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INTUITIONISTIC FUZZY MONOIDS IN AN INTUITIONISTIC FUZZY FINITE AUTOMATON WITH UNIQUE MEMBERSHIP TRANSITION ON AN INPUT SYMBOL

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Abstract

An intuitionistic fuzzy finite state automaton assigns a membership and nonmembership values in which there is a unique membership transition on an input symbol ($IFAUM$) is considered.It is proved and illustrated the existence of two different intuitionistic fuzzy monoids $F(\mathscr{A})$ and $S_{\mathscr{A}}$ from an intuitionistic fuzzy transition function of the given IFAUM $\mathscr{A}$. Also it is proved that $F(\mathscr{A})$ and $S_{\mathscr{A}}$ are anti-isomorphic as monoids.

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1. Introduction

Zadeh [21] was the first to propose the theory of fuzzy sets, as an effective generalization of classical sets, which has been widely used in dealing with problems with imprecision and uncertainty. Fuzzy set theory has become more and more mature in many fields such as fuzzy relation, fuzzy logic, fuzzy decision-making, fuzzy classification, fuzzy pattern recognition, fuzzy control and fuzzy optimization. The concept of fuzzy automaton in the late 1960s was presented by Malik, Mordeson, Sen, Chowdhry, Wee and Fu, as in [12, 17, 19].

Lee, Zadeh, Thomason and Marinos [9, 18, 22] originated the research on fuzzy languages accepted by fuzzy finite-state machines in the early 1970s. Also, the fuzzy finite automaton can be applied in many areas such as learning systems, the model computing with words, pattern recognition, lattice-valued fuzzy finite automaton and data base theory by Li, Shi, Pedryez and Ying, as in [10, 11, 20].

Finite state automaton, deterministic finite state automaton, nondeterministic finite state automaton and regular expression were introduced by Hopcroft and Ullman, as in [13]. The usual fuzzy finite state automaton can have more than one transition with a membership value on an input symbol. So, the uniqueness in the membership transition are introduced, to reduce the number of transitions to at most one transition where the fuzzy behavior need not be the same, as in [16]. However, it only acts as a deterministic fuzzy recognizer, so to retain the same fuzzy behavior a condition is incorporated that the membership function has a unique transition on an input symbol, as in [14].

Intuitionistic fuzzy sets (*IFS*) introduced in 1983 are generalization of fuzzy sets, in which membership and nonmembership values for every elements are defined by Atanassov, as in [1–5]. Jun, [6–8] presented the concept of intuitionistic fuzzy finite state machines (*IFFSM*) as a generalization of fuzzy finite state machines using the notions of IFSs and fuzzy finite automaton." The notions of intuitionistic fuzzy recognizer, complete accessible intuitionistic fuzzy recognizer, intuitionistic fuzzy finite automaton, deterministic intuitionistic fuzzy finite automaton and intuitionistic fuzzy languages are introduced by Zhang, Li, as in [23]. Samuel Eilenburg, [15] introduced the notion of an automaton and of a set recognized by an automaton.

In this paper, the authors proved that an intuitionistic fuzzy monoid is closed under the composition of functions and some properties of monoid.

2. Basic definitions

These preliminaries have been found to be essential to build the parameters of the present work [1, 2].

Definition 1. An Intuitionistic fuzzy sets (IFS) A in a nonempty set Σ is an object having the form $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in \Sigma\}$, where the functions $\mu_A : \Sigma \to [0, 1]$ and $\nu_A : \Sigma \to [0, 1]$ denote the degree of membership and nonmembership of each element $x \in \Sigma$ to the set A respectively, and $0 \le \mu_A(x)+\nu_A(x) \le 1$ for each $x \in \Sigma$. For the sake of simplicity, we use the notation $A = (\mu_A, \nu_A)$ instead of $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in \Sigma\}$.

Definition 2. Intuitionistic fuzzy finite automaton with unique membership transition on an input symbol is an ordered 5-tuple (IFAUM) $\mathscr{A} = (Q, \Sigma, A, i, f)$, where

(i) Q is a finite non-empty set of states.

(ii) Σ is a finite non-empty set of input symbols.

(iii) $A = (\mu_A, \nu_A)$, each is an intuitionistic fuzzy subset of $Q \times \Sigma \times Q$.

(a) the fuzzy subset $\mu_A : Q \times \Sigma \times Q \to [0,1]$ denotes the degree of membership such that $\mu_A(p,a,q) = \mu_A(p,a,q')$ for some $q, q' \in Q$ then $q = q'$.

(b) $\nu_A : Q \times \Sigma \times Q \to [0,1]$ denotes the degree of nonmembership is a fuzzy subset of Q.

(iv) $i = (i_{\mu_A}, i_{\nu_A})$, each is an intuitionistic fuzzy subset of Q, i.e., $i_{\mu_A} : Q \to [0,1]$ and $i_{\nu_A} : Q \to [0,1]$ called the intuitionistic fuzzy subset of initial states.

