Discussiones Mathematicae
General Algebra and Applications 42 (2022) 159–178
https://doi.org/10.7151/dmgaa.1384

DISJUNCTIVE IDEALS OF ALMOST DISTRIBUTIVE LATTICES

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Abstract

The concept of disjunctive ideals is introduced in an Almost Distributive Lattice (ADL). It is proved that the set of all disjunctive ideals of an ADL forms a complete lattice. A necessary and sufficient condition is derived for an inverse homomorphic image of a disjunctive ideal of an ADL to be again a disjunctive ideal. Later, the concept of strongly disjunctive ideals is introduced in an ADL and their properties are studied. Some equivalent conditions are established for the set of all strongly disjunctive ideals to convert into a sublattice of the ideal lattice.

Keywords: Almost Distributive Lattice (ADL), normal ADL, disjunctive ideal, strongly disjunctive ideal, normal prime ideal, minimal prime ideal.

2010 Mathematics Subject Classification: 06D99, 06D15.

1. Introduction

An Almost Distributive Lattice (ADL) was introduced by Swamy and Rao [8] as a common abstraction of many existing ring theoretic generalizations of a Boolean algebra on one hand and the class of distributive lattices on the other. In that

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paper, the concept of an ideal in an ADL was introduced analogous to that in a distributive lattice and it was observed that the set $PI(L)$ of all principal ideals of L forms a distributive lattice. This provided a path to extend many existing concepts of lattice theory to the class of ADLs. In [5], Rao and Ravi Kumar proved that some important results on minimal prime ideals of an ADL. In [4], Rao and Ravi Kumar, characterized the normal ADL in terms of its prime ideals, minimal prime ideals and annihilator ideals. In [7], Sambasiva Rao introduced the concepts of disjunctive ideals, strongly disjunctive ideals and normal prime ideals in distributive Lattices. In that paper, he derived a set of equivalent conditions for every ideal of a lattice to become a disjunctive ideal. In this paper, we have introduced the concept of disjunctive ideals in an ADL, analogous to that in a distributive lattice. We have proved that the set of all disjunctive ideals of an ADL can be made into a complete lattice. For any ideal I of an ADL, we have derived that the extension I^e is a disjunctive ideal containing I. We have provided a necessary and sufficient condition for an inverse homomorphic image of a disjunctive ideal of an ADL to be again a disjunctive ideal. We have introduced the concept of normal prime ideals in an ADL and studied their properties. We have given an equivalent condition for every minimal prime ideal to convert into a normal prime ideal. We have derived a set of equivalent conditions for every ideal of an ADL to convert into a strongly disjunctive ideal. Finally, we have established some equivalent conditions for the set of all strongly disjunctive ideals to convert into a sublattice of the ideal lattice.

2. Preliminaries

In this section we give some important definitions and results that are frequently used for ready reference.

Definition 2.1 [8]**.** An Almost Distributive Lattice with zero or simply ADL is an algebra $(L, \vee, \wedge, 0)$ of type $(2, 2, 0)$ satisfying:

1. $a \vee 0 = a$
2. $0 \wedge a = 0$
3. $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$
4. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
5. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
6. $(a \vee b) \wedge b = b$, for all $a, b, c \in L$.

Every nonempty set X can be regarded as an ADL as follows. Let $x_0 \in X$. Define the binary operations $\vee, \wedge$ on X by

$$x \vee y = \begin{cases} x & \text{if } x \neq x_0 \\ y & \text{if } x = x_0 \end{cases} \qquad x \wedge y = \begin{cases} y & \text{if } x \neq x_0 \\ x_0 & \text{if } x = x_0. \end{cases}$$

Then $(X, \vee, \wedge, x_0)$ is an ADL (where x_0 is the zero) and is called a discrete ADL. If $(L, \vee, \wedge, 0)$ is an ADL, for any $a, b \in L$, define $a \leq b$ if and only if $a = a \wedge b$ (or equivalently, $a \vee b = b$), then $\leq$ is a partial ordering on L.

Theorem 2.2 [8]**.** *If $(L, \vee, \wedge, 0)$ is an ADL, for any $a, b, c \in L$, we have the following:*

1. $a \vee b = a \Leftrightarrow a \wedge b = b$
2. $a \vee b = b \Leftrightarrow a \wedge b = a$
3. $\wedge$ *is associative in* L
4. $a \wedge b \wedge c = b \wedge a \wedge c$
5. $(a \vee b) \wedge c = (b \vee a) \wedge c$
6. $a \wedge b = 0 \Leftrightarrow b \wedge a = 0$
7. $a \wedge (a \vee b) = a \vee (b \wedge a) = (a \vee b) \wedge a = a \vee (a \wedge b) = a$ *and* $(a \wedge b) \vee b = b$
8. $a \leq a \vee b$ *and* $a \wedge b \leq b$
9. $a \wedge a = a$ *and* $a \vee a = a$
10. $0 \vee a = a$ *and* $a \wedge 0 = 0$.
11. *If* $a \leq c$, $b \leq c$, *then* $a \wedge b = b \wedge a$ *and* $a \vee b = b \vee a$
12. $a \vee b = (a \vee b) \vee a$.

It can be observed that an ADL L satisfies almost all the properties of a distributive lattice except the right distributivity of $\vee$ over $\wedge$, commutativity of $\vee$, commutativity of $\wedge$. Any one of these properties make an ADL L a distributive lattice.

Theorem 2.3 [8]**.** *Let $(L, \vee, \wedge, 0)$ be an ADL with 0. Then the following are equivalent:*

1. $(L, \vee, \wedge, 0)$ *is a distributive lattice*
2. $a \vee b = b \vee a$, *for all* $a, b \in L$
3. $a \wedge b = b \wedge a$, *for all* $a, b \in L$
4. $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$, *for all* $a, b, c \in L$.

As usual, an element $m \in L$ is called maximal if it is a maximal element in the partially ordered set $(L, \leq)$. That is, for any $a \in L$, $m \leq a \Rightarrow m = a$.

Theorem 2.4 [8]**.** *Let L be an ADL and $m \in L$. Then the following are equivalent:*

1. *m is maximal with respect to $\leq$*
2. *$m \vee a = m$, for all $a \in L$*
3. *$m \wedge a = a$, for all $a \in L$*
4. *$a \vee m$ is maximal, for all $a \in L$.*

As in distributive lattices [1, 2], a nonempty subset I of an ADL L is called an ideal of L if $a \vee b \in I$ and $a \wedge x \in I$ for any $a, b \in I$ and $x \in L$. Also, a nonempty subset F of L is said to be a filter of L if $a \wedge b \in F$ and $x \vee a \in F$ for $a, b \in F$ and $x \in L$.

The set $I(L)$ of all ideals of L is a bounded distributive lattice with least element $\{0\}$ and greatest element L under set inclusion in which, for any $I, J \in I(L)$, $I \cap J$ is the infimum of I and J while the supremum is given by $I \vee J := \{a \vee b \mid a \in I, b \in J\}$. A proper ideal P of L is called a prime ideal if, for any $x, y \in L$, $x \wedge y \in P \Rightarrow x \in P$ or $y \in P$. A proper ideal M of L is said to be maximal if it is not properly contained in any proper ideal of L. It can be observed that every maximal ideal of L is a prime ideal. Every proper ideal of L is contained in a maximal ideal. For any subset S of L the smallest ideal containing S is given by $(S] := \{(\bigvee_{i=1}^{n} s_i) \wedge x \mid s_i \in S, x \in L \text{ and } n \in N\}$. If $S = \{s\}$, we write $(s]$ instead of $(S]$. Similarly, for any $S \subseteq L$, $[S) := \{x \vee (\bigwedge_{i=1}^{n} s_i) \mid s_i \in S, x \in L$ and $n \in N\}$. If $S = \{s\}$, we write $[s)$ instead of $[S)$.

