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FUZZY DISTRIBUTIVE PAIRS IN FUZZY LATTICES

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Abstract

We generalize the concept of a fuzzy distributive lattice by introducing the concepts of a fuzzy join-distributive pair and a fuzzy join-semidistributive pair in a fuzzy lattice. A relationship among a fuzzy join-distributive pair, a fuzzy join-semidistributive pair and a fuzzy join-modular pair is proved. It is shown that for a pair of fuzzy atoms, the notions of a fuzzy join-distributive pair and a fuzzy join-semidistributive pair coincide.

Keywords: fuzzy lattices, fuzzy modular pair, fuzzy distributive pair, fuzzy semi-distributive pair.

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1. Introduction

The concept of a Boolean algebra has its origin in the work of George Boole on "Laws of Thoughts"(1854). A Boolean algebra is a complemented, distributive lattice. The concept of a distributive lattice is generalized in various directions, such as distributive pairs (Maeda [4]), distributive triples (Maeda [5]), modular pairs in semilattices (Thakare *et al.* [10]).

We know that a lattice $\langle L; \vee; \wedge \rangle$ is distributive iff it satisfies any of the following equivalent conditions for all $a, b, c \in L$.

(i) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$,

(ii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

A well known example of a distributive lattice is the power set of a set (i.e. the collection of all subsets of a set) which is a lattice with set inclusion as the partial order relation.

There are lattices which are not distributive, e.g. the lattice of all normal subgroups of a group is not distributive in general. Also, there are lattices which are not distributive but one of the conditions (i) or (ii) of distributivity may hold for some triplets of elements a, b, c. This gives rise to the concept of a distributive triple (Maeda and Maeda [5, p. 15]). Also, in some lattices, one of the conditions (i) or (ii) may hold for a pair of elements $a, b \in L$ and for all choices of c. This gives rise to the concept of a distributive pair. Maeda [4] has studied the concept of a distributive pair in a lattice. There are nonmodular lattices, in which a pair of elements may exist which satisfies the condition of modularity. It gives rise to the concept of a modular pair of elements. Thakare *et al.* [10] have studied this concept in the context of semilattices.

The concept of a fuzzy set is introduced by Zadeh [13]. Many researchers have introduced fuzzy algebraic structures such as fuzzy groups by Rosenfeld [9], fuzzy lattices by Ajmal *et al.* [1], Chon [2], Mezzomo *et al.* [6, 8]. Wasadikar and Khubchandani [11] have introduced the concept of a fuzzy modular pair in a fuzzy lattice.

In this paper, we define a fuzzy distributive pair, a fuzzy semi-distributive pair in a fuzzy lattice and study relationships among them. The motivation is from the work of Maeda [4]. We also prove that for a pair of fuzzy atoms, the concepts of a fuzzy distributive pair and a fuzzy semi-distributive pair coincide.

2. Preliminaries

Zadeh [14] introduced the concept of a fuzzy binary relation and a fuzzy partial order relation. Throughout this paper, (X, A) denotes a fuzzy lattice, where A is a binary ordering relation on a nonempty set X.

We recall some concepts.

Definition (Chon [2, Definition 2.1]). A mapping $A : X \times X \to [0, 1]$ is called a fuzzy binary relation on X.

Definition (Chon [2, Definition 2.1]). A fuzzy binary relation A on X is called:

(i) fuzzy reflexive: if $A(a, a) = 1$, for all $a \in X$;

(ii) fuzzy symmetric: if $A(a, b) = A(b, a)$, for all $a, b \in X$;

(iii) fuzzy transitive: if $A(a, c) \geq \sup_{b \in X} \min[A(a, b),\ A(b, c)]$;

(iv) fuzzy antisymmetric: if $A(a,b) > 0$ and $A(b,a) > 0$ implies $a = b$.

Definition (Chon [2, Definition 2.1]). Let A be a fuzzy binary relation on X.

(i) A is called a fuzzy equivalence relation on X, if A is fuzzy reflexive, fuzzy symmetric and fuzzy transitive.

(ii) A is called a fuzzy partial order relation, if A is fuzzy reflexive, fuzzy antisymmetric and fuzzy transitive. The pair (X, A) is called a fuzzy partially ordered set or a fuzzy poset.

(iii) A is called a fuzzy total order relation, if it is a fuzzy partial order relation and $A(a,b) > 0$ or $A(b,a) > 0$, for all $a, b \in X$. In this case, the fuzzy poset (X, A) is called a fuzzy totally ordered set or a fuzzy chain.

Several researchers have studied fuzzy lattices. Chon [2] and some others use the terms *upper bound, lower bound* and use the notations $a \vee b$ and $a \wedge b$ to denote the supremum and the infimum of two elements a, b in a fuzzy lattice X in the fuzzy sense. Since the set X is arbitrary, this gives the impression that X itself is a partially ordered set or a lattice. So we use the notations $a \vee_F b$ and $a \wedge_F b$ to denote the fuzzy supremum and the fuzzy infimum of $a, b \in X$.

Definition (Chon [2, Definition 3.1]). Let (X, A) be a fuzzy poset and let $Y \subseteq X$. An element $b \in X$ is said to be a fuzzy upper bound for Y iff $A(a,b) > 0$ for all $a \in Y$. A fuzzy upper bound b_0 for Y is called a least upper bound (or supremum) of Y iff $A(b_0, b) > 0$ for every fuzzy upper bound b for Y. We then write $b_0 = \sup_F Y = \vee_F Y$. If $Y = \{a, b\}$, then we write $\vee_F Y = a \vee_F b$.

Similarly, an element $c \in X$ is said to be a fuzzy lower bound for Y iff $A(c,a) > 0$, for all $a \in Y$. A fuzzy lower bound c_0 for Y is called a fuzzy greatest lower bound (or infimum) of Y iff $A(c, c_0) > 0$ for every fuzzy lower bound c for Y. We then write $c_0 = \inf_F Y = \wedge_F Y$. If $Y = \{a, b\}$, then we write $\wedge_F Y = a \wedge_F b$.

Since A is fuzzy antisymmetric, the fuzzy least upper (fuzzy greatest lower) bound, if it exists, is unique, see Mezzomo *et al.* [6, Remark 3.2].

Definition (Chon [2, Definition 3.2]). Let (X, A) be a fuzzy poset. Then, (X, A) is called a fuzzy lattice if and only if $a \vee_F b$ and $a \wedge_F b$ exist, for all $a, b \in X$.

Definition (Mezzomo *et al.* [7, Definition 3.4]). A fuzzy lattice (X, A) is said to be bounded if there exist elements $\perp$ and $\top$ in X, such that $A(\perp, a) > 0$ and $A(a, \top) > 0$, for every $a \in X$. In this case, $\perp$ and $\top$ are respectively, called bottom and top elements of X.

We illustrate these concepts in the following example. In this example, the fuzzy poset (X, A) is a fuzzy lattice.