(v) $f = (f_{\mu_A}, f_{\nu_A})$, each is an intuitionistic fuzzy subset of Q, i.e., $f_{\mu_A} : Q \to [0,1]$ and $f_{\nu_A} : Q \to [0,1]$ called the intuitionistic fuzzy subset of final states.

Definition 3. Let $\mathscr{A} = (Q, \Sigma, A, i, f)$ be an IFAUM. Then the fuzzy behavior of IFAUM is $L_{\mathscr{A}} = (L_{\mu_{\mathscr{A}}}, L_{\nu_{\mathscr{A}}})$.

Definition 4. Let $\mathscr{A} = (Q, \Sigma, A, i, f)$ be an IFAUM. Define an IFS $A^* = (\mu_A^*, \nu_A^*)$ in $Q \times \Sigma^* \times Q$ as follows $\forall p, q \in Q, x \in \Sigma^*, a \in \Sigma$.

$$\mu_A^*(p,\epsilon,q) = \begin{cases} 1, & \text{if } p = q \\ 0, & \text{if } p \neq q \end{cases}$$

$$\nu_A^*(p,\epsilon,q) = \begin{cases} 0, & \text{if } p = q \\ 1, & \text{if } p \neq q \end{cases}$$

$$\mu_A^*(p,xa,q) = \vee \{\mu_A^*(p,x,r) \wedge \mu_A(r,a,q) \mid r \in Q\}$$

$$\nu_A^*(p,xa,q) = \wedge \{\nu_A^*(p,x,r) \vee \nu_A(r,a,q) \mid r \in Q\}.$$

Definition 5. Let $\mathscr{A} = (Q, \Sigma, A, i, f)$ be an IFAUM and $x \in \Sigma^*$. Then x is recognized by $\mathscr{A}$ if $\vee\{i_{\mu_A}(p) \wedge \mu_A^*(p,x,q) \wedge f_{\mu_A}(q) \mid p, q \in Q\} > 0$ and $\wedge\{i_{\nu_A}(p) \vee \nu_A^*(p,x,q) \vee f_{\nu_A}(q) \mid p, q \in Q\} < 1$.

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Definition 6. Let $\mathscr{A} = (Q, \Sigma, A, i, f)$ be an IFAUM. For $a \in \Sigma$, we define $f_a : Q \to Q$ by

$$f_a(p) = \begin{cases} q, \text{if } \mu_A(p,a,q) = \vee\{\mu_A(p,a,r) \mid r \in Q\} > 0 \\ q', \text{if } \mu_A(p,a,q) = \mu_A(p,a,q_i) = \vee\{\mu_A(p,a,r) \mid r \in Q\} > 0 \ i = 1,2,\ldots,n \\ \text{and } \nu_A(p,a,q') = \nu_A(p,a,q) \wedge \nu_A(p,a,q_i) \\ p, \text{if } \vee\{\mu_A(p,a,r) \mid r \in Q\} = 0. \end{cases}$$

For $x \in \Sigma^*$, we define $f_x : Q \to Q$ by

(i) $f_\lambda(p) = p$, and inductively

(ii) $f_{ax}(p) = f_x(q)$, where q is such that
$\mu_A(p,a,q) = \vee\{\mu_A(p,a,r) \mid r \in Q\} \forall p \in Q$ and
$\nu_A(p,a,q') = \nu_A(p,a,q) \wedge \nu_A(p,a,q_i) \forall p \in Q$.

Theorem 1. *Let $\mathscr{A} = (q, \Sigma, A, i, f)$ be an IFAUM and let $F(\mathscr{A}) = \{f_x \mid x \in \Sigma^*\}$, then $F(\mathscr{A})$ is an intuitionistic fuzzy finite monoid (submonoid of Q^Q) under $\circ$, the composition of functions.*

Proof. Let $f_x, f_y \in F(\mathscr{A}), x, y \in \Sigma^*$. For $p \in Q$,

$$\begin{aligned}(f_x \circ f_y)(p) &= f_x(f_y(p)) \\ &= f_x(q), \text{where q is such that} f_y(p) = q \\ &= s, \text{where s is such that} f_x(q) = s.\end{aligned}$$

Now $f_{yx}(p) = f_x(q) = s$. Therefore $(f_x \circ f_y)(p) = f_{yx}(p)$. Since p is arbitrary, $f_x \circ f_y = f_{yx} \in F(\mathscr{A})$, a unique function. Therefore $F(\mathscr{A})$ is closed under composition $\circ$. Let $f_x, f_y, f_z \in F(\mathscr{A})$.