Theorem 2.5 [8]**.** *For any x, y in L the following are equivalent:*

1. $(x] \subseteq (y]$
2. $y \wedge x = x$
3. $y \vee x = y$
4. $[y) \subseteq [x)$.

For any $x, y \in L$, it can be verified that $(x] \vee (y] = (x \vee y]$ and $(x] \wedge (y] = (x \wedge y]$. Hence the set $PI(L)$ of all principal ideals of L is a sublattice of the distributive lattice $I(L)$ of ideals of L.

Definition 2.6 [6]**.** For any nonempty subset A of an ADL L, define $A^* = \{x \in L \mid a \wedge x = 0 \text{ for all } a \in A\}$. Here A^* is called the annihilator of A in L.

For any $a \in L$, we have $\{a\}^* = (a]^*$, where $(a]$ is the principal ideal generated by a. For any $a \in L$, we denote $(\{a\})^* = (a)^*$.

Annulets have many important properties. We give some of them in the following.

Theorem 2.7 [6]**.** *Let L be an ADL. For any $x, y \in L$, we have:*

1. $x \leq y \Rightarrow (y]^* \subseteq (x]^*$

2. $(x \wedge y]^* = (y \wedge x]^*$
3. $(x \vee y]^* = (y \vee x]^*$
4. $(x \vee y]^* = (x]^* \cap (y]^*$
5. $(x]^* \vee (y]^* \subseteq (x \wedge y]^*$
6. $x = 0 \Leftrightarrow (x]^* = L$.

Definition 2.8 [5]**.** A prime ideal of L is called a minimal prime ideal if it is a minimal element in the set of all prime ideals of L ordered by set inclusion.

Theorem 2.9 [5]**.** *Let L be an ADL. Then a prime ideal P is minimal if and only if for any $x \in P$, there exist an element $y \notin P$ such that $x \wedge y = 0$.*

3. Disjunctive ideals of ADLs

In this section, we have introduced the concept of disjunctive ideals and normal prime ideals in an ADL, analogous to that in a distributive lattice. We have proved that the set of all disjunctive ideals of an ADL can be made into a complete lattice. For any ideal I of an ADL, we have derived that the extension I^e is a disjunctive ideal containing I. We have derived a necessary and sufficient condition for an inverse homomorphic image of a disjunctive ideal of an ADL to be again a disjunctive ideal. We have derived that a set of all equivalent conditions for every ideal to become a disjunctive ideal.

We start this section with the following definition.

Definition 3.1. For any nonempty subset A of an ADL L, define $A^\circ = \{x \in L \mid (a)^* \vee (x)^* = L$, for all $a \in A\}$.

For any $a \in L$, we denote $(\{a\})^\circ = (a)^\circ$.

Lemma 3.2. *Let L be an ADL with maximal element m. For any nonempty subsets A, B of L, we have:*

1. *A° is an ideal of L*
2. *$A \cap A^\circ \subseteq \{0\}$*
3. *If $A \subseteq B$, then $B^\circ \subseteq A^\circ$*
4. *$A \subseteq A^{\circ\circ}$*
5. *$A^{\circ\circ\circ} = A^\circ$*
6. *$A^\circ = L$ if and only if $A = \{0\}$.*

Proof. Let A, B be any two nonempty subsets of L.

1. Clearly, we have that $(0)^* \vee (a)^* = L$, for all $a \in A$. Then $0 \in A^\circ$ and hence A° is a nonempty set. Let $x, y \in A^\circ$. Then $(x)^* \vee (a)^* = L$ and $(y)^* \vee (a)^* = L$, for

all $a \in A$. Now $(x \vee y)^* \vee (a)^* = ((x)^* \cap (y)^*) \vee (a)^* = ((x)^* \vee (a)^*) \cap ((y)^* \vee (a)^*) = L$. Therefore $x \vee y \in A^\circ$. Let $x \in A^\circ$. Then $(x)^* \vee (a)^* = L$, for all $a \in A$. Let $r \in L$. Since $r \wedge x \leq x$, we have that $(x)^* \subseteq (r \wedge x)^* = (x \wedge r)^*$. Now $L = (x)^* \vee (a)^* \subseteq (x \wedge r)^* \vee (a)^*$. Therefore $(x \wedge r)^* \vee (a)^* = L$ and hence $x \wedge r \in A^\circ$. Thus A° is an ideal of L.

2. Let $x \in A \cap A^\circ$. Then $x \in A$ and $x \in A^\circ$. Since $x \in A^\circ$, we have $(x)^* \vee (a)^* = L$, for all $a \in A$. Since $x \in A$, we get that $(x)^* \vee (x)^* = L$. That implies $(x)^* = L$ and hence $x = 0$. Therefore $A \cap A^\circ \subseteq \{0\}$.

3. Assume that $A \subseteq B$. Let $x \in B^\circ$. Then $(x)^* \vee (b)^* = L$, for all $b \in B$. Since $A \subseteq B$, we get that $(x)^* \vee (a)^* = L$, for all $a \in A$. Therefore $x \in A^\circ$. Hence $B^\circ \subseteq A^\circ$.

4. Let $x \in A$ and $y \in A^\circ$. Since $y \in A^\circ$, we have that $(y)^* \vee (a)^* = L$, for all $a \in A$. That implies $(y)^* \vee (x)^* = L$, for all $y \in A^\circ$. Therefore $x \in A^{\circ\circ}$ and hence $A \subseteq A^{\circ\circ}$.

5. By 4, we have that $A \subseteq A^{\circ\circ}$ and hence $A^{\circ\circ\circ} \subseteq A^\circ$. Let x be any element of A° and $t \in A^{\circ\circ}$. Then $(t)^* \vee (s)^* = L$, for all $s \in A^\circ$. That implies $(x)^* \vee (t)^* = L$, for all $t \in A^{\circ\circ}$. That implies $x \in A^{\circ\circ\circ}$ and hence $A^\circ \subseteq A^{\circ\circ\circ}$. Therefore $A^\circ = A^{\circ\circ\circ}$.

6. Assume that $A^\circ = L$. Let m be any maximal element of L. Then $m \in A^\circ$. That implies $(m)^* \vee (a)^* = L$, for all $a \in A$. That implies $(m \wedge a)^* = L$ and hence $(a)^* = L$, for all $a \in A$. Therefore $a = 0$. Thus $A = \{0\}$. Conversely assume that $A = \{0\}$. Since $(0)^* = L$, we get that $(x)^* \vee (0)^* = L$, for all $x \in L$. Therefore $x \in A^\circ$, for all $x \in L$. Hence $A^\circ = L$. ■

Theorem 3.3. *Let I, J be any two ideals of ADL L. Then we have the following:*

1. $(I \vee J)^\circ = I^\circ \cap J^\circ$
2. $(I \cap J)^{\circ\circ} \subseteq I^{\circ\circ} \cap J^{\circ\circ}$
3. $I^{\circ\circ} \cap J^{\circ\circ} \subseteq (I \vee J)^{\circ\circ}$
4. $I \subseteq J^\circ \Rightarrow I \cap J = \{0\}$.