Example 1. Let $X = \{\bot, a, b, c, d, e, f, \top\}$. Define a fuzzy relation $A : X \times X \longrightarrow [0,1]$ on X as follows such that

$A(\bot,\bot) = A(a,a) = A(b,b) = A(c,c) = A(d,d) = A(e,e) = A(f,f) = 1$
$A(\top,\top) = 1$,
$A(\bot,a) = 0.1$, $A(\bot,b) = 0.1$, $A(\bot,c) = 0.1$, $A(\bot,d) = 0.1$, $A(\bot,e) = 0.1$, $A(\bot,f) = 0.1$, $A(\bot,\top) = 0.1$,
$A(a,\bot) = 0$, $A(a,b) = 0.5$, $A(a,c) = 0$, $A(a,d) = 0.5$, $A(a,e) = 0.5$, $A(a,f) = 0$, $A(a,\top) = 0.01$,
$A(b,\bot) = 0$, $A(b,a) = 0$, $A(b,c) = 0$, $A(b,d) = 0.5$, $A(b,e) = 0$, $A(b,f) = 0$, $A(b,\top) = 0.01$,
$A(c,\bot) = 0$, $A(c,a) = 0$, $A(c,b) = 0.5$, $A(c,d) = 0.5$, $A(c,e) = 0$, $A(c,f) = 0.5$, $A(c,\top) = 0.01$,
$A(d,\bot) = 0$, $A(d,a) = 0$, $A(d,b) = 0$, $A(d,c) = 0$, $A(d,e) = 0$, $A(d,f) = 0$, $A(d,\top) = 0.01$,
$A(e,\bot) = 0$, $A(e,a) = 0$, $A(e,b) = 0$, $A(e,c) = 0$, $A(e,d) = 0$, $A(e,f) = 0$, $A(e,\top) = 0.01$,
$A(f,\bot) = 0$, $A(f,a) = 0$, $A(f,b) = 0$, $A(f,c) = 0$, $A(f,d) = 0$, $A(f,e) = 0$, $A(f,\top) = 0.01$,
$A(\top,\bot) = 0$, $A(\top,a) = 0$, $A(\top,b) = 0$, $A(\top,c) = 0$, $A(\top,d) = 0$, $A(\top,e) = 0$, $A(\top,f) = 0$.

This fuzzy relation is shown in the following table:

A	$\bot$	a	b	c	d	e	f	$\top$
$\bot$	1.0	0.1	0.1	0.1	0.1	0.1	0.1	0.1
a	0.0	1.0	0.5	0.0	0.5	0.5	0.0	0.01
b	0.0	0.0	1.0	0.0	0.5	0.0	0.0	0.01
c	0.0	0.0	0.5	1.0	0.5	0.0	0.5	0.01
d	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.01
e	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.01
f	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.01
$\top$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0

The fuzzy join and fuzzy meet tables are as follows:

$\vee_F$	$\bot$	a	b	c	d	e	f	$\top$
$\bot$	$\bot$	a	b	c	d	e	f	$\top$
a	a	a	b	b	d	e	$\top$	$\top$
b	b	b	b	b	d	$\top$	$\top$	$\top$
c	c	b	b	c	d	$\top$	f	$\top$
d	d	d	d	d	d	$\top$	$\top$	$\top$
e	e	e	$\top$	$\top$	$\top$	e	$\top$	$\top$
f	f	$\top$	$\top$	f	$\top$	$\top$	f	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

$\wedge_F$	$\bot$	a	b	c	d	e	f	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	a	$\bot$	a	a	$\bot$	a
b	$\bot$	a	b	c	b	a	c	b
c	$\bot$	$\bot$	c	c	c	$\bot$	c	c
d	$\bot$	a	b	c	d	a	c	d
e	$\bot$	a	a	$\bot$	a	e	$\bot$	e
f	$\bot$	$\bot$	c	c	c	$\bot$	f	f
$\top$	$\bot$	a	b	c	d	e	f	$\top$

We note that (X, A) is a fuzzy lattice.

The following result is from Chon [2, Proposition 3.3] and Mezzomo *et al.* [6, Proposition 3.5].

Proposition 2. *Let (X, A) be a fuzzy lattice. For $a, b, c \in X$. The following statements hold:*

(i) $A(a, a \vee_F b) > 0$, $A(b, a \vee_F b) > 0$, $A(a \wedge_F b, a) > 0$, $A(a \wedge_F b, b) > 0$.

(ii) $A(a, c) > 0$ *and* $A(b, c) > 0$ *implies* $A(a \vee_F b, c) > 0$.

(iii) $A(c, a) > 0$ *and* $A(c, b) > 0$ *implies* $A(c, a \wedge_F b) > 0$.

(iv) $A(a, b) > 0$ *iff* $a \vee_F b = b$.

(v) $A(a, b) > 0$ *iff* $a \wedge_F b = a$.

(vi) *If* $A(b, c) > 0$, *then* $A(a \wedge_F b, a \wedge_F c) > 0$ *and* $A(a \vee_F b, a \vee_F c) > 0$.

(vii) *If* $A(a \vee_F b, c) > 0$, *then* $A(a, c) > 0$ *and* $A(b, c) > 0$.

(viii) *If* $A(a, b \wedge_F c) > 0$, *then* $A(a, b) > 0$ *and* $A(a, c) > 0$.

Corollary 3. *Let (X, A) be a fuzzy lattice and $a, b, c, d \in X$. If $A(c, a) > 0$ and $A(d, b) > 0$, then $A(c \wedge_F d, a \wedge_F b) > 0$ and $A(c \vee_F d, a \vee_F b) > 0$.*

Proof. Since $A(c, a) > 0$ holds, by (vi) of Proposition 2, we get $A(c \wedge_F d, a \wedge_F d) > 0$. Similarly, from $A(d, b) > 0$ we get $A(a \wedge_F d, a \wedge_F b) > 0$. Hence by fuzzy transitivity of A, we conclude that $A(c \wedge_F d, a \wedge_F b) > 0$. Similarly, we can show that $A(c \vee_F d, a \vee_F b) > 0$. ■

Chon [2] has considered fuzzy distributivity and fuzzy modularity in fuzzy lattices.

Definition (Chon [2])**.** Let (X, A) be a fuzzy lattice. (X, A) is called a fuzzy distributive lattice, if $a \wedge_F (b \vee_F c) = (a \wedge_F b) \vee_F (a \wedge_F c)$ and $a \vee_F (b \wedge_F c) = (a \vee_F b) \wedge_F (a \vee_F c)$ for all $a, b, c \in X$.

Definition (Chon [2])**.** A fuzzy lattice (X, A) is called fuzzy modular if $A(a, c) > 0$ implies $a \vee_F (b \wedge_F c) = (a \vee_F b) \wedge_F c$, for all $a, b, c \in X$.

3. FUZZY DISTRIBUTIVE PAIRS AND FUZZY SEMIDISTRIBUTIVE PAIRS

Chon [2] and Yuan and Wu [12] have proved results related to fuzzy distributive lattices.

The notion of a distributive pair in a lattice, introduced by Maeda [4], has motivated us to introduce and study fuzzy join-distributive pairs and fuzzy meet-distributive pairs in fuzzy lattices.

Definition. Let (X, A) be a fuzzy lattice. A pair of elements $a, b \in X$ is called a fuzzy join-distributive pair, denoted by $(a, b)_F D_j$, if the following condition holds:
$(a, b)_F D_j$: If $c \in X$ is such that $A(c, a \vee_F b) > 0$, then there exist $a_1, b_1 \in X$ satisfying $A(a_1, a) > 0$, $A(b_1, b) > 0$ and $a_1 \vee_F b_1 = c$.

Dually we can define the concept of a fuzzy meet-distributive pair, $(a, b)_F D_m$. We prove some equivalent forms of $(a, b)_F D_j$.