Now $f_x \circ (f_y \circ f_z) = f_x \circ (f_{zy}) = f_{zyx} = (f_{yx}) \circ f_z = (f_x \circ f_y) \circ f_z$. Thus $\circ$ is associative. For $p \in Q, (f_x \circ f_\lambda)(p) = f_x(f_\lambda(p)) = f_x(p) = f_\lambda(f_x(p)) = (f_\lambda \circ f_x)(p)$. Since p is arbitrary, $f_x \circ f_\lambda = f_x = f_\lambda \circ f_x$. Hence f_λ is the identity element which is in $F(\mathscr{A})$. Therefore $(F(\mathscr{A}), \circ)$ is an intuitionistic finite fuzzy monoid. Since $\mathrm{Im}(\mu)$ and $\mathrm{Im}(\nu)$ are finite, $F(\mathscr{A})$ is finite. ■

Corollary 1. $(F(\mathscr{A}), \circ)$ *is an intuitionistic finite fuzzy monoid.*

Proof. Let $x \in \Sigma^*, p \in Q, x = a_1 a_2 \cdots a_n, a_i \in \Sigma, i = 1, 2, \ldots, n$. Let $f_x(p) = q$ and the sequence of membership values in the path be $\mu_A(p, a_1, p_1), \mu_A(p_1, a_2, p_2), \ldots, \mu_A(p_{n-1}, a_n, p_n)$, where $p_n = q$. Define $\mu_{A_1} : F(\mathscr{A}) \to [0, 1]$ by

$\mu_{A_1}(f_{a_1 a_2 \cdots a_n}) = \vee\{\mu_A(p, a_1, p_1) \wedge \mu_A(p_1, a_2.p_2) \wedge \cdots \wedge \mu_A(p_{n-1}, a_n, p_n) \mid p \in Q\}$.

Therefore $\mu_{A_1}(f_x \circ f_y) = \mu_{A_1}(f_{yx}) \geq \wedge(\mu_{A_1}(f_y), \mu_{A_1}(f_x))$. Define $\nu_{A_1} : F(\mathscr{A}) \to [0, 1]$ by $\nu_{A_1}(f_{a_1 a_2 \cdots a_n}) = \wedge\{\nu_A(p, a_1, p_1) \vee \nu_A(p_1, a_2.p_2) \vee \cdots \vee \nu_A(p_{n-1}, a_n, p_n) \mid p \in Q\}$.

Therefore $\nu_{A_1}(f_x \circ f_y) = \nu_{A_1}(f_{yx}) \geq \wedge(\nu_{A_1}(f_y), \nu_{A_1}(f_x))$. Thus $(F(\mathscr{A}), \circ)$ is an intuitionistic finite fuzzy monoid. ■

Theorem 2. *$F(\mathscr{A})$ is an anti-homomorphic image of Σ^*.*

Proof. Let $\cdot$ be the concatenation operator. Then $(\Sigma^*, \cdot)$ is a semigroup with identity elememt λ. Define $\phi : (\Sigma^*, \cdot) \to (F(\mathscr{A}), \circ)$ by $\phi(x) = f_x \forall x \in \Sigma^*$.

Clearly ϕ is well-defined, since $x = y \in \Sigma^*$, implies that $f_x(p) = f_y(p) \forall p \in Q$, therefore $f_x = f_y$. Hence $\phi(x) = \phi(y)$. Again,for $x, y \in \Sigma^*$, we have $\phi(xy) = f_{xy} = f_y \circ f_x = \phi(y) \circ \phi(x)$ and $\phi(\lambda) = f_\lambda$. Also for each $y \in F(\mathscr{A})$, there exists $x \in \Sigma^*$ such that $y = f_x = \phi(x)$, which implies ϕ is onto. Thus $F(\mathscr{A})$ ia an anti-homomorphic image of Σ^*. ■

Theorem 3. *Let* $\mathscr{A} = (Q, \Sigma, A, i, f)$ *be an IFAUM. Define a relation* $\equiv$ *on* Σ^* *by* $x \equiv y$ *if and only if* $f_x = f_y, \forall x, y \in \Sigma^*$. *Then* $\equiv$ *is a congruence relation on* Σ^*.

Proof. Let $x \in \Sigma^*, f_x = f_y$, implies that $x \equiv x$. Therefore $\equiv$ is reflexive. Let $x, y \in \Sigma^*$ and $x \equiv y$. Therefore $f_x = f_y$, implies that $f_y = f_x$. Therefore $y \equiv x$. Hence $\equiv$ is symmetric. Let $x, y, z \in \Sigma^*, x \equiv y$ and $y \equiv z$. Therefore $f_x = f_y$ and $f_y = f_z$, implies that $f_x = f_z$. Therefore $x \equiv z$. Thus $\equiv$ is transitive.