Proof. 1. Clearly, we have that $(I \vee J)^\circ \subseteq I^\circ$ and $(I \vee J)^\circ \subseteq J^\circ$. That implies $(I \vee J)^\circ \subseteq I^\circ \cap J^\circ$. Let $x \in I^\circ \cap J^\circ$. Then $x \in I^\circ$ and $x \in J^\circ$. Then $(x)^* \vee (i)^* = L$, for all $i \in I$ and $(x)^* \vee (j)^* = L$, for all $j \in J$. Now, $(x)^* \vee (i \vee j)^* = (x)^* \vee ((i)^* \cap (j)^*) = ((x)^* \vee (i)^*) \cap ((x)^* \vee (j)^*) = L$. That implies $(x)^* \vee (i \vee j)^* = L$, for all $i \vee j \in I \vee J$. That implies $x \in (I \vee J)^\circ$ and hence $I^\circ \cap J^\circ \subseteq (I \vee J)^\circ$. Therefore $(I \vee J)^\circ = I^\circ \cap J^\circ$.

2, 3, 4 are Clear. ■

Corollary 3.4. *Let L be an ADL with maximal elements. If $\{I_i \mid i \in \Delta\}$ is a family of ideals of L, then $(\bigcap_{i \in \Delta} I_i)^{\circ\circ} = \bigcap_{i \in \Delta} (I_i)^{\circ\circ}$.*

The following result can be verified easily.

Theorem 3.5. *Let L be an ADL with maximal elements. For any $a, b \in L$, we have the following:*

1. $((a])^\circ = (a)^\circ$
2. $(0)^\circ = L$.
3. *For any maximal element m of L, $(m)^\circ = \{0\}$.*
4. *If $a \leq b$, then $(b)^\circ \subseteq (a)^\circ$*
5. $(a \vee b)^\circ = (a)^\circ \cap (b)^\circ$
6. $(a)^\circ \vee (b)^\circ \subseteq (a \wedge b)^\circ$
7. $(a \vee b)^\circ = (b \vee a)^\circ$
8. $(a \wedge b)^\circ = (b \wedge a)^\circ$
9. $(a)^\circ = L$ *if and only if* $a = 0$.

Theorem 3.6. *Let L be an ADL with maximal elements. For any nonempty subset A of L, we have $A^\circ = \bigcap_{a \in A} (a)^\circ$.*

Proof. Let $x \in A^\circ$. Then $(x)^* \vee (a)^* = L$, for all $a \in A$. That implies $x \in (a)^\circ$, for all $a \in A$. and hence $x \in \bigcap_{a \in A} (a)^\circ$. Therefore $A^\circ \subseteq \bigcap_{a \in A} (a)^\circ$. Conversely let $x \in \bigcap_{a \in A} (a)^\circ$. Then $x \in (a)^\circ$, for all $a \in A$. That implies $(x)^* \vee (a)^* = L$, for all $a \in A$. That implies $x \in A^\circ$. Therefore $\bigcap_{a \in A} (a)^\circ \subseteq A^\circ$. Hence $A^\circ = \bigcap_{a \in A} (a)^\circ$. ■

Lemma 3.7. *Let I be any ideal of an ADL L with maximal elements. Then $I^\circ \subseteq I^*$.*

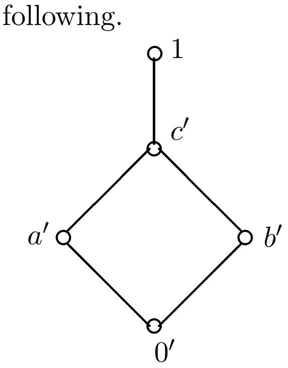
Proof. Let $x \in I^\circ$. Then $(x)^* \vee (i)^* = L$, for all $i \in I$. Let m be any maximal element of L such that $m \in (x)^* \vee (i)^*$. Then there exist elements $y \in (x)^*$ and $a \in (i)^*$ such that $m = y \vee a$. Since $y \in (x)^*$ and $a \in (i)^*$, we have that $y \wedge x = 0$ and $a \wedge i = 0$. Now $x \wedge i = m \wedge x \wedge i = (y \vee a) \wedge x \wedge i = (y \wedge x \wedge i) \vee (a \wedge x \wedge i) = 0$. Therefore $x \wedge i = 0$, for all $i \in I$ and hence $x \in I^*$. Thus $I^\circ \subseteq I^*$. ■

Example 3.8. Let $A = \{0, a\}$ be a discrete ADL and $B = \{0', a', b', c', 1\}$ be a distributive lattice whose Hasse diagram is given in the following.

Take $L = A \times B = \{(0, 0'), (0, a'), (0, b'), (0, c'),$

$(0, 1), (a, 0'), (a, a'), (a, b'), (a, c'), (a, 1)\}$.

Then $(L, \vee, \wedge, \overline{0})$ is an ADL with zero $\overline{0} = (0, 0')$ under point-wise operations.



Here

$(0,0')^* = L,$
$(0,a')^* = \{(0,0'),(0,b'),(a,b'),(a,0')\},$
$(0,b')^* = \{(0,0'),(0,a'),(a,a'),(a,0')\},$
$(0,c')^* = (0,1)^* = \{(0,0'),(a,0')\},$
$(a,0')^* = \{(0,0'),(0,a'),(0,b'),(0,c'),(0,1)\},$
$(a,a')^* = \{(0,0'),(0,b')\},$
$(a,b')^* = \{(0,0'),(0,a')\},$
$(a,c')^* = (a,1)^* = \{(0,0')\}.$

Now

$(0,0')^\circ = L,$
$(0,a')^\circ = (0,b')^\circ = (0,c')^\circ = (0,1)^\circ = \{(0,0'),(a,0')\},$
$(a,0')^\circ = \{(0,0'),(0,a'),(0,b'),(0,c'),(0,1)\},$
$(a,a')^\circ = (a,b')^\circ = (a,c')^\circ = (a,1)^\circ = \{(0,0')\}.$

Consider an ideal $I = \{(0,0'),(0,a')\}$. Clearly, we have that $I^* = \{(0,0'),\ (0,b'),\ (a,b'),(a,0')\}$ and $I^\circ = \{(0,0'),(a,0')\}$. Hence $I^\circ \subseteq I^*$ but not $I^* \nsubseteq I^\circ$.

Definition 3.9 [4]**.** An ADL L is said to be normal if every prime ideal of L contains unique minimal prime ideal.

Definition 3.10 [4]**.** Two ideals $I,\ J$ of an ADL L are said to be co-maximal if $I \vee J = L$.

Theorem 3.11 [4]**.** *Let L be an ADL with maximal elements. Then the following are equivalent:*

1. *L is normal.*
2. *Any two distinct minimal prime ideals are co-maximal.*
3. *For any prime ideal P, $O(P) = \{x \in L \mid x \wedge y = 0,$ for some $y \notin P\}$ is prime.*
4. *For any $x, y \in L$, $x \wedge y = 0$ implies $(x)^* \vee (y)^* = L$.*
5. *For any $x, y \in L$, $(x)^* \vee (y)^* = (x \wedge y)^*$.*

Now we derive a set of equivalent conditions for every ideal I of L satisfy $I^* \subseteq I^\circ$ which is not true in general.

Theorem 3.12. *Let L be an ADL with maximal elements. Then the following conditions are equivalent:*

1. *L is a normal ADL.*
2. *For any ideals I, J of L, $I \cap J = \{0\}$ if and only if $I \subseteq J^\circ$.*
3. *For any ideal I of L, $I^\circ = I^*$.*
4. *For any $a \in L$, $(a)^\circ = (a)^*$.*

Proof. $1 \Rightarrow 2:$ Assume that L is a normal ADL. Let I, J be any two ideals of L. Now we prove that $I \cap J = \{0\}$ if and only if $I \subseteq J^{\circ}$. Suppose $I \cap J = \{0\}$. Let $x \in I$. Then $x \wedge a \in I$, for all $a \in J$. Since $a \in J$, we have that $x \wedge a \in J$. That implies $x \wedge a \in I \cap J = \{0\}$. So that $x \wedge a = 0$. Since L is a normal ADL, we have that $(x)^* \vee (a)^* = L$, for all $a \in J$. That implies $x \in J^{\circ}$ and hence $I \subseteq J^{\circ}$. From Theorem-3.3(4), we have the converse part.

