.org/10.2140/pjm.1976.63.555

[10] P.S. Venkatesan, *On right unipotent semigroups* II, Glasgow Math. J. **19** (1978) 63–68.
https://doi.org/10.1017/S0017089500003384

[11] P.S. Venkatesan, *Bisimple left inverse semigroups*, Semigroup Forum **4** (1972) 34–45.
https://doi.org/10.1007/BF02570767

Received 27 May 2018
Revised 9 August 2020
Accepted 31 July 2021