Hence $\equiv$ is an equivalence relation. We show that $\equiv$ is a congruence relation. Let $x, y \in \Sigma^*$ and $x \equiv y$. Then $f_x = f_y$, that is, $f_x(p) = f_y(p) \forall p \in Q$. Now for any $z \in \Sigma^*, f_{zx}(p) = f_x(f_z(p)) = f_y(f_z(p)) = f_{zy}(p) \forall p \in Q$. Therefore $f_{zx} = f_{zy}$, and so $zx \equiv zy$. Similarly $xz \equiv yz$. Therefore $\equiv$ ia a congruence relation on Σ^*. ∎

Example 1. Let $\mathscr{A} = (Q, \Sigma, A, i, f)$ be an IFAUM, where $Q = \{q_1, q_2, q_3\}$ and $\Sigma = \{a, b\}$. Define $A = (\mu_A, \nu_A)$ such that $\mu_A : Q \times \Sigma \times Q \to [0, 1]$ and $\nu_A : Q \times \Sigma \times Q \to [0, 1]$ by

$$\begin{array}{llll} \mu_A(q_1, a, q_1) & = 0.3 & \nu_A(q_1, a, q_1) & = 0.4 \\ \mu_A(q_1, a, q_2) & = 0.3 & \nu_A(q_1, a, q_2) & = 0.5 \\ \mu_A(q_1, b, q_1) & = 0.7 & \nu_A(q_1, b, q_1) & = 0.2 \\ \mu_A(q_2, b, q_1) & = 0.5 & \nu_A(q_2, b, q_1) & = 0.2 \\ \mu_A(q_2, b, q_2) & = 0.5 & \nu_A(q_2, b, q_2) & = 0.3 \\ \mu_A(q_2, a, q_3) & = 0.4 & \nu_A(q_2, a, q_3) & = 0.3 \end{array}$$

$i : Q \to [0, 1]$ such that $i_{\mu_A}(q_1) = 1$ and $i_{\nu_A}(q_1) = 0.1$
$f : Q \to [0, 1]$ such that $f_{\mu_A}(q_3) = 1$ and $f_{\nu_A}(q_3) = 0$.
An intuitionistic fuzzy behavior of $\mathscr{A}$ is $L_{\mu_{\mathscr{A}}} : \Sigma^* \to [0, 1]$ and $L_{\nu_{\mathscr{A}}} : \Sigma^* \to [0, 1]$ such that

$$L_{\mu_{\mathscr{A}}}(x) = \begin{cases} 0.3, & \text{if } x \in \{a, b\}^* aa \\ 0.3, & \text{if } x \in \{a, b\}^* aba \\ 0, & \text{otherwise.} \end{cases}$$

Now, $aba \equiv aaba$. Since

$$\begin{array}{llllll} f^{\mu}_{aba}(q_1) & = q_1 & f^{\mu}_{aba}(q_2) & = q_3 & f^{\mu}_{aba}(q_3) & = q_3 \\ f^{\mu}_{aaba}(q_1) & = q_1 & f^{\mu}_{aaba}(q_2) & = q_3 & f^{\mu}_{aaba}(q_3) & = q_3. \end{array}$$

However, aba is not congruent to ba, since $f^{\mu}_{aba}(q_2) = q_3$ while $f^{\mu}_{ba}(q_2) = q_1$. Note that,any two strings beginning with a are equivalent and any two strings ending with b are equivalent and any other two strings are not equivalent.Therefore, there are only three equivalence classes namely, $[\lambda], [a\{a, b\}^*], [b\{a, b\}^*]$.

Also, $\Sigma^* = [\lambda] \cup [a\{a, b\}^*] \cup [b\{a, b\}^*]$. Next we compute $F(\mathscr{A}) = \{f_x \mid x \in \Sigma^*\}$. We have $f_a = f_{a\{a,b\}^*}, f_b = f_{b\{a,b\}^*}$. Therefore $E(\mathscr{A}) = \{f_\lambda, f_a, f_b\}$. The following is the required table for operations.

$*$	f_λ	f_a	f_b
f_λ	f_λ	f_a	f_b
f_a	f_a	f_{aa}	f_{ba}
f_b	f_b	f_{ab}	f_{bb}

But $f_{aa} = f_a, f_{bb} = f_b, f_{ba} = f_b, f_{ab} = f_a$. Therefore the operation table changes as follows.

$*$	f_λ	f_a	f_b
f_λ	f_λ	f_a	f_b
f_a	f_a	f_a	f_b
f_b	f_b	f_a	f_b

Thus $(F(\mathscr{A}), *)$ is an intuitionistic fuzzy finite semigroup with identity.