Theorem 4. *In a fuzzy lattice (X, A), the following conditions are equivalent:*

(i) *A pair (a, b) of elements in X satisfies $(a, b)_F D_j$.*

(ii) *$A((a \vee_F b) \wedge_F c, (a \wedge_F c) \vee_F (b \wedge_F c)) > 0$ for every $c \in X$.*

(iii) *$(a \vee_F b) \wedge_F c = (a \wedge_F c) \vee_F (b \wedge_F c)$ for every $c \in X$.*

Proof. (i)⇒(ii) Suppose that $(a, b)_F D_j$ holds, i.e., if $x \in X$ is such that $A(x, a \vee_F b) > 0$ holds, then there exist $a_1, b_1 \in X$ such that

$$A(a_1, a) > 0,\ A(b_1, b) > 0 \text{ and } a_1 \vee_F b_1 = x. \tag{3.1}$$

By (i) of Proposition 2, for any $x \in X$ we have

$$A((a \vee_F b) \wedge_F x, a \vee_F b) > 0 \text{ and } A((a \vee_F b) \wedge_F x, x) > 0. \tag{3.2}$$

Let $c \in X$. Taking $x = c$ in (3.2), we get

$$A((a \vee_F b) \wedge_F c, a \vee_F b) > 0.$$

As $(a, b)_F D_j$ holds, by (3.1), there exist $a_1, b_1 \in X$ such that

$$A(a_1, a) > 0, A(b_1, b) > 0 \text{ and } a_1 \vee_F b_1 = (a \vee_F b) \wedge_F c.$$

From $a_1 \vee_F b_1 = (a \vee_F b) \wedge_F c$ we conclude that $A(a_1, c) > 0$ and $A(b_1, c) > 0$. Using $A(a_1, a) > 0$, $A(a_1, c) > 0$, and (iii) of Proposition 2, we get $A(a_1, a \wedge_F c) > 0$. Similarly, from $A(b_1, c) > 0$ and $A(b_1, b) > 0$, we get $A(b_1, b \wedge_F c) > 0$. Hence by Corollary 3 we have

$$A(a_1 \vee_F b_1, (a \wedge_F c) \vee_F (b \wedge_F c)) > 0.$$

Since $a_1 \vee_F b_1 = (a \vee_F b) \wedge_F c$, we conclude that

$$A((a \vee_F b) \wedge_F c, (a \wedge_F c) \vee_F (b \wedge_F c)) > 0.$$

(ii)⇒(i) Let $c \in X$ be such that $A(c, a \vee_F b) > 0$. Since (ii) holds for any $c \in X$, we have

$$A((a \vee_F b) \wedge_F c, (a \wedge_F c) \vee_F (b \wedge_F c)) > 0. \tag{3.3}$$

By (i) of Proposition 2 we have

$$A(a \wedge_F c, a) > 0 \text{ and } A(b \wedge_F c, b) > 0.$$

This implies by Corollary 3 that

$$A((a \wedge_F c) \vee_F (b \wedge_F c), a \vee_F b) > 0. \tag{3.4}$$

Also, from

$$A(a \wedge_F c, c) > 0 \text{ and } A(b \wedge_F c, c) > 0,$$

we have by Corollary 3 that

$$A((a \wedge_F c) \vee_F (b \wedge_F c), c) > 0. \tag{3.5}$$

Thus from (3.4), (3.5) and by Corollary 3 we get

$$A((a \wedge_F c) \vee_F (b \wedge_F c), (a \vee_F b) \wedge_F c) > 0. \tag{3.6}$$

From (3.3) and (3.6) by fuzzy antisymmetry of A, we get

$$(a \wedge_F c) \vee_F (b \wedge_F c) = (a \vee_F b) \wedge_F c. \tag{3.7}$$

From (v) of Proposition 2 and $A(c, a \vee_F b) > 0$, we get $(a \vee_F b) \wedge_F c = c$. Putting $a_1 = a \wedge_F c$ and $b_1 = b \wedge_F c$ and using (3.7) we note that a_1, b_1 satisfy

$$A(a_1, a) > 0,\ A(b_1, b) > 0 \text{ and } a_1 \vee_F b_1 = c.$$

Thus $(a, b)_F D_j$ holds.

(ii)⇒(iii) We note that for any $a, b, c \in X$, $A(a \wedge_F c, (a \vee_F b) \wedge_F c) > 0$ and $A(b \wedge_F c, (a \vee_F b) \wedge_F c) > 0$. Hence by using Corollary 3, we get

$$A((a \wedge_F c) \vee_F (b \wedge_F c), (a \vee_F b) \wedge_F c) > 0.$$

By (ii) we have

$$A((a \vee_F b) \wedge_F c, (a \wedge_F c) \vee_F (b \wedge_F c)) > 0.$$

Hence by fuzzy antisymmetry of A, we get

$$(a \vee_F b) \wedge_F c = (a \wedge_F c) \vee_F (b \wedge_F c).$$

(iii)⇒(ii) The proof is obvious. ∎

Lemma 5. *Let (X, A) be a fuzzy lattice. Then the following statements hold:*

(i) *For any $a, b \in X$, if $A(a, b) > 0$, then $(a, b)_F D_j$.*

(ii) *For any* $a, b \in X$*, the following statements hold:* $(a \wedge_F b, a)_F D_j$, $(a \wedge_F b, b)_F D_j$, $(a, a \vee_F b)_F D_j$, $(b, a \vee_F b)_F D_j$, $(a \wedge_F b, a \vee_F b)_F D_j$.

(iii) *If* X *has the elements* $\perp$ *and* $\top$*, then for every* $a \in X$*,* $(\perp, a)_F D_j$ *and* $(a, \top)_F D_j$ *hold.*

Proof. (i) Let $A(a, b) > 0$. This implies that $a \vee_F b = b$. To show $(a, b)_F D_j$, we need to show that if $x \in X$ is such that $A(x, a \vee_F b) > 0$, then there exist $a_1, b_1 \in X$ such that

$$A(a_1, a) > 0,\ A(b_1, b) > 0 \text{ and } a_1 \vee_F b_1 = x.$$

Let $c \in X$ be such that $A(c, a \vee_F b) > 0$. Since $a \vee_F b = b$, we get $A(c, b) > 0$ and so $b \wedge_F c = c$. For any $a \in X$, we always have $c = c \vee_F (a \wedge_F c)$. Since $c = b \wedge_F c$, we get

$$c = (b \wedge_F c) \vee_F (a \wedge_F c).$$

Put $a_1 = a \wedge_F c$ and $b_1 = b \wedge_F c$. Then by (v) of Proposition 2, $A(a_1, a) > 0$ and $A(b_1, b) > 0$ hold. Thus we have $a_1 \vee_F b_1 = c$. Hence $(a, b)_F D_j$ holds.

(ii) Follows from (i) of Proposition 2 and (i).

(iii) Follows from (i). ∎

Now we give a generalization of the concept of a fuzzy join-distributive pair.

Definition. Let (X, A) be a fuzzy lattice. A pair of elements $a, b \in X$ is called a fuzzy join-semidistributive pair, denoted by $(a, b)_F SD_j$, if the following condition holds:

$(a, b)_F SD_j$: Let $c \in X$ be such that $A(c, a \vee_F b) > 0$. Then there exists $a_1 \in X$ satisfying $A(a_1, a) > 0$, $A(a_1, c) > 0$ and $A(c, a_1 \vee_F b) > 0$.

Dually, we have the concept of a fuzzy meet-semidistributive pair, denoted by $(a, b)_F SD_m$.

Lemma 6. *In any fuzzy lattice* (X, A)*, if* $(a, b)_F D_j$ *holds, then* $(a, b)_F SD_j$ *holds.*

Proof. Suppose that for some $a, b \in X$, $(a, b)_F D_j$ holds, i.e., if $c \in X$ is such that $A(c, a \vee_F b) > 0$, then there exist $a_1, b_1 \in X$ such that

$$A(a_1, a) > 0,\ A(b_1, b) > 0 \text{ and } a_1 \vee_F b_1 = c. \tag{3.8}$$

To show that $(a, b)_F SD_j$ holds, i.e., to show that, if $x \in X$ is such that $A(x, a \vee_F b) > 0$ holds, then there exists $d \in X$ such that

$$A(d, a) > 0,\ A(d, x) > 0 \text{ and } A(x, d \vee_F b) > 0. \tag{3.9}$$

Suppose that $c \in X$ is such that $A(c, a \vee_F b) > 0$ holds. Since $(a,b)_F D_j$ holds, by (3.8), there exist $a_1, b_1 \in X$ such that

$$A(a_1, a) > 0,\ A(b_1, b) > 0 \text{ and } a_1 \vee_F b_1 = c. \tag{3.10}$$

As $a_1 \vee_F b_1 = c$ we get $A(a_1, c) > 0$. Since $A(b_1, b) > 0$ and $A(a_1, a_1) > 0$, by Corollary 3, we get $A(a_1 \vee_F b_1, a_1 \vee_F b) > 0$. Hence $A(c, a_1 \vee_F b) > 0$. Thus a_1 satisfies

$$A(a_1, a) > 0,\ A(a_1, c) > 0 \text{ and } A(c, a_1 \vee_F b) > 0.$$

Hence $(a,b)_F SD_j$ holds. ■

The following example shows that in a fuzzy lattice,

(i) there may exist a pair of elements which is not a fuzzy semi-distributive pair

(ii) there may exist a pair of elements x, y for which $(x,y)_F SD_j$ holds but $(x,y)_F D_j$ does not hold.