$2 \Rightarrow 3:$ Assume 2. Clearly, we have that $I^{\circ} \subseteq I^*$. Let $x \in I^*$. Then $x \wedge i = 0$, for all $i \in I$. That implies $(x \wedge i] = (0]$, for all $i \in I$. That implies $(x] \cap (i] = \{0\}$, for all $i \in I$. By our assumption, we get that $(x] \subseteq (i]^{\circ}$, for all $i \in I$. That implies $(x] \subseteq \bigcap_{i \in I} (i]^{\circ} = I^{\circ}$ and hence $x \in I^{\circ}$. Therefore $I^* \subseteq I^{\circ}$. Thus $I^{\circ} = I^*$.

$3 \Rightarrow 4:$ Assume 3. We prove that $(a)^{\circ} = (a)^*$, for all $a \in L$. By our assumption, we have that $(a]^{\circ} = (a]^*$. Clearly, we have that $(a)^* = (a]^*$ and $(a)^{\circ} = (a]^{\circ}$. Therefore $(a)^{\circ} = (a)^*$.

$4 \Rightarrow 1:$ Assume $(a)^{\circ} = (a)^*$, for all $a \in L$. Let $x, y \in L$ with $x \wedge y = 0$. Then $x \in (y)^*$. By our assumption, we get that $x \in (y)^{\circ}$. That implies $(x)^* \vee (y)^* = L$. Hence L is a normal ADL. ∎

Definition 3.13. An ideal I of an ADL L is said to be disjunctive if for any $x, y \in L$, $(x)^{\circ} = (y)^{\circ}$ and $x \in I$ implies $y \in I$.

Lemma 3.14. *Let L be an ADL with maximal elements. Then we have the following:*

1. *$(x)^{\circ}$ is a disjunctive ideal of L, for all $x \in L$.*
2. *If I is an ideal of L such that $x \in L$, $x \in I$ implies $(x)^{\circ\circ} \subseteq I$, then I is a disjunctive ideal of L.*

Proof. 1. Clearly, $(x)^{\circ}$ is an ideal of L. Let $a, b \in L$ with $(a)^{\circ} = (b)^{\circ}$ and $a \in (x)^{\circ}$. Since $a \in (x)^{\circ}$, we have $(x)^* \vee (a)^* = L$. That implies $x \in (a)^{\circ} = (b)^{\circ}$. That implies $(x)^* \vee (b)^* = L$ and hence $b \in (x)^{\circ}$. Therefore $(x)^{\circ}$ is a disjunctive ideal of L.

2. Let x be any element of an ideal I with $(x)^{\circ\circ} \subseteq I$. We prove that I is a disjunctive ideal of L. Let $a, b \in L$ with $(a)^{\circ} = (b)^{\circ}$ and $a \in I$. Then $(a)^{\circ\circ} = (b)^{\circ\circ}$ and $(a)^{\circ\circ} \subseteq I$. That implies $(b)^{\circ\circ} \subseteq I$. Let $x \in (b)^{\circ}$. Then $(x)^* \vee (b)^* = L$. Since $x \in (b)^{\circ}$, we get that $b \in (b)^{\circ\circ}$. Therefore I is a disjunctive ideal of L. ∎

Theorem 3.15. *Let L be an ADL and S a multiplicatively closed subset of L (i.e., a subset S of L in which $a \wedge b \in S$ for all $a, b \in S$). Then the set $I = \{x \in L \mid (x)^* \vee (a)^* = L$, for some $a \in S\}$ is a disjunctive ideal of L.*

Proof. Clearly, $0 \in I$ and hence I is a nonempty set. Let $x, y \in I$. Then there exist elements $a, b \in S$ such that $(x)^* \vee (a)^* = L$ and $(x)^* \vee (b)^* = L$. Since $a, b \in S$, we have that $a \wedge b \in S$. Now $(x \vee y)^* \vee (a)^* \vee (b)^* = ((x)^* \cap (y)^*) \vee (a)^* \vee (b)^* = ((x)^* \vee (a)^* \vee (b)^*) \cap ((y)^* \vee (a)^* \vee (b)^*) = L$. Since $(a)^* \vee (b)^* \subseteq (a \wedge b)^*$, we get that $(x\vee y)^*\vee(a\wedge b)^*=L$. Since $a\wedge b\in S$, we get that $x\vee y\in I$. Let $x\in I$. Then there exists an element $a\in S$ such that $(x)^*\vee(a)^*=L$. Let r be any element of L. Clearly, we have that $(x)^*\vee(r)^*\vee(a)^*=L$. That implies $(x\wedge r)^*\vee(a)^*=L$ and hence $x\wedge r\in I$. Therefore I is an ideal of L. It can be easily observed that $I=\bigcup_{a\in\ S}(a]^\circ$. Now, let $x\in I$. Then there exists an element $a\in S$ such that $x\in(a]^\circ$. That implies $(x]^{\circ\circ}\subseteq(a]^\circ$. Hence $(x]^{\circ\circ}\subseteq\bigcup_{a\in S}(a]^\circ=I$. Therefore I is a disjunctive ideal of L. ■

Corollary 3.16. *Let L be an ADL. Then for any prime ideal P of L, the set $\ell(P)=\{x\in L\mid(x)^*\vee(a)^*=L$, for some $a\notin P\}$ is a disjunctive ideal of L.*

Proof. It is clear by taking $S=L\setminus P$, in the above theorem. ■

We now prove the following.

Lemma 3.17. *Let L be an ADL. If the set-theoretic union of disjunctive ideals of L is an ideal, then it is also a disjunctive ideal in L.*

Proof. Let $\{J_i\}_{i\in\Delta}$ be an arbitrary family of disjunctive ideals of L. By hypothesis $\bigcup_{i\in\Delta}J_i$ is an ideal of L. Let $x,y\in L$ with $(x)^\circ=(y)^\circ$ and $x\in\bigcup_{i\in\Delta}J_i$. Since $x\in\bigcup_{i\in\Delta}J_i$, there exists $k\in\Delta$ such that $x\in J_k$. Since J_k is a disjunctive ideal of L and $(x)^\circ=(y)^\circ$, we get that $y\in J_k$. Therefore $y\in\bigcup_{i\in\Delta}J_i$. Hence $\bigcup_{i\in\Delta}J_i$ is a disjunctive ideal of L. ■

We define an extension of an ideal in an ADL.

Definition 3.18. Let L be an ADL. For any ideal I of L, define $I^e=\{x\in L\mid(a)^\circ\subseteq(x)^\circ$ for some $a\in I\}$.

We first prove some elementary properties of I^e in the following.

Lemma 3.19. *Let L be an ADL with maximal elements. Then for any ideals I,J of L, we have the following:*

1. $I\subseteq I^e$
2. $I\subseteq J\Rightarrow I^e\subseteq J^e$
3. $(I\cap J)^e\subseteq I^e\cap J^e$
4. $I^e\vee J^e\subseteq(I\vee J)^e$
5. $(I^\circ)^e=I^\circ$.

Proof. 1. Clear.

2. Assume that $I\subseteq J$. Let $x\in I^e$. Then there exists an element $a\in I$ such that $(a)^\circ\subseteq(x)^\circ$. Since $a\in I\subseteq J$, we get that $x\in J^e$. Therefore $I^e\subseteq J^e$.