Theorem 4. *Let $\mathscr{A} = (Q, \Sigma, A, i, f)$ be an IFAUM, then $\Sigma^*/\equiv$ is a monoid.*

Proof. Let $x \in \Sigma^*, [x] = \{y \in \Sigma^* \mid x \equiv y\}, \Sigma^*/\equiv = \{[x] \mid x \in \Sigma^*\}$. Define a binary operation $*$ on $\Sigma^*/\equiv$ by $\forall\, [x], [y] \in \Sigma^*/\equiv, [x] * [y] = [yx]$. Clearly $*$ is well defined. Let $[x], [y], [z] \in \Sigma^*/\equiv$. Now $[x] * ([y] * [z]) = [x] * [zy] = [zyx] = [yx] * [z] = ([x] * [y]) * [z]$. Therefore $*$ is associative. For $\lambda \in \Sigma^*$, we have $[\lambda] \in \Sigma^*/\equiv$, and for any $[x] \in \Sigma^*/\equiv$, $[x] * [\lambda] = [\lambda x] = [x\lambda] = [\lambda] * [x]$. Thus $[\lambda]$ is the identity element which is in $\Sigma^*/\equiv$. Hence $\Sigma^*/\equiv$. is a monoid. ∎

Theorem 5. *Let $\mathscr{A} = (Q, \Sigma, A, i, f)$ be an IFAUM, then $\Sigma^*/\equiv$ is isomorphic to $F(\mathscr{A})$.*

Proof. Let $\mathscr{A} = (Q, \Sigma, A, i, f)$ be an IFAUM. $\Sigma^*/\equiv = \{[x] \mid x \in \Sigma^*\}$ and $F(\mathscr{A}) = \{f_x \mid x \in \Sigma^*.\}$ $(\Sigma^*/\equiv, *)$ and $(F(\mathscr{A}), *)$ are monoids. Define $\phi : (\Sigma^*, \cdot) \to (F(\mathscr{A}), *)$ by $\phi(x) = f_x \forall x \in \Sigma^*$. By Theorem 4, ϕ is an anti-epimorphism. Define $g : (\Sigma^*/\equiv, *) \to (F(\mathscr{A}), *)$ by $g[x] = \phi(x) \forall [x] \in \Sigma^*/\equiv$.

We show that, g is well defined. Let $[x], [y] \in \Sigma^*/\equiv$. Now $[x] = [y]$, implies that $x \equiv y$. Therefore $f_x = f_y$. Since ϕ is onto, there exists $x, y \in \Sigma^*$ such that $\phi(x) = f_x$ and $\phi(y) = f_y$.

Therefore $\phi(x) = \phi(y)$, implies that $g[x] = g[y]$. Thus g is well defined. Now we prove g is a homomorphism. Let $[x], [y] \in \Sigma^*/\equiv$. $g([x] * [y]) = g[yx] = \phi(yx) = f_x * f_y = g[x] * g[y]$ and $g([\lambda]) = \phi(\lambda) = f_\lambda$. Therefore g is a homomorphism of monoids. To prove g is one-one, take $[x], [y] \in \Sigma^*/\equiv$ and $g[x] = g[y]$. Now $\phi(x) = \phi(y)$, implies that $f_x = f_y$. Therefore $x \equiv y$, implies that $[x] = [y]$ Therefore g is one-one. Finally we prove, g is onto. Let $f_x \in F(\mathscr{A})$. Since ϕ is onto there exists an $x \in \Sigma^*$ such that $\phi(x) = f_x$, which implies that $g[x] = f_x$. Hence g is an isomorphism of monoids. ■

Definition 7. Let $\mathscr{A} = (Q, \Sigma, A, i, f)$ be an IFAUM. For $p, q \in Q$, $x \in \Sigma^*$, $x = a_1 a_2 \cdots a_n$. Let $f_x(p) = q$ and the sequence of membership values in the path be $\mu_A(p, a_1, p_1), \mu_A(p_1, a_2, p_2), \ldots, \mu_A(p_{n-1}, a_n, p_q)$. Define $\mu_A(p, x, q) = \mu_A(p, a_1, p_1) \wedge \mu_A(p_1, a_2, p_2) \wedge \cdots \wedge \mu_A(p_{n-1}, a_n, q)$. Therefore $\mu_A^* : Q \times \Sigma^* \times Q \to [0, 1]$ is defined as follows:

$$\mu_A(p, a, q) = \begin{cases} \mu_A(p, a, q), & \text{if } \mu_A(p, a, q) = \vee\{\mu_A(p, a, r) \mid r \in Q\} \\ 0, & \text{otherwise.} \end{cases}$$

Inductively $\mu_A^*(p, xa, q) = \mu_A^*(p, x, r) \wedge \mu_A(r, a, q)$, for some $r \in Q$ and $\nu_A^*(p, xa, q) = \nu_A^*(p, x, r) \vee \nu_A(r, a, q)$, for some $r \in Q$. Since $\mathscr{A}$ is an IFAUM, r will be unique.

Lemma 1. Let $\mathscr{A} = (Q, \Sigma, A, i, f)$ be an IFAUM. Then for any $x, y \in \Sigma^*$, $\mu_A^*(p, xy, q) = \mu_A^*(p, x, r) \wedge \mu_A^*(r, y, q)$ and $\nu_A^*(p, xy, q) = \nu_A^*(p, x, r) \vee \nu_A^*(r, y, q)$, some $r \in Q \forall p, q \in Q$.