Thus "fuzzy semi-distributivity"is weaker than "fuzzy distributivity".

Example 7. Consider the fuzzy lattice in Example 1.

(i) We show that the pair (e,b) in Example 1 is not a fuzzy semi-distributive pair. If $(e,b)_F SD_j$ holds, then we need to show that for every x satisfying $A(x, e \vee_F b) > 0$, there exists a_1 such that

$$A(a_1, e) > 0,\ A(a_1, x) > 0 \text{ and } A(x, a_1 \vee_F b) > 0.$$

We note that for $x = f$, $A(f, e \vee_F b) = A(f, \top) > 0$ holds. The only element a_1 satisfying both $A(a_1, e) > 0$, $A(a_1, f) > 0$ is $a_1 = \perp$. For $a_1 = \perp$ we have $A(f, a_1 \vee_F b) = A(f, b)$ but $A(f, b) > 0$ does not hold. Thus $(e,b)_F SD_j$ does not hold.

(ii) We show that $(e,f)_F SD_j$ holds but $(e,f)_F D_j$ does not hold.

(I) To show that $(e,f)_F SD_j$ holds, we need to show that if $x \in X$ is such that $A(x, e \vee_F f) > 0$, then there exists $a_1 \in X$ such that

$$A(a_1, x) > 0, A(a_1, e) > 0 \text{ and } A(x, a_1 \vee_F f) > 0. \tag{3.11}$$

We note $e \vee_F f = \top$.

Hence the possible choices for x satisfying $A(x, e \vee_F f) > 0$ are $x = \perp, a, b, c, d, e, f, \top$.

For $x = \perp$, we take $a_1 = \perp$ and it satisfies all the conditions in (3.11).
For $x = a$, we take $a_1 = a$ and it satisfies all the conditions in (3.11).
For $x = b$, we take $a_1 = a$ and it satisfies all the conditions in (3.11).
For $x = c$, we take $a_1 = \perp$ and it satisfies all the conditions in (3.11).

For $x = d$, we take $a_1 = a$ and it satisfies all the conditions in (3.11).
For $x = e$, we take $a_1 = a$ and it satisfies all the conditions in (3.11).
For $x = f$, we take $a_1 = \bot$ and it satisfies all the conditions in (3.11).
For $x = \top$, we take $a_1 = a$ and it satisfies all the conditions in (3.11).

Thus $(e, f)_F SD_j$ holds.

(II) We claim that $(e, f)_F D_j$ does not hold. We observe that $e \vee_F f = \top$ and $A(d, e \vee_F f) = A(d, \top) > 0$. Suppose that there exist $a_1, b_1 \in X$ such that $A(a_1, e) > 0$, $A(b_1, f) > 0$. We note that $A(a_1, e) > 0$ is satisfied only for $a_1 = \bot$ or a or e. The only elements satisfying $A(b_1, f) > 0$ are $b_1 = \bot$ or c or f. Hence the possible choices for $a_1 \vee_F b_1$ are $\bot$ or a or b or e or f or $a \vee_F c = b$ or $a \vee_F f = \top$, $e \vee_F c = \top$ or $e \vee_F f = \top$. Thus there do not exist $a_1, b_1 \in X$ such that

$$A(a_1, e) > 0,\ A(b_1, f) > 0 \text{ and } a_1 \vee_F b_1 = d.$$

Hence $(e, f)_F D_j$ does not hold.

We prove some equivalent forms of $(a, b)_F SD_j$.

Theorem 8. *Let $a, b \in X$. The following conditions are equivalent:*

(i) *(a, b) is a fuzzy join-semidistributive pair.*

(ii) *$A(\{(a \vee_F b) \wedge_F c\} \vee_F b, (a \wedge_F c) \vee_F b) > 0$ for every $c \in X$.*

(iii) *$\{(a \vee_F b) \wedge_F c\} \vee_F b = (a \wedge_F c) \vee_F b$ for every $c \in X$.*

Proof. (i)$\Rightarrow$(ii) Suppose that (a, b) is a fuzzy join-semidistributive pair, i.e., if $x \in X$ is such that $A(x, a \vee_F b) > 0$ holds. Then there exists $a_1 \in X$ such that

$$A(a_1, a) > 0,\ A(a_1, x) > 0 \text{ and } A(x, a_1 \vee_F b) > 0.$$

Let $c \in X$. We note that $A((a \vee_F b) \wedge_F c, a \vee_F b) > 0$. By (i) there exists $a_1 \in X$ such that

$$A(a_1, a) > 0,\ A(a_1, (a \vee_F b) \wedge_F c) > 0 \text{ and } A((a \vee_F b) \wedge_F c, a_1 \vee_F b) > 0.$$

Hence we get

$$A(a_1 \vee_F b, [(a \vee_F b) \wedge_F c] \vee_F b) > 0 \text{ and } A([(a \vee_F b) \wedge_F c] \vee_F b, a_1 \vee_F b) > 0.$$

Hence by fuzzy antisymmetry, we get

$$[(a \vee_F b) \wedge_F c] \vee_F b = a_1 \vee_F b.$$

From $A(a_1, (a \vee_F b) \wedge_F c) > 0$ and $A((a \vee_F b) \wedge_F c, c) > 0$ by fuzzy transitivity of A we have $A(a_1, c) > 0$. Now $A(a_1, a) > 0$, $A(a_1, c) > 0$ imply by (vi) of Proposition 2 that $A(a_1, a \wedge_F c) > 0$ and by (vi) of Proposition 2 we get

$$A(a_1 \vee_F b, (a \wedge_F c) \vee_F b) > 0.$$

Using $[(a \vee_F b) \wedge_F c] \vee_F b = a_1 \vee_F b$, we get

$$A([(a \vee_F b) \wedge_F c] \vee_F b, (a \wedge_F c) \vee_F b) > 0.$$

Since $A((a \vee_F b) \wedge_F c, [(a \vee_F b) \wedge_F c] \vee_F b) > 0$ holds, by fuzzy transitivity of A we have

$$A((a \vee_F b) \wedge_F c, (a \wedge_F c) \vee_F b) > 0.$$

By (vi) of Proposition 2 we get

$$A(\{(a \vee_F b) \wedge_F c\} \vee_F b, (a \wedge_F c) \vee_F b) > 0.$$

Thus, (ii) holds.