3. Clear.

4. Clear.

5. Clearly, $I^{\circ} \subseteq (I^{\circ})^{e}$. Let $x \in (I^{\circ})^{e}$. Then there exists an element $a \in I^{\circ}$ such that $(a)^{\circ} \subseteq (x)^{\circ}$. Since $a \in I^{\circ}$, we get that $(a)^{*} \vee (i)^{*} = L$, for all $i \in I$. That implies $i \in (a)^{\circ} \subseteq (x)^{\circ}$, for all $i \in I$. That implies $(x)^{*} \vee (i)^{*} = L$, for all $i \in I$. Therefore $x \in I^{\circ}$ and hence $(I^{\circ})^{e} \subseteq I^{\circ}$. Thus $(I^{\circ})^{e} = I^{\circ}$. ∎

Theorem 3.20. *Let L be an ADL with maximal elements and I an ideal of L. Then I^{e} is a disjunctive ideal of L containing I.*

Proof. Clearly, $0 \in I^{e}$ and hence $I^{e} \neq \emptyset$. Let $x, y \in I^{e}$. There there exist elements $a, b \in I$ such that $(a)^{\circ} \subseteq (x)^{\circ}$ and $(b)^{\circ} \subseteq (y)^{\circ}$. Since $a, b \in I$, we get that $a \vee b \in I$. Now, $(a \vee b)^{\circ} = (a)^{\circ} \cap (b)^{\circ} \subseteq (x)^{\circ} \cap (y)^{\circ} = (x \vee y)^{\circ}$. Since $a \vee b \in I$, we get that $x \vee y \in I^{e}$. Let $x \in I^{e}$. Then there exits an element $a \in I$ such that $(a)^{\circ} \subseteq (x)^{\circ}$. Let r be any element of L. Clearly, we have that $(a)^{\circ} \subseteq (x)^{\circ} \subseteq (x \wedge r)^{\circ}$. That implies $x \wedge r \in I^{e}$. Therefore I^{e} is an ideal of L. Let $x \in I^{e}$. Then there exists an element $a \in I$ such that $(a)^{\circ} \subseteq (x)^{\circ}$. Now we prove that $(x)^{\circ\circ} \subseteq I^{e}$. Let $t \in (x)^{\circ\circ}$. Then $(x)^{\circ} \subseteq (t)^{\circ}$ and hence $(a)^{\circ} \subseteq (x)^{\circ} \subseteq (t)^{\circ}$. That implies $t \in I^{e}$. Therefore $(x)^{\circ\circ} \subseteq I^{e}$, for all $x \in I^{e}$. Thus I^{e} is a disjunctive ideal of L containing I. ∎

Theorem 3.21. *Let I be a disjunctive ideal of an ADL L. Then $I^{e} = I$.*

Proof. Clearly, we have that $I \subseteq I^{e}$. Let $x \in I^{e}$. Then there exists an element $a \in I$ such that $(a)^{\circ} \subseteq (x)^{\circ}$. That implies $(a)^{\circ} = (a)^{\circ} \cap (x)^{\circ} = (a \vee x)^{\circ}$. Since I is a disjunctive ideal of L and $a \in I$, We get that $a \vee x \in I$. That implies $(a \vee x) \wedge x \in I$ and hence $x \in I$. Therefore $I^{e} \subseteq I$. Thus $I = I^{e}$. ∎

Theorem 3.22. *Let L be an ADL with maximal elemnts. Then the set $\mathcal{I}_D(L)$ of all disjunctive ideals of L forms a complete lattice on its own.*

Proof. For $I, J \in \mathcal{I}_D(L)$, define $I \wedge J = I \cap J$ and $I \tilde{\vee} J = (I \vee J)^{e}$. Then $I \cap J$ is a disjunctive ideal and the infimum of I and J is in $\mathcal{I}_D(L)$. Therefore $I \cap J \in \mathcal{I}_D(L)$. Also $I \tilde{\vee} J$ is a disjunctive ideal. Clearly $I, J \subseteq I \vee J \subseteq (I \vee J)^{e} = I \tilde{\vee} J$. Let K be any upper bound for I, J in $\mathcal{I}_D(L)$. Hence $I \vee J \subseteq K$, which implies that $(I \vee J)^{e} \subseteq K^{e} = K$ (since $K \in I_D(L)$). Therefore $I \tilde{\vee} J$ is the supremum of both I and J in $\mathcal{I}_D(L)$. Hence $(\mathcal{I}_D(L), \wedge, \tilde{\vee})$ is a lattice. For $I, J \in \mathcal{I}_D(L)$, define $I \leq J \Leftrightarrow I \subseteq J$. Clearly $\langle \mathcal{I}_D(L), \leq \rangle$ is a partially ordered set. Clearly, $(0]$ and L are the disjunctive ideals in L and they are the bounds for $\mathcal{I}_D(L)$. Let $\{I_i\}_{i \in \Delta}$ be a family of disjunctive ideals in $\mathcal{I}_D(L)$. Since $\bigcap_{i \in \Delta} I_i$ is the ideal, we have $\bigcap_{i \in \Delta} I_i \subseteq (\bigcap_{i \in \Delta} I_i)^{e}$. Again, we have $\bigcap_{i \in \Delta} I_i \subseteq I_i$ for all $i \in \Delta$. That implies $(\bigcap_{i \in \Delta} I_i)^{e} \subseteq (I_i)^{e}$ for all $i \in \Delta$. That implies $(\bigcap_{i \in \Delta} I_i)^{e} \subseteq I_i$ for all $i \in \Delta$(since $I_i \in \mathcal{I}_D(L)$). That implies $(\bigcap_{i \in \Delta} I_i)^{e} \subseteq \bigcap_{i \in \Delta} I_i$. Hence $\{\bigcap_{i \in \Delta} I_i\}^{e} = \bigcap_{i \in \Delta} I_i$. Clearly $\bigcap_{i \in \Delta} I_i$ is the infimum of $\{I_i\}_{i \in \Delta}$ in $\mathcal{I}_D(L)$. Therefore $\mathcal{I}_D(L)$ is a complete lattice. ∎

Theorem 3.23. *Let L be an ADL with maximal elements. Then the following conditions are equivalent:*

1. *Every ideal is a disjunctive ideal.*
2. *Every principal ideal is a disjunctive ideal.*
3. *Every prime ideal is a disjunctive ideal.*
4. *For any $a, b \in L$, $(a)^\circ = (b)^\circ$ implies $(a] = (b]$.*

Proof. $1 \Rightarrow 2:$ Clear.

$2 \Rightarrow 3:$ Assume that every principal ideal is a disjunctive ideal. Let $x, y \in L$ and P, any prime ideal of L with $(x)^\circ = (y)^\circ$ and $x \in P$. Since $x \in P$, we get that $(x] \subseteq P$. By our assumption, we have that $(x]$ is a disjunctive ideal of L. Since $(x)^\circ = (y)^\circ$, we get that $y \in (x]$ and hence $y \in P$. Therefore P is a disjunctive ideal of L.

$3 \Rightarrow 4:$ Assume that every prime ideal is a disjunctive ideal. Let $x, y \in L$ with $(x)^\circ = (y)^\circ$. Now we prove that $(x] = (y]$. Suppose $(x] \neq (y]$. With out loss of generality we can assume that $(x] \nsubseteq (y]$. Then there exists an element $a \in (x]$ such that $a \notin (y]$. Since $a \notin (y]$, there exists a maximal ideal M such that $a \notin M$ and $(y] \subseteq M$. Since M is a maximal ideal, we get that M is prime ideal of L. By our assumption, we get that M is a disjunctive ideal of L. Since $(x)^\circ = (y)^\circ$ and $y \in M$, we get that $x \in M$. That implies $(x] \subseteq M$ and hence $a \in M$, which is a contradiction to $a \notin M$. Therefore $(x] = (y]$.