Proof. Let $p, q \in Q$ and $x, y \in \Sigma^*$. We prove the result by induction on $|y| = n$. Let $n = 0$, then $y = \lambda$ and hence $xy = x\lambda = x$. Thus

$\mu_A^*(p, xy, q) = \mu_A^*(p, x\lambda, q) = \mu_A^*(p, x, q) \wedge \mu_A^*(q, \lambda, q)$, since $\mu_A^*(q, \lambda, q) = 1$.

$\nu_A^*(p, xy, q) = \nu_A^*(p, x\lambda, q) = \nu_A^*(p, x, q) \vee \nu_A^*(q, \lambda, q)$, since $\nu_A^*(q, \lambda, q) = 0$.

Therefore $\mu_A^*(p, xy, q) = \mu_A^*(p, x, r) \wedge \mu_A^*(r, y, q)$, such that $r = q \in Q$. and $\nu_A^*(p, xy, q) = \nu_A^*(p, x, r) \vee \nu_A^*(r, y, q)$, such that $r = q \in Q$. Thus the result is true for $n = 0$. Suppose the result is true for all $y \in \Sigma^*$ such that $|y| \leq n - 1$. Let $y = ua$ where $u \in \Sigma^*$ and $|u| = n - 1, n > 0$. Now

$$\begin{aligned} \mu_A^*(p, xy, q) &= \mu_A^*(p, xua, q) \\ &= \mu_A^*(p, xu, s) \wedge \mu_A^*(s, a, q), \textit{some } s \in Q \\ &= \mu_A^*(p, x, r) \wedge \mu_A^*(r, u, s) \wedge \mu_A^*(s, a, q), \textit{some } s, r \in Q \\ &= \mu_A^*(p, x, r) \wedge \mu_A^*(r, ua, q) \\ &= \mu_A^*(p, x, r) \wedge \mu_A^*(r, y, q), r \in Q \end{aligned}$$

and

$$\begin{aligned}\nu_A^*(p,xy,q) &= \mu_A^*(p,xua,q)\\ &= \nu_A^*(p,xu,s)\vee\nu_A^*(s,a,q), some\ s\in Q\\ &= \nu_A^*(p,x,r)\vee\nu_A^*(r,u,s)\vee\nu_A^*(s,a,q), some\ s,r\in Q\\ &= \nu_A^*(p,x,r)\vee\nu_A^*(r,ua,q)\\ &= \nu_A^*(p,x,r)\vee\nu_A^*(r,y,q), r\in Q.\end{aligned}$$

Thus the result is true for $|y| = n$. ■

Definition 8. Let $\mathscr{A} = (Q, \Sigma, A, i, f)$ be an IFAUM. For any $x \in \Sigma^*$ define an intuitionistic fuzzy subset $x^A : Q \times Q \to [0,1]$ by $x_\mu^A(s,t) = \mu_A^*(s,x,t)$ and $x_\nu^A(s,t) = \nu_A^*(s,x,t)\forall s,t \in Q$.

Theorem 6. *Let $\mathscr{A} = (Q, \Sigma, A, i, f)$ be an IFAUM. Let $S_{\mathscr{A}} = \{x^A \mid x \in \Sigma^*\}$. Then $(S_{\mathscr{A}}, \circ)$ is an intuitionistic fuzzy finite monoid.*

Proof. Let $x^A, y^A, z^A \in S_{\mathscr{A}}, \forall p,q \in Q$. Then

$$\begin{aligned}(x^A\circ y^A)_\mu(s,t) &= \vee\{x^A(s,q)\wedge y^A(q,t)\mid r\in Q\}\\ &= \mu_A^*(s,x,q)\wedge\mu_A^*(q,y,t), forsomer\in Q\\ &= \mu_A^*(s,xy,t)\\ &= (x\circ y)^A(s,t)\end{aligned}$$

and

$$\begin{aligned}(x^A\circ y^A)_\nu(s,t) &= \wedge\{x^A(s,q)\vee y^A(q,t)\mid r\in Q\}\\ &= \nu_A^*(s,x,q)\vee\nu_A^*(q,y,t), forsomer\in Q\\ &= \nu_A^*(s,xy,t)\\ &= (x\circ y)^A(s,t).\end{aligned}$$

Thus $(x^A \circ y^A) = (x \circ y)^A$. Therefore $S_{\mathscr{A}}$ is closed under $\circ$. In fact, $(x^A \circ y^A) \circ z^A = (x \circ y)^A \circ z^A = (x \circ y \circ z)^A = x^A \circ (y \circ z)^A = x^A \circ (y^A \circ z^A)$ $\lambda^A \in S_{\mathscr{A}}$, and $(x^A \circ \lambda^A) = (x \circ \lambda)^A = x^A = (\lambda \circ x)^A = (\lambda^A \circ x^A)$. Therefore λ^A is the identity element which is in $S_{\mathscr{A}}$. $\mathrm{Im}(\mu)$ and $\mathrm{Im}(\nu)$ is finite, implies that $S_{\mathscr{A}}$ is finite. Thus $(S_{\mathscr{A}}, \circ)$ is an intuitionistic fuzzy finite group with identity. ■