(ii)⇒(i) Suppose that (ii) holds. We need to show that (a, b) is a fuzzy join-semidistributive pair, i.e., to show that, if $x \in X$ is such that $A(x, a \vee_F b) > 0$ holds, then there exists $a_1 \in X$ such that

$$A(a_1, a) > 0,\ A(a_1, x) > 0 \text{ and } A(x, a_1 \vee_F b) > 0.$$

Let $c \in X$ be such that $A(c, a \vee_F b) > 0$. Hence by (v) of Proposition 2, $c = (a \vee_F b) \wedge_F c$. By (ii) $A(\{(a \vee_F b) \wedge_F c\} \vee_F b, (a \wedge_F c) \vee_F b) > 0$. Hence by (vii) of Proposition 2, (as $(a \vee_F b) \wedge_F c \leq_F \{(a \vee_F b) \wedge_F c\} \vee_F b$) we get

$$A((a \vee_F b) \wedge_F c, (a \wedge_F c) \vee_F b) > 0. \tag{3.12}$$

Putting $c = (a \vee_F b) \wedge_F c$ in (3.12), we get

$$A(c, (a \wedge_F c) \vee_F b) > 0.$$

Put $a \wedge_F c = a_1$. Then $A(a_1, a) > 0$ and $A(a_1, c) > 0$ and $A(c, a_1 \vee_F b) > 0$. Thus, (i) holds.

(ii)⇒(iii) By using Proposition 2, we note that for all $a, b, c \in X$, $A((a \wedge_F c) \vee_F b, \{(a \vee_F b) \wedge_F c\} \vee_F b) > 0$ always holds. By (ii) for every $c \in X$, $A(\{(a \vee_F b) \wedge_F c\} \vee_F b, (a \wedge_F c) \vee_F b) > 0$ holds. Hence by fuzzy antisymmetry of A, for every $c \in X$ we get

$$\{(a \vee_F b) \wedge_F c\} \vee_F b = (a \wedge_F c) \vee_F b.$$

(iii)⇒(ii) Obvious. ■

We recall some definitions from Wasadikar and Khubchandani [11]. In [11], the term modular pair (respectively, dual modular pair) is used. But we use the terminology from Maeda [4].

Definition [11, Definition 3.1]**.** Let (X, A) be a fuzzy lattice. We say that (a, b) is a fuzzy join-modular pair and we write $(a, b)_F M_j$, if whenever $A(b, c) > 0$ for some $c \in X$, then $(c \wedge_F a) \vee_F b = c \wedge_F (a \vee_F b)$.

Dually, we can define a fuzzy meet-modular pair $(a, b)_F M_m$.

Definition. A fuzzy lattice (X, A) is called a fuzzy modular lattice, if $(a, b)_F M_j$ (equivalently, $(a, b)_F M_m$) holds for all $a, b \in X$.

We prove a characterization for a pair of elements to be a fuzzy join-modular pair.

Theorem 9. *Let (X, A) be a fuzzy lattice. For a pair (a, b) of elements in X, the following conditions are equivalent:*

(i) *(a, b) is a fuzzy join-modular pair.*

(ii) *If $A(b, c) > 0$ and $A(c, a \vee_F b) > 0$, then there exists $a_1 \in X$ such that $A(a_1, a) > 0$ and $a_1 \vee b = c$.*

Proof. (i)⇒(ii) Suppose that (a, b) is a fuzzy join-modular pair, i.e., if $A(b, x) > 0$ for some $x \in X$, then $(x \wedge_F a) \vee_F b = x \wedge_F (a \vee_F b)$. Let $c \in X$ be such that $A(b, c) > 0$ and $A(c, a \vee_F b) > 0$. Since (a, b) is a fuzzy join-modular pair and $A(b, c) > 0$, by (i),

$$(c \wedge_F a) \vee_F b = c \wedge_F (a \vee_F b). \tag{3.13}$$

Since $A(c, a \vee_F b) > 0$, it follows from (v) of Proposition 2, that

$$c \wedge_F (a \vee_F b) = c. \tag{3.14}$$

Put $a_1 = c \wedge_F a$. Then $A(a_1, a) > 0$. From (3.13) and (3.14), we get $a_1 \vee_F b = c$. Thus, (ii) holds.

(ii)⇒(i) To show that (a, b) is a fuzzy join-modular pair, i.e., to show that if $A(b, x) > 0$ for some $x \in X$, then $(x \wedge_F a) \vee_F b = x \wedge_F (a \vee_F b)$. Let $c \in X$ be such that $A(b, c) > 0$. By (i) of Proposition 2, $A(b, a \vee_F b) > 0$ holds. Hence by Corollary 3, $A(b, c \wedge_F (a \vee_F b)) > 0$. Similarly, we get $A(c \wedge_F (a \vee_F b), a \vee_F b) > 0$. By applying (ii) to $A(b, c \wedge_F (a \vee_F b)) > 0$, there exists $d \in X$ such that $A(d, a) > 0$ and $d \vee_F b = c \wedge_F (a \vee_F b)$. From this, we have $A(d, c \wedge_F (a \vee_F b)) > 0$. This implies $A(d, c) > 0$. By (vi) of Proposition 2, we have $A(d \wedge_F a, c \wedge_F (a \vee_F b) \wedge_F a) > 0$, i.e., $A(d \wedge_F a, c \wedge_F a) > 0$. From $A(d, a) > 0$ and $A(d, c) > 0$, by Corollary 3, we have $A(d, c \wedge_F a) > 0$. We have

$$\begin{aligned}(c \wedge_F a) \vee_F \{c \wedge_F (a \vee_F b)\} &= (c \wedge_F a) \vee_F d \vee_F b \\ &= (c \wedge_F a) \vee_F b, \text{ as } A(d, c \wedge_F a) > 0.\end{aligned} \tag{3.15}$$

Since $A(a, a \vee_F b) > 0$ by (vi) of Proposition 2 we have

$$A(c \wedge_F a, c \wedge_F (a \vee_F b)) > 0.$$

Again by (iv) of Proposition 2 we have

$$(c \wedge_F a) \vee_F \{c \wedge_F (a \vee_F b)\} = c \wedge_F (a \vee_F b).$$

Hence from (3.15) we conclude that

$$c \wedge_F (a \vee_F b) = (c \wedge_F a) \vee_F b.$$

Thus, (i) holds. ∎

In the following lemma we prove relationships among a fuzzy distributive pair, a fuzzy semi-distributive pair and a fuzzy join-modular pair.

Lemma 10. *For a pair (a, b) of elements in X, consider the following statements:*

(i) *(a, b) is a fuzzy join-distributive pair.*

(ii) *(a, b) is a fuzzy join-semidistributive pair.*

(iii) *(a, b) is a fuzzy join-modular pair.*

Then the following implications hold:
(i)⇒(ii)⇒(iii).

Proof. (i)⇒(ii) This implication is shown in Lemma 6.

(ii)⇒(iii) Suppose that (a, b) is a fuzzy join-semidistributive pair. To show that (a, b) is a fuzzy join-modular pair, we need to show that if $A(b, x) > 0$ for some $x \in X$, then $(x \wedge_F a) \vee_F b = x \wedge_F (a \vee_F b)$.

Let $c \in X$ be such that $A(b, c) > 0$ and $A(c, a \vee_F b) > 0$. Since (a, b) is a fuzzy join-semidistributive pair, it follows from (ii) that there exists $a_1 \in X$ such that $A(a_1, a) > 0$, $A(a_1, c) > 0$ and $A(c, a_1 \vee_F b) > 0$. By (vi) of Proposition 2 we get $A(a_1 \vee_F b, c \vee_F b) > 0$ and $A(c \vee_F b, a_1 \vee_F b) > 0$. Therefore, by fuzzy antisymmetry of A we get $a_1 \vee_F b = c \vee_F b$. As $A(b, c) > 0$ by (iv) of Proposition 2 we have $a_1 \vee_F b = c$. Thus, by Theorem 9, (iii) holds. ∎

In the following remark we show that the implications in Lemma 10 are not reversible.

Remark 11. (i) We have shown in Example 1, that $(e, f)_F SD_j$ holds but $(e, f)_F D_j$ does not hold.