$4 \Rightarrow 1:$ Assume 4. Let $x, y \in L$ and I, any ideal of L with $(x)^\circ = (y)^\circ$ and $x \in I$. By our assumption, we get that $(x] = (y]$ and $x \in I$. That implies $y \in I$. Therefore I is a disjunctive ideal of L. ∎

Now we introduce the concept of normal prime ideal to an ADL.

Definition 3.24. Let L be an ADL with maximal elements. A prime ideal P of L is said to be normal if to each $x \in P$, there exists $y \notin P$ such that $(x)^\circ \vee (y)^\circ = L$.

Theorem 3.25. *Every normal prime ideal of an ADL L is a minimal prime ideal.*

Proof. Let P be a normal prime ideal of L. Let $x \in P$. Then there exists an element $y \notin P$ such that $(x)^\circ \vee (y)^\circ = L$. That implies $(x \wedge y)^\circ = L$. That implies $x \wedge y = 0$. Therefore P is a minimal prime ideal of L. ∎

In general, the converse of the above theorem is not true. It can see in the following example.

Example 3.26. From the Example 3.8, we have that $I = \{(0, 0'), (0, a')\}$ is an ideal of L. Clearly, I is a minimal prime ideal of L and $(0, a')^\circ \vee (x, y)^\circ \neq L$, for all $(x, y) \notin I$. Hence I is not a normal prime ideal of L.

We derive a sufficient condition for every minimal prime ideal to become a normal prime ideal.

Theorem 3.27. *Let L be a normal ADL with maximal elements. Then every minimal prime ideal of L is normal prime ideal.*

Proof. Let P be a minimal prime ideal of L with $x \in P$. Then there exists an element $y \notin P$ such that $x \wedge y = 0$. Since L is normal, we get that $(x)^* \vee (y)^* = L$ and hence $(x)^\circ \vee (y)^\circ = L$. Therefore P is a normal prime ideal of L. ∎

Theorem 3.28. *Let P be a normal prime ideal of an ADL L with maximal elements. Then for each $x \in L$, we have $x \notin P$ if and only if $(x)^\circ \subseteq P$.*

Proof. Assume $x \notin P$. Let $a \in (x)^\circ$. Then $(a)^* \vee (x)^* = L$. Let m be any maximal element of L. Then $m \in (a)^* \vee (x)^*$. Then there exist elements $s \in (a)^*$ and $t \in (x)^*$ such that $m = s \vee t$. Since $s \in (a)^*$ and $t \in (x)^*$, we have that $s \wedge a = 0$ and $t \wedge x = 0$. That implies $t \wedge x = 0 \in P$. Since $x \notin P$, we get that $t \in P$. Since $m = s \vee t$, we get that $s \notin P$. Since $s \wedge a = 0$, we get that $s \wedge a \in P$. Since $s \notin S$, we get that $a \in P$. Therefore $(x)^\circ \subseteq P$. Conversely, assume that $(x)^\circ \subseteq P$. We prove that $x \notin P$. Suppose $x \in P$. Since P is normal prime ideal of L, there exists an element $y \notin P$ such that $(x)^\circ \vee (y)^\circ = L$. That implies $(x)^* \vee (y)^* = L$. That implies $y \in (x)^\circ \subseteq P$. That implies $y \in P$, which is a contradiction to $y \notin P$. Therefore $x \notin P$. ∎

Corollary 3.29. *Let L be an ADL with maximal elements. Then for any $x \in L$, $(x]^\circ = \bigcap\{P \mid P$ is a normal prime ideal and $x \notin P\}$.*

Theorem 3.30. *Every normal prime ideal of L is a disjunctive ideal.*

Proof. Let $x, y \in L$ and P, a normal prime ideal of L with $(x)^\circ = (y)^\circ$ and $x \in P$. We prove that $y \in P$. Suppose $y \notin P$. Then by the above result we get that $(y)^\circ \subseteq P$. That implies $(x)^\circ \subseteq P$. Again by the above result, we get that $x \notin P$, which is a contradiction. Therefore $y \in P$ and hence P is a disjunctive ideal of L. ∎

However in the following we derive a necessary and sufficient condition for the contraction of a disjunctive ideal of an ADL L_1 to become a disjunctive ideal.

Theorem 3.31. *Let L_1 and L_2 be any two ADLs with maximal elements and f, a homomorphism from L_1 onto L_2. If I is a disjunctive ideal of L_2, then the following are equivalent:*

1. *$f^{-1}(I)$ is a disjunctive ideal of L_1*
2. *for any $x \in L_2$, $f^{-1}((x)^\circ)$ is a disjunctive ideal of L_1.*

Proof. $1 \Rightarrow 2$: Assume 1. Let $x \in L_2$. Clearly, we have that $(x)^\circ$ is a disjunctive ideal of L_2. By our assumption, we get that $f^{-1}((x)^\circ)$ is a disjunctive ideal of L_1.

$2 \Rightarrow 1$: Assume 2. Clearly, $f^{-1}(I)$ is an ideal of L_1. Now we prove that $f^{-1}(I)$ is a disjunctive ideal of L_1. Let $x, y \in L$ with $(x)^\circ = (y)^\circ$ and $x \in f^{-1}(I)$. Since $x \in f^{-1}(I)$, we have that $f(x) \in I$. We show that $(f(x))^\circ = (f(y))^\circ$. Now,

$$\begin{aligned} a \in (f(x))^\circ &\Leftrightarrow (a)^* \vee (f(x))^* = L_2 \\ &\Leftrightarrow f(x) \in (a)^\circ \\ &\Leftrightarrow x \in f^{-1}((a)^\circ) \\ &\Leftrightarrow y \in f^{-1}((a)^\circ) \\ &\Leftrightarrow f(y) \in (a)^\circ \\ &\Leftrightarrow (f(y))^* \vee (a)^* = L_2 \\ &\Leftrightarrow a \in (f(y))^\circ. \end{aligned}$$

Therefore $(f(x))^\circ = (f(y))^\circ$. Since $f(x) \in I$ and I is a disjunctive ideal of L_2, we get that $f(y) \in I$ and hence $y \in f^{-1}(I)$. Therefore $f^{-1}(I)$ is a disjunctive ideal of L_1. ■

Theorem 3.32. *Let L, L' be two ADLs. If $f : L \longrightarrow L'$ is an epimorphism with $f((a]^\circ) = \{f((a])\}^\circ$, for all $a \in L$, then every disjunctive ideal of L' contracts to a disjunctive ideal in L.*

Proof. Suppose J is a disjunctive ideal of L'. Let $a, b \in L$ such that $(a]^\circ = (b]^\circ$. Now $(a]^\circ = (b]^\circ \Leftrightarrow f((a]^\circ) = f((b]^\circ) \Leftrightarrow \{f((a])\}^\circ = \{f((b])\}^\circ$. That implies $(f(a)]^\circ = (f(b)]^\circ$. Suppose $a \in f^{-1}(J)$. Then $f(a) \in J$. Since J is a disjunctive ideal in L', we get $f(b) \in J$. Hence $b \in f^{-1}(J)$. Therefore $f^{-1}(J)$ is a disjunctive ideal in L. ■

Corollary 3.33. *Let L, L' be two ADLs. If f is an epimorphism with $f((a]^\circ) = \{f((a])\}^\circ$, for all $a \in L$ then for any nonempty subset A of L', $f^{-1}(A^\circ)$ is a disjunctive ideal of L containing $\{f^{-1}(A)\}^\circ$.*