Theorem 7. *Let $\mathscr{A} = (Q, \Sigma, A, i, f)$. Then $S_{\mathscr{A}}$ and $E(\mathscr{A})$ are anti-isomorphic as monoids.*

Proof. Define $\phi : (S_{\mathscr{A}}, \circ) \to (E(\mathscr{A}), \circ)$ by $\phi(x^A) = f_x \forall x^A \in S_{\mathscr{A}}$. Let $x^A, y^A \in S_{\mathscr{A}}$. Then $x^A = y^A$ if and only if $x^A(s,t) = y^A(s,t)\forall s,t \in Q$ if and only if

$\mu_A^*(s,x,t) = \mu_A^*(s,y,t)$ and $\nu_A^*(s,x,t) = \nu_A^*(s,y,t) \forall s,t \in Q$ if and only if $f_x = f_y$. Hence f is well defined. Now,

$$\begin{aligned}\phi(x^A \circ y^A) &= \phi\left((x \circ y)^A\right) \\ &= f_{xy} \\ &= f_y \circ f_x \\ &= \phi\left(x^A\right) \circ \phi\left(y^A\right).\end{aligned}$$

Also $\phi(\lambda^A) = f_\lambda$. Thus, ϕ is homomorphism. Next, we prove ϕ is one-one. Let $x^A, y^A \in S_{\mathscr{A}}$ and $\phi(x^A) = \phi(y^A)$. Therefore $f_x = f_y$, implies that $f_x(p) = f_y(p) = q \in Q$. Therefore $\mu^*(s,x,t) = \mu^*(s,y,t)$. In IFAUM for $r \neq q, \mu(s,x,r) = \mu(t,y,r) = -0$. Therefore, $\mu(s,x,t) = \mu(s,y,t) \forall s,t \in Q$, implies that $x^A(s,t) = y^A(s.t)$. Hence $x^A = y^A$. Therefore ϕ is one-one. Finally we prove f is onto. Let $f_x \in E(\mathscr{A}), x \in \Sigma^*, x^A \in S_{\mathscr{A}}$. Therefore we have $\phi(x^A) = f_x$. Thus ϕ is onto. Hence $S_{\mathscr{A}} \simeq E(\mathscr{A})$. ■

Definition 9. Let $\mathscr{A} = (Q, \Sigma, A, i, f)$ be an IFAUM. For any $x \in \Sigma^*$ define an intuitionistic fuzzy subset $x^{\mathscr{A}}$ of $Q \times Q$ by $x_\mu^{\mathscr{A}}(s,t) = \mu_A^*(s,x,t)$ and $x_\nu^{\mathscr{A}}(s,t) = \nu_A^*(s,x,t) \forall s,t \in Q$.

Theorem 8. *Let $\mathscr{A} = (Q, \Sigma, A, i, f)$ be an IFAUM. Let $S_{\mathscr{A}} = \{x^{\mathscr{A}} \mid x \in \Sigma^*\}$. Then $(S_{\mathscr{A}}, \circ)$ is an intuitionistic fuzzy finite monoid.*

Proof. Let $x^{\mathscr{A}}, y^{\mathscr{A}}, z^{\mathscr{A}} \in S_{\mathscr{A}}, \forall s,t \in Q$. Then

$$\begin{aligned}(xy)_\mu^{\mathscr{A}}(s,t) &= \mu_A^*(s,xy,t) \\ &= \vee\left\{\mu_A^*(s,x,q) \wedge \mu_A^*(q,y,t) \mid q \in Q\right\} \\ &= \vee\left\{x^{\mathscr{A}}(s,q) \wedge y^{\mathscr{A}}(q,t) \mid q \in Q\right\} \\ &= \left(x^{\mathscr{A}} \circ y^{\mathscr{A}}\right)_\mu(s,t)\end{aligned}$$

and

$$\begin{aligned}(xy)_\nu^{\mathscr{A}}(s,t) &= \nu_A^*(s,xy,t) \\ &= \wedge\left\{\nu_A^*(s,x,q) \vee \nu_A^*(q,y,t) \mid q \in Q\right\} \\ &= \wedge\left\{x^{\mathscr{A}}(s,q) \vee y^{\mathscr{A}}(q,t) \mid q \in Q\right\} \\ &= \left(x^{\mathscr{A}} \circ y^{\mathscr{A}}\right)_\nu(s,t).\end{aligned}$$