(ii) We show that in the fuzzy lattice in Example 1 $(e, d)_F M_j$ holds but $(e, d)_F SD_j$ does not hold. To show that $(e, d)_F M_j$ holds, we need to show that for every $x \in X$ satisfying $A(d, x) > 0$, $A(x, e \vee_F d) > 0$, there exists $a_1 \in X$ such that

$$A(a_1, e) > 0 \text{ and } a_1 \vee_F d = x. \quad (3.16)$$

We note that $e \vee_F d = \top$. The possible choices for x satisfying $A(d,x) > 0$, $A(x, e \vee_F d) > 0$ are $x = d, \top$. For $x = d$, we take $a_1 = \bot$ and it satisfies all the conditions in (3.16). For $x = \top$, we take $a_1 = e$ and it satisfies all the conditions in (3.16). Thus $(e,d)_F M_j$ holds. We claim that $(e,d)_F SD_j$ does not hold. We need to show that for some $x \in X$ satisfying $A(x, e \vee_F d) > 0$, there does not exist $a_1 \in X$ such that

$$A(a_1, x) > 0, \ A(a_1, e) > 0 \text{ and } A(x, a_1 \vee_F d) > 0. \tag{3.17}$$

Since $e \vee_F d = \top$, every element $x \in X$ satisfies $A(x, \top) > 0$.

If we consider $x = f$, then we have to find a_1 satisfying $A(a_1, f) > 0$, $A(a_1, e) > 0$ and $A(f, a_1 \vee_F d) > 0$. The only elements satisfying $A(a_1, e) > 0$ are $a_1 = \bot, a, e$. We note that if we take $a_1 = \bot$, then the condition $A(f, a_1 \vee_F d) > 0$ becomes $A(f, d) > 0$ but it does not hold. If we take $a_1 = a$, then the condition $A(a_1, f) > 0$ becomes $A(a, f) > 0$, a contradiction to $A(a, f) = 0$. If we take $a_1 = e$, then the condition $A(a_1, f) > 0$ becomes $A(e, f) > 0$, a contradiction to $A(e, f) = 0$. Thus, there does not exist a_1 satisfying $A(a_1, f) > 0$, $A(a_1, e) > 0$ and $A(f, a_1 \vee_F d) > 0$. Hence $(e,d)_F SD_j$ does not hold.

This shows that the implications in Lemma 10 are not reversible.

Lemma 12. *If (a,b) is a fuzzy join-semidistributive pair and if (b, a_1) is a fuzzy join-modular pair for every a_1 satisfying $A(a_1, a) > 0$, then $(a,b)_F D_j$.*

Proof. Let $A(c, a \vee_F b) > 0$. Since (a,b) is a fuzzy join-semidistributive pair, there exists $a_1 \in X$ such that $A(a_1, a) > 0$, $A(a_1, c) > 0$ and $A(c, a_1 \vee_F b) > 0$. By assumption, (b, a_1) is a fuzzy join-modular pair. Hence there exists $b_1 \in X$ such that $A(b_1, b) > 0$ and $b_1 \vee_F a_1 = c$. Thus (a,b) is a fuzzy join-distributive pair. ∎

From Lemma 12 we conclude that in a fuzzy modular lattice if (a,b) is a fuzzy join-semidistributive pair, then (a,b) is a fuzzy join-distributive pair. We note some elementary properties.

Proposition 13. *In a fuzzy lattice (X, A), the following statements hold:*

(i) *Let $a, b, c \in X$ be such that $(a,b)_F D_j$ and $(a \vee_F b, c)_F D_j$ hold. If $a_1 \in X$ satisfies $A(a, a_1) > 0$ and $A(a_1, a \vee_F c) > 0$, then $(a_1, b \vee_F c)_F D_j$ holds.*

(ii) *Let $a, b, c \in X$ be such that $(a,b)_F SD_j$ and $(a \vee_F b, c)_F SD_j$ hold. If $a_1 \in X$ satisfies $A(a, a_1) > 0$ and $A(a_1, a \vee_F c) > 0$, then $(a_1, b \vee_F c)_F SD_j$ holds.*

Proof. (i) Suppose that $a, b, c \in X$ are such that $(a,b)_F D_j$ and $(a \vee_F b, c)_F D_j$ hold. Let $a_1 \in X$ be such that $A(a, a_1) > 0$ and $A(a_1, a \vee_F c) > 0$. To show that

$(a_1, b \vee_F c)_F D_j$ holds, i.e., to show that if $x \in X$ is such that $A(x, a_1 \vee_F b \vee_F c) > 0$, then there exist $\alpha, \beta \in X$ satisfying

$$A(\alpha, a_1) > 0, A(\beta, b \vee_F c) > 0 \text{ and } \alpha \vee_F \beta = x. \tag{3.18}$$

Suppose that for some $d \in X$, $A(d, a_1 \vee_F b \vee_F c) > 0$. Since $A(a, a_1) > 0$ holds, by (vi) of Proposition 2, we get $A(a \vee_F b, a_1 \vee_F b) > 0$. From this by applying (vi) of Proposition 2, we get $A(a \vee_F b \vee_F c, a_1 \vee_F b \vee_F c) > 0$. Similarly, by applying (vi) of Proposition 2, to $A(a_1, a \vee_F c) > 0$ we get $A(a_1 \vee_F b, a \vee_F c \vee_F b) > 0$ and from this by (vi) of Proposition 2 we get $A(a_1 \vee_F b \vee_F c, a \vee_F c \vee_F b) > 0$. Thus by fuzzy antisymmetry of A we get

$$a_1 \vee_F b \vee_F c = a \vee_F b \vee_F c.$$

Hence we conclude that $A(d, a \vee_F b \vee_F c) > 0$. Since $(a \vee_F b, c)_F D_j$ holds and $A(d, a \vee_F b \vee_F c) > 0$, there exist $e, f \in X$ such that

$$A(e, a \vee_F b) > 0, \ A(f, c) > 0 \text{ and } e \vee_F f = d.$$

Since $(a, b)_F D_j$ and $A(e, a \vee_F b) > 0$, there exist $g, h \in X$ such that

$$A(g, a) > 0, \ A(h, b) > 0 \text{ and } g \vee_F h = e.$$

From $A(h, b) > 0$ and $A(f, c) > 0$ we get by Corollary 3, $A(h \vee_F f, b \vee_F c) > 0$. Put $i = f \vee_F h$. Then $A(i, b \vee_F c) > 0$ and $g \vee_F i = d$. From $A(g, a) > 0$ and $A(a, a_1) > 0$, by fuzzy transitivity of A we get $A(g, a_1) > 0$. Thus we have shown the existence of $g, i \in X$ such that $A(g, a_1) > 0$, $A(i, b \vee_F c) > 0$ and $g \vee_F i = d$. Thus (3.18) holds with $\alpha = g$, $\beta = i$ and $x = d$. Hence $(a_1, b \vee_F c)_F D_j$ holds.