Proof. Let A be a nonempty subset of L'. Then for any $x \in A^\circ$, we have $(x]^{\circ\circ} \subseteq A^{\circ\circ\circ} = A^\circ$. Hence A° is a disjunctive ideal of L'. Therefore by above theorem, $f^{-1}(A^\circ)$ is a disjunctive ideal of L. Let $x \notin f^{-1}(A^\circ)$. Then $f(x) \notin A^\circ$. There there exists an element $f(y) \notin A$ such that $(f(x))^* \vee (f(y)^*) \neq L'$. That implies $(f(x) \wedge f(y))^* \neq L'$. That implies $f(x) \wedge f(y) \neq 0'$, where $0'$ is the zero element of L'. Since f is homomorphism, we have that $f(x \wedge y) \neq 0'$. That implies $x \wedge y \neq 0$, where 0 is the zero element of L. That implies $(x \wedge y)^* \neq L$, for some $y \notin f^{-1}(A)$. Therefore $x \notin (f^{-1}(A))^\circ$. Hence $\{f^{-1}(A)\}^\circ \subseteq f^{-1}(A^\circ)$. ■

Corollary 3.34. *If f is an epimorphism from L to L' with $f((a]^\circ) = \{f((a])\}^\circ$, for all $a \in L$ then $Ker f$ is a disjunctive ideal of L.*

Proof. We have $Kerf = f^{-1}(\{0'\})$. Since $\{0'\}$ is a disjunctive ideal in L', $f^{-1}(\{0'\})$ is a disjunctive ideal in L. Thus $Kerf$ is a disjunctive ideal of L. ■

The following result can be verified easily.

Lemma 3.35. *Let L_1 and L_2 be any two ADLs with maximal elements. For any $(a, b) \in L_1 \times L_2$, we have the following:*

1. $(a, b)^* = (a)^* \times (b)^*$
2. $(a, b)^* \vee (c, d)^* = (a \vee c,\ b \vee d)^*$
3. $(a, b)^\circ = (a)^\circ \times (b)^\circ$.

We conclude this section with the following theorem.

Theorem 3.36. *Let $L = L_1 \times L_2$ be the product of ADLs L_1 and L_2. If I_1 and I_2 are disjunctive ideals of L_1 and L_2, respectively, then $I_1 \times I_2$ is a disjunctive ideal of L. Conversely, every disjunctive ideal of L can be expressed as $I = I_1 \times I_2$, where I_1 and I_2 are disjunctive ideals of L_1 and L_2, respectively.*

Proof. Let I_1 and I_2 be the disjunctive ideals of L_1 and L_2, respectively. Clearly, we have that $I_1 \times I_2$ is an ideal of L. Now we prove that $I_1 \times I_2$ is a disjunctive ideal of L. Let $(a, b), (c, d) \in L_1 \times L_2$ with $((a, b))^\circ = ((c, d))^\circ$ and $(a,\ b) \in I_1 \times I_2$. Then $(a)^\circ \times (b)^\circ = (c)^\circ \times (d)^\circ$ and $a \in I_1,\ b \in I_2$. That implies $(a)^\circ = (c)^\circ, (b)^\circ = (d)^\circ$ and $a \in I_1, b \in I_2$. Since I_1 and I_2 are disjunctive ideals of L_1 and L_2, we get that $c \in I_1, d \in I_2$. That implies $(c, d) \in I_1 \times I_2$. Therefore $I_1 \times I_2$ is a disjunctive ideal of L. Let I be any disjunctive ideal of L. Consider $I_1 = \{a \in L_1 \mid (a, b) \in I$, for some $b \in L_2\}$. Clearly, I_1 is an ideal of L_1. Let $x, y \in L_1$ with $(x)^\circ = (y)^\circ$ and $x \in I_1$. Since $x \in I_1$, there exists an element $a \in L_2$ such that $(x, a) \in I$. Since $(x)^\circ = (y)^\circ$, we get that $(x)^\circ \times (a)^\circ = (y)^\circ \times (a)^\circ$. That implies $((x, a))^\circ = ((y, a))^\circ$. Since I is a disjunctive ideal of L and $(x, a) \in I$, we get that $(y, a) \in I$. That implies $y \in I_1$. Therefore I_1 is a disjunctive ideal of L_1. Consider $I_2 = \{b \in L_2 \mid (a,\ b) \in I$, for some $a \in L_1\}$. Clearly, I_2 is a disjunctive ideal of L_2. We prove that $I = I_1 \times I_2$. We get easily that $I \subseteq I_1 \times I_2$. Let $(a,\ b) \in I_1 \times I_2$. Then $a \in I_1$ and $b \in I_2$. Then there exist elements $a_1, b_1 \in L_1$ such that $(a, a_1) \in I$ and $(b_1, b) \in I$. Since I is an ideal of L, we get that $(a, 0) \wedge (a, a_1) \in I$ and $(0, b) \wedge (b_1, b) \in I$. That implies $(a, 0) \in I$ and $(0, b) \in I$. That implies $(a, 0) \vee (0,\ b) \in I$. That implies $(a, b) \in I$. Therefore $I_1 \times I_2 \subseteq I$ and hence $I = I_1 \times I_2$. ■

4. Strongly disjunctive ideals of ADLs

In this section, we have introduced the concept of strongly disjunctive ideals in an ADL and studied their properties. We have derived a set of equivalent conditions for every ideal of an ADL to become a strongly disjunctive ideal. Finally, A set of all equivalent conditions are established for the set of all strongly disjunctive ideals of an ADL to become a sublattice of the ideal lattice of an ADL.

Now we have the following definition.

Definition 4.1. Let I be an ideal of an ADL L with maximal elements. Define $\beta(I) = \{x \in L \mid (x)^\circ \vee I = L\}$.

Lemma 4.2. *Let L be an ADL with maximal elements. For any ideals I, J of L, we have the following:*

1. $\beta(I) \subseteq I$
2. *If* $I \subseteq J$, *then* $\beta(I) \subseteq \beta(J)$
3. $\beta(I \cap J) = \beta(I) \cap \beta(J)$
4. $\beta(I)$ *is an ideal of* L.

Proof. 1. Let $x \in \beta(I)$. Then $(x)^\circ \vee I = L$. That implies $(x)^* \vee I = L$. Let m be any maximal element of L. Since $(x)^* \vee I = L$, we get that $m \in (x)^* \vee I$. Then there exist elements $y \in (x)^*$ and $a \in I$ such that $m = y \vee a$. Since $y \in (x)^*$, we have that $y \wedge x = 0$. Now $x = m \wedge x = (y \vee a) \wedge x = (y \wedge x) \vee (a \wedge x) = a \wedge x$. Since $a \in I$, we get that $a \wedge x \in I$ and hence $x \in I$. Therefore $\beta(I) \subseteq I$.

2. Suppose $I \subseteq J$. Let $x \in \beta(I)$. Then $(x)^\circ \vee I = L$. By our assumption, we get that $(x)^\circ \vee J = L$. That implies $x \in \beta(J)$. Therefore $\beta(I) \subseteq \beta(J)$.

3. Clearly, we have that $\beta(I \cap J) \subseteq \beta(I) \cap \beta(J)$. Let $x \in \beta(I) \cap \beta(J)$. Then $x \in \beta(I)$ and $x \in \beta(J)$. That implies $(x)^\circ \vee I = L$ and $(x)^\circ \vee J = L$. Now $(x)^\circ \vee (I \cap J) = ((x)^\circ \vee I) \cap ((x)^\circ \vee J) = L$. That implies $x \in \beta(I \cap J)$. Therefore $\beta(I) \cap \beta(J) \subseteq \beta(I \cap J)$. Hence $\beta(I \cap J) = \beta(I) \cap \beta(J)$.