Thus $(xy)^{\mathscr{A}} = (x^{\mathscr{A}} * y^{\mathscr{A}})$. Therefore $S_{\mathscr{A}}$ is closed under $\circ$. In fact, $(x^{\mathscr{A}} \circ y^{\mathscr{A}}) \circ z^{\mathscr{A}} = (x \circ y)^{\mathscr{A}} \circ z^{\mathscr{A}} = (x \circ y \circ z)^{\mathscr{A}} = x^{\mathscr{A}} \circ (y \circ z)^{\mathscr{A}} = x^{\mathscr{A}} \circ (y^{\mathscr{A}} \circ z^{\mathscr{A}})$ $\lambda^{\mathscr{A}} \in S_{\mathscr{A}}$, and $(x^{\mathscr{A}} \circ \lambda^{\mathscr{A}}) = (x \circ \lambda)^{\mathscr{A}} = x^{\mathscr{A}} = (\lambda \circ x)^{\mathscr{A}} = (\lambda^{\mathscr{A}} \circ x^{\mathscr{A}})$. Therefore $\lambda^{\mathscr{A}}$ is the identity element which is in $S_{\mathscr{A}}$. Im(μ) and Im(ν) is finite, implies that $S_{\mathscr{A}}$ is finite. Thus $(S_{\mathscr{A}}, \circ)$ is an intuitionistic fuzzy finite monoid. ■

Theorem 9. *Let $\mathscr{A} = (Q, \Sigma, A, i, f)$. Then $S_{\mathscr{A}} \simeq S(\mathscr{A})$, i.e., $S_{\mathscr{A}}$ and $S(\mathscr{A})$ are anti- isomorphic as intuitionistic fuzzy finite monoids.*

Proof. Define $\phi : (S_{\mathscr{A}}, \circ) \to (S(\mathscr{A}), \circ)$ by $\phi(x^{\mathscr{A}}) = f_x \forall x^{\mathscr{A}} \in S_{\mathscr{A}}$. We show that, ϕ is well-defined. Let $x^{\mathscr{A}}, y^{\mathscr{A}} \in S_{\mathscr{A}}$. Now $x^{\mathscr{A}} = y^{\mathscr{A}}$, implies that $x^{\mathscr{A}}(s,t) = y^{\mathscr{A}}(s,t) \forall s, t \in Q$. That is $\mu^*_A(s,x,t) = \mu^*_A(s,y,t)$ and $\nu^*_A(s,x,t) = \nu^*_A(s,y,t) \forall s, t \in Q$ and this implies that $f_x(s) = f_y(s)\ \forall s \in Q$. Therefore $f_x = f_y$, and so $\phi(x^{\mathscr{A}}) = \phi(y^{\mathscr{A}})$. Hence ϕ is well defined. To prove, ϕ is an anti-homomorphism of monoids, let $x^{\mathscr{A}}, y^{\mathscr{A}} \in S_{\mathscr{A}}$. Now,

$$\begin{aligned}
\phi\big(x^{\mathscr{A}} \circ y^{\mathscr{A}}\big) &= \phi\big((x \circ y)^{\mathscr{A}}\big) \\
&= f_{xy} \\
&= f_y \circ f_x \\
&= \phi\big(y^{\mathscr{A}}\big) \circ \phi\big(x^{\mathscr{A}}\big).
\end{aligned}$$

Also $\phi(\lambda^{\mathscr{A}}) = f_\lambda, \lambda^{\mathscr{A}} \in S_{\mathscr{A}}$. Therefore, ϕ is an anti-homomorphism of monoids. Next, we prove ϕ is one-one. Let $x^{\mathscr{A}}, y^{\mathscr{A}} \in S_{\mathscr{A}}$ and $\phi(x^{\mathscr{A}}) = \phi(y^{\mathscr{A}})$. Therefore $f_x = f_y$, implies that $f_x(s) = f_y(s) = t \in Q$. Therefore $\mu^*(s,x,t) = \mu^*(s,y,t)$ and $\nu^*(s,x,t) = \nu^*(s,y,t)$. In IFAUM for $r \neq q, \mu(s,x,r) = \mu(t,y,r) = 0$ and $\nu(s,x,r) = \nu(t,y,r) = 1$.

Therefore, $\mu(s,x,t) = \mu(s,y,t)$ and $\nu(s,x,t) = \nu(s,y,t)\ \forall s,t \in Q$, implies that $x^{\mathscr{A}}(s,t) = y^{\mathscr{A}}(s,t)$. Hence $x^{\mathscr{A}} = y^{\mathscr{A}}$. Therefore ϕ is one-one.

Finally we prove ϕ is onto. Let $f_x \in F(\mathscr{A}), x \in \Sigma^*, x^{\mathscr{A}} \in S_{\mathscr{A}}$. Therefore we have $\phi(x^{\mathscr{A}}) = f_x$. Thus ϕ is onto. Hence $S_{\mathscr{A}} \simeq S(\mathscr{A})$. ∎

4. Conclusion

In this paper, the authors have made an attempt to study the properties of an intuitionistic fuzzy finite monoid. We have made a humble beginning in this direction, however, many concepts are yet to be fuzzyfied in the context of *IFAUM*.

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