(ii) Suppose that $(a, b)_F SD_j$ and $(a \vee_F b, c)_F SD_j$ hold. Let $a_1 \in X$ be such that $A(a, a_1) > 0$ and $A(a_1, a \vee_F c) > 0$. We have to show that $(a_1, b \vee_F c)_F SD_j$ holds, i.e., to show that if $x \in X$ satisfies $A(x, a_1 \vee_F b \vee_F c) > 0$, then there exists $\alpha \in X$ such that

$$A(\alpha, a_1) > 0, \ A(\alpha, x) > 0 \text{ and } A(x, \alpha \vee_F b \vee_F c) > 0. \tag{3.19}$$

Suppose that for some $d \in X$, $A(d, a_1 \vee_F b \vee_F c) > 0$. Since $A(a, a_1) > 0$, $A(a_1, a \vee_F c) > 0$, by (vi) of Proposition 2, as shown in (i), we get

$$A(a \vee_F b \vee_F c, a_1 \vee_F b \vee_F c) > 0 \text{ and } A(a_1 \vee_F b \vee_F c, a \vee_F b \vee_F c) > 0.$$

By fuzzy antisymmetry of A we get

$$a_1 \vee_F b \vee_F c = a \vee_F b \vee_F c.$$

Hence $A(d, a \vee_F b \vee_F c) > 0$. Since $(a \vee_F b, c)_F SD_j$ and $A(d, a \vee_F b \vee_F c) > 0$ hold, there exists $f \in X$ such that

$$A(f, a \vee_F b) > 0,\ A(f, d) > 0 \text{ and } A(d, f \vee_F c) > 0.$$

Since $(a, b)_F SD_j$ and $A(f, a \vee_F b) > 0$ hold, there exists $e \in X$ such that

$$A(e, a) > 0,\ A(e, f) > 0 \text{ and } A(f, e \vee_F b) > 0.$$

From $A(e, a) > 0$ and $A(a, a_1) > 0$ by fuzzy transitivity of A we get $A(e, a_1) > 0$. From $A(e, f) > 0$ and $A(f, d) > 0$ by fuzzy transitivity of A we get $A(e, d) > 0$. As $A(f, e \vee_F b) > 0$ holds, by (vi) of Proposition 2, we get

$$A(f \vee_F c, e \vee_F b \vee_F c) > 0. \tag{3.20}$$

We have noted above that

$$A(d, f \vee_F c) > 0. \tag{3.21}$$

From (3.20) and (3.21) by fuzzy transitivity of A we get

$$A(d, e \vee_F b \vee_F c) > 0.$$

Thus we have shown the existence of $e \in X$ such that

$$A(e, a_1) > 0, A(e, d) > 0 \text{ and } A(d, e \vee_F b \vee_F c) > 0.$$

Thus (3.19) holds with $x = d$, $\alpha = e$. Hence $(a_1, b \vee_F c)_F SD_j$ holds. ■

Proposition 14. *Let (X, A) be a fuzzy lattice and $a, a_1, b, b_1 \in X$.*

(i) *If $(a_1, b)_F D_j$ and $(a_2, b)_F D_j$ hold, then $(a_1 \wedge_F a_2, b)_F D_j$ holds.*

(ii) *If $(a_1, b)_F SD_j$ and $(a_2, b)_F SD_j$ hold, then $(a_1 \wedge_F a_2, b)_F SD_j$ holds.*

Proof. (i) Suppose that $(a_1, b)_F D_j$ and $(a_2, b)_F D_j$ hold. To show that $(a_1 \wedge_F a_2, b)_F D_j$ holds, i.e., to show that if $x \in X$ is such that $A(x, (a_1 \wedge_F a_2) \vee_F b) > 0$, then there exist $\alpha, \beta \in X$ such that

$$A(\alpha, a_1 \wedge_F b_1) > 0,\ A(\beta, b) > 0 \text{ and } \alpha \vee_F \beta = x. \tag{3.22}$$

Let $c \in X$ satisfy $A(c, (a_1 \wedge_F a_2) \vee_F b) > 0$. Since $A((a_1 \wedge_F a_2) \vee_F b, a_1 \vee_F b) > 0$ and $A((a_1 \wedge_F a_2) \vee_F b, a_2 \vee_F b) > 0$ always hold, by fuzzy transitivity of A, from $A(c, (a_1 \wedge_F a_2) \vee_F b) > 0$ we get

$$A(c, a_1 \vee_F b) > 0 \text{ and } A(c, a_2 \vee_F b) > 0.$$

Since $A(c, a_1 \vee_F b) > 0$ and $(a_1, b)_F D_j$ hold, there exist $c_1, b_1 \in X$ such that

$$A(c_1, a_1) > 0,\ A(b_1, b) > 0 \text{ and } c_1 \vee_F b_1 = c.$$

From $c_1 \vee_F b_1 = c$, we conclude that $A(c_1, c) > 0$. Using the fuzzy transitivity of A and from

$$A(c_1, c) > 0 \text{ and } A(c, a_2 \vee_F b) > 0$$

we have

$$A(c_1, a_2 \vee_F b) > 0.$$

Since $A(c_1, a_2 \vee_F b) > 0$ and $(a_2, b)_F D_j$ hold, there exist $c_2, b_2 \in X$ such that

$$A(c_2, a_2) > 0,\ A(b_2, b) > 0 \text{ and } c_2 \vee_F b_2 = c_1.$$

We have $A(c_1, a_1) > 0$, $A(c_2, a_2) > 0$. From Corollary 3, this implies that

$$A(c_1 \wedge_F c_2, a_1 \wedge_F a_2) > 0.$$

From $c_2 \vee_F b_2 = c_1$, we have $A(c_2, c_1) > 0$ and so $c_1 \wedge_F c_2 = c_2$. This implies

$$A(c_2, a_1 \wedge_F a_2) > 0.$$

From $A(b_1, b) > 0$ and $A(b_2, b) > 0$, from Corollary 3, we have $A(b_1 \vee_F b_2, b) > 0$. Thus we have $A(c_2, a_1 \wedge_F a_2) > 0$, $A(b_1 \vee_F b_2, b) > 0$ and $c_2 \vee_F (b_1 \vee_F b_2) = c$. Hence (3.22) holds with $\alpha = c_2$, $\beta = b_1 \vee_F b_2$ and so $(a_1 \wedge_F a_2, b)_F D_j$ holds.

(ii) Suppose that $(a_1, b)_F SD_j$ and $(a_2, b)_F SD_j$ hold. To show that $(a_1 \wedge_F a_2, b)_F SD_j$ holds, i.e., to show that if $x \in X$ is such that $A(x, (a_1 \wedge_F a_2) \vee_F b) > 0$, then there exists $\alpha \in X$ such that

$$A(\alpha, a_1 \wedge_F a_2) > 0,\ A(\alpha, b) > 0, \text{ and } A(x, \alpha \vee_F b) > 0. \tag{3.23}$$

Let $c \in X$ be such that $A(c, (a_1 \wedge_F a_2) \vee_F b) > 0$. As shown in the proof of (i), this implies that $A(c, a_1 \vee_F b) > 0$ and $A(c, a_2 \vee_F b) > 0$. As $A(c, a_1 \vee_F b) > 0$ and $(a_1, b)_F SD_j$ hold, there exists $c_1 \in X$ such that

$$A(c_1, c) > 0,\ A(c_1, a_1) > 0 \text{ and } A(c, c_1 \vee_F b) > 0.$$

Since $A(c_1, c) > 0$ and $A(c, a_2 \vee_F b) > 0$, we get by fuzzy transitivity of A, that $A(c_1, a_2 \vee_F b) > 0$. From $A(c_1, a_2 \vee_F b) > 0$ and $(a_2, b)_F SD_j$, there exists $c_2 \in X$ such that

$$A(c_2, a_2) > 0,\ A(c_2, c_1) > 0 \text{ and } A(c_1, c_2 \vee_F b) > 0.$$

By the fuzzy transitivity of A, we get the following. From $A(c_2, c_1) > 0$ and $A(c_1, c) > 0$, we get $A(c_2, c) > 0$. From $A(c_2, c_1) > 0$ and $A(c_1, a_1) > 0$ we get $A(c_2, a_1) > 0$. By Corollary 3, we get the following. Using $A(c_2, a_2) > 0$ and

$A(c_2, a_1) > 0$ we get $A(c_2, a_1 \wedge_F a_2) > 0$. From $A(c_1, c_2 \vee_F b) > 0$ and $A(b, b) > 0$, we get $A(c_1 \vee_F b, c_2 \vee_F b) > 0$. From $A(c, c_1 \vee_F b) > 0$ and $A(c_1 \vee_F b, c_2 \vee_F b) > 0$, by fuzzy transitivity, we get $A(c, c_2 \vee_F b) > 0$. Thus c_2 satisfies

$$A(c_2, a_1 \wedge_F a_2) > 0,\ A(c_2, c) > 0 \text{ and } A(c, c_2 \vee_F b) > 0.$$

Hence (3.23) is satisfied with $\alpha = c_2, x = c$. Thus, $(a_1 \wedge_F a_2, b)_F SD_j$ holds. ■

Definition. Let (X, A) be a fuzzy lattice with $\perp$. An element $a \in X$ is called a fuzzy atom, if $A(b, a) > 0$ holds for some $b \in X$, then either $b = \perp$ or $b = a$.