4. Clearly, $(0)^\circ \vee I = L$. That implies $0 \in \beta(I)$ and hence $\beta(I) \neq \emptyset$. Let $x, y \in \beta(I)$. Then $(x)^\circ \vee I = L$ and $(y)^\circ \vee I = L$. Now, $(x \vee y)^\circ \vee I = ((x)^\circ \cap (y)^\circ) \vee I = ((x)^\circ \vee I) \cap ((y)^\circ \vee I) = L$. That implies $x \vee y \in \beta(I)$. Let $x \in \beta(I)$. Then $(x)^\circ \vee I = L$. Let r be any element of L. Now, $L = (x)^\circ \vee I \subseteq (x \wedge r)^\circ \vee I$. That implies $(x \wedge r)^\circ \vee I = L$ and hence $x \wedge r \in \beta(I)$. Therefore $\beta(I)$ is an ideal of L. ■

Now, we define the concept of strongly disjunctive ideal in an ADL.

Definition 4.3. An ideal I of an ADL L is said to be a strongly disjunctive if $\beta(I) = I$.

Lemma 4.4. *Every strongly disjunctive ideal of an ADL L is disjunctive.*

Proof. Let I be any strongly disjunctive ideal of L. Let $x, y \in L$ with $(x)^\circ = (y)^\circ$ and $x \in I$. Since $x \in I$, we get that $x \in \beta(I)$. Then $(x)^\circ \vee I = L$ and hence $(y)^\circ \vee I = L$. Therefore $y \in \beta(I) = I$. Thus I is a disjunctive ideal of L. ■

Theorem 4.5. *If every prime ideal of an ADL L is normal, then every ideal is strongly disjunctive.*

Proof. Assume that every prime ideal of L is normal. Let I be any ideal of L. Clearly, we have that $\beta(I) \subseteq I$. Let $x \in I$. Now we prove that $x \in \beta(I)$. Suppose $x \notin \beta(I)$. Then $(x)^\circ \vee I \neq L$. Then there exists a prime ideal P of an ADL L such that $(x)^\circ \vee I \subseteq P$. That implies $(x)^\circ \subseteq P$ and $I \subseteq P$. Since $x \in I$, we have $x \in P$. By our assumption, P is a normal prime ideal of L. Since $(x)^\circ \subseteq P$, we get that $x \notin P$, which is a contradiction to $x \in P$. Therefore $x \in \beta(I)$ and hence $I \subseteq \beta(I)$. Therefore $\beta(I) = I$. Thus I is a strongly disjunctive ideal of L. ∎

Theorem 4.6. *Let P be a prime ideal of a normal ADL L with maximal elements. If P is strongly disjunctive, then P is normal.*

Proof. Assume that P is a strongly disjunctive ideal of L. We prove that P is normal. Let $x \in P$. Then $x \in \beta(P)$. That implies $(x)^\circ \vee P = L$. Let m be any maximal element of L such that $m \in (x)^\circ \vee P$. Then there exist elements $a \in (x)^\circ$ and $b \in P$ such that $m = a \vee b$. Since $a \in (x)^\circ$, we have that $(a)^* \vee (x)^* = L$. Since L is normal, we get that $(a)^\circ \vee (x)^\circ = L$. Since $a \vee b = m$ and $b \in P$, we get that $a \notin P$. Therefore P is normal. ∎

Theorem 4.7. *Let L be an ADL with maximal elements. Then the following are equivalent:*

1. $(x)^\circ \vee (x)^{\circ\circ} = L$, *for all* $x \in L$.
2. *Every ideal I of the form $I = I^{\circ\circ}$, is strongly disjunctive.*
3. *For each $x \in L$, $(x)^{\circ\circ}$ is strongly disjunctive.*

Proof. $1 \Rightarrow 2:$ Assume that $(x)^\circ \vee (x)^{\circ\circ} = L$, for all $x \in L$. Let I be an ideal of L with $I = I^{\circ\circ}$. We prove that I is strongly disjunctive. Clearly, we have that $\beta(I) \subseteq I$. Let $x \in I$. Then $(x)^{\circ\circ} \subseteq I^{\circ\circ} = I$. By our assumption, we get that $(x)^\circ \vee I = L$. That implies $x \in \beta(I)$. Therefore $I \subseteq \beta(I)$ and hence $\beta(I) = I$. Thus I is a strongly disjunctive ideal of L.

$2 \Rightarrow 3:$ Clear.

$3 \Rightarrow 1:$ Assume that $(x)^{\circ\circ}$ is strongly disjunctive, for all $x \in L$. Then $\beta((x)^{\circ\circ}) = (x)^{\circ\circ}$. Since $x \in (x)^{\circ\circ}$, we get that $x \in \beta((x)^{\circ\circ})$. Therefore $(x)^\circ \vee (x)^{\circ\circ} = L$. ∎

Definition 4.8. For any maximal ideal M of an ADL L, define $\Omega(M) = \{x \in L \mid (x)^\circ \nsubseteq M\}$.

Lemma 4.9. *Let M be a maximal ideal of an ADL L with maximal elements. Then $\Omega(M)$ is an ideal of L contained in M.*

Proof. Clearly, we have that $0 \in \Omega(M)$ and hence $\Omega(M) \neq \emptyset$. Let $x, y \in \Omega(M)$. Then $(x)^{\circ} \nsubseteq M$ and $(y)^{\circ} \nsubseteq M$. Since M is prime, we get that $(x \vee y)^{\circ} = (x)^{\circ} \cap (y)^{\circ} \nsubseteq M$. Therefore $x \vee y \in \Omega(M)$. Let $x \in \Omega(M)$. Then $(x)^{\circ} \nsubseteq M$. Let r be any element of L. Since $(x)^{\circ} \subseteq (x \wedge r)^{\circ}$, we get that $(x \wedge r)^{\circ} \nsubseteq M$. Therefore $x \wedge r \in \Omega(M)$. Hence $\Omega(M)$ is an ideal of L. Let $x \in \Omega(M)$. Then $(x)^{\circ} \nsubseteq M$. Choose an element $a \in (x)^{\circ}$ such that $a \notin M$. That implies $(a)^{*} \vee (x)^{*} = L$. Let m be any maximal element of L such that $m \in (a)^{*} \vee (x)^{*}$. Then there exist elements $b \in (a)^{*}$ and $y \in (x)^{*}$ such that $m = b \vee y$. Since $b \in (a)^{*}$ and $y \in (x)^{*}$, we have that $a \wedge b = 0$ and $y \wedge x = 0$. Since $a \notin M$ and $a \wedge b = 0$, we get that $b \in M$. Since $b \in M$ and $m = b \vee y$, we get that $y \notin M$. Since $y \wedge x = 0$ and $y \notin M$, we get that $x \in M$. Therefore $\Omega(M) \subseteq M$. ∎

We denote the set of all maximal ideals of an ADL by $Max\ L$. For any ideal I of an ADL, $K(I) = \{M \in Max\ L \mid I \subseteq M\}$.

Theorem 4.10. *Let I be an ideal of an ADL L. Then $\beta(I) = \bigcap_{M \in K(I)} \Omega(M)$.*

Proof. Let $x \in \beta(I)$ and $M \in K(I)$. Then $(x)^{\circ} \vee I = L$ and $I \subseteq M$. That implies $(x)^{\circ} \vee M = L$. Now we prove that $(x)^{\circ} \nsubseteq M$. Suppose $(x)^{\circ} \subseteq M$. Then $M = L$, which is a contradiction. Therefore $(x)^{\circ} \nsubseteq M$ and hence $x \in \Omega(M)$, for all $M \in K(I)$. Thus $\beta(I) \subseteq \bigcap_{M \in K(I)} \Omega(M)$. Conversely, let $x \in \bigcap_{M \in K(I)} \Omega(M)$. Then $x \in \Omega(M)$, for all $M \in K(I)$. That implies $(x)^{\circ} \nsubseteq M$, for all $M \in K(I)$. We prove that $x \in \beta(I)$. Suppose $