Theorem 15. *Let (X, A) be a fuzzy lattice with $\perp$. Let a be a fuzzy atom and $b \in X$. Then the following statements are equivalent:*

(i) $(a, b)_F SD_j$.

(ii) *If $A(c, a \vee_F b) > 0$, for some $c \in X$, then either $A(a, c) > 0$ or $A(c, b) > 0$.*

Proof. (i)⇒(ii) Suppose that $(a, b)_F SD_j$ holds, i.e., if $x \in X$ is such that $A(x, a \vee_F b) > 0$, then there exists $\alpha \in X$ satisfying

$$A(\alpha, a) > 0,\ A(\alpha, b) > 0, \text{ and } A(x, \alpha \vee_F b) > 0. \tag{3.24}$$

Let $c \in X$ be such that $A(c, a \vee_F b) > 0$. Since $(a, b)_F SD_j$ holds, $A(c, a \vee_F b) > 0$ implies from (3.24) that there exists $a_1 \in X$ such that

$$A(a_1, a) > 0,\ A(a_1, c) > 0 \text{ and } A(c, a_1 \vee_F b) > 0.$$

Since a is a fuzzy atom, $A(a_1, a) > 0$ implies that either $a_1 = \perp$ or $a_1 = a$. If $a_1 = \perp$, then $A(c, a_1 \vee_F b) > 0$ implies that $A(c, b) > 0$. If $a_1 = a$, then $A(a_1, c) > 0$ implies that $A(a, c) > 0$. Thus, (ii) holds.

(ii)⇒(i) Let a be a fuzzy atom. Suppose that for some $c \in X$, $A(c, a \vee_F b) > 0$ holds. By (ii), either $A(a, c) > 0$ or $A(c, b) > 0$. If $A(a, c) > 0$, we take $\alpha = a$ and $x = c$ and all the conditions in (3.24) are satisfied. If $A(c, b) > 0$, we take $\alpha = \perp$ and $x = c$ and all the conditions in (3.24) are satisfied. Thus, $(a, b)_F SD_j$ holds. ■

In the next theorem, we prove that for a pair of fuzzy atoms, the concepts of a fuzzy join-distributive pair and a fuzzy join semi-distributive pair are equivalent.

Theorem 16. *Let (X, A) be a fuzzy lattice with $\perp$. Let p and q be fuzzy atoms of X. Then the following statements are equivalent.*

(i) $(p, q)_F D_j$.

(ii) $(p, q)_F SD_j$ *and* $(q, p)_F SD_j$.

(iii) *Let* $a \neq \perp$. *If* $0 < A(\perp, a) < 1$ *and* $0 < A(a, p \vee_F q) < 1$, *then either* $a = p$ *or* $a = q$.

Proof. (i)⇒(ii) Suppose that $(p, q)_F D_j$ holds, i.e., if $x \in X$ is such that $A(x, p \vee_F q) > 0$, then there exist a_1, b_1 such that

$$A(a_1, p) > 0 \text{ and } A(b_1, q) > 0 \text{ such that } a_1 \vee_F b_1 = x. \tag{3.25}$$

To show that $(p, q)_F SD_j$ holds, i.e., to show that if $x \in X$ is such that $A(x, p \vee_F q) > 0$, then there exists $\alpha \in X$ satisfying

$$A(\alpha, p) > 0, \ A(\alpha, x) > 0 \text{ and } A(x, p \vee_F q) > 0. \tag{3.26}$$

Let $c \in X$ be such that $A(c, p \vee_F q) > 0$. Since $(p, q)_F D_j$ holds and $A(c, p \vee_F q) > 0$, by (3.25), there exist c_1, d_1 satisfying

$$A(c_1, p) > 0 \text{ and } A(d_1, q) > 0 \text{ such that } c_1 \vee_F d_1 = c.$$

Since p is a fuzzy atom, either $c_1 = \perp$ or $c_1 = p$. Thus we note that taking $x = c$, $\alpha = \perp$ or $\alpha = p$ all the conditions in (3.26) are satisfied and thus $(p, q)_F SD_j$ holds. Similarly, we can prove that $(q, p)_F SD_j$ holds.

(ii)⇒(iii) Suppose that both $(p, q)_F SD_j$ and $(q, p)_F SD_j$ hold. Let $0 < A(\perp, a) < 1$ and $0 < A(a, p \vee_F q) < 1$. It follows from Lemma 15 that $A(p, a) > 0$ or $A(a, q) > 0$.

Case (1). Suppose that $A(p, a) > 0$. Since $(q, p)_F SD_j$ holds. It follows from $A(a, q \vee_F p) > 0$ and Theorem 15 that either $A(q, a) > 0$ or $A(a, p) > 0$. Suppose that $A(q, a) > 0$. This together with $A(p, a) > 0$ implies by Corollary 3 that $A(p \vee_F q, a) > 0$.

Hence it follows from $A(p \vee_F q, a) > 0$, $A(a, p \vee_F q) > 0$ and fuzzy antisymmetry of A, that $p \vee_F q = a$ and so $A(a, p \vee_F q) = 1$, a contradiction. Hence $A(q, a) > 0$ is not possible. Thus $A(a, p) > 0$. This together with $A(p, a) > 0$ implies that $a = p$.

Case (2). Suppose that $A(a, q) > 0$. Since q is a fuzzy atom, this implies that $a = q$. Thus, (iii) holds.

(iii)⇒(i) Let $a \neq \perp$ and $A(a, p \vee_F q) > 0$. Then either $0 < A(a, p \vee_F q) < 1$ or $A(a, p \vee_F q) = 1$. We have to show that $(p, q)_F D_j$ holds, i.e., we have to show that there exist $b, c \in X$ satisfying $A(b, p) > 0$, $A(c, q) > 0$ and $b \vee_F c = a$. We have the following two cases.

Case (1). Suppose that $0 < A(a, p \vee_F q) < 1$. Then by (iii) either $a = p$ or $a = q$. If $a = p$, we take $b = p$ and $c = \perp$, then $A(p, p) > 0$, $A(\perp, q) > 0$ and $p \vee_F \perp = a$ hold. If $a = q$, we take $b = \perp$ and $c = q$, then $A(\perp, p) > 0$, $A(q, q) > 0$ and $\perp \vee_F q = a$ hold. Thus in any case $(p, q)_F D_j$ holds.

Case (2). Suppose that $A(a, p \vee_F q) = 1$. Then $a = p \vee_F q$. We take $b = p$ and $c = q$. We then have $A(p, p) > 0$, $A(q, q) > 0$ and $p \vee_F q = a$. Thus, $(p, q)_F D_j$ holds. ■

4. Conclusion and future work

In this paper, we have introduced the concepts of fuzzy distributive pairs, fuzzy semi-distributive pairs and fuzzy modular pairs in a fuzzy lattice and have given some characterizations and implications of these pairs.

In future, we shall introduce these types of pairs in fuzzy α-lattices and shall try to prove their properties.

There is a vast study of distributive pairs, semi-distributive pairs and modular pairs in the context of posets. We shall try to introduce the concepts of fuzzy distributive pairs, fuzzy semi-distributive pairs and fuzzy modular pairs in fuzzy partially ordered sets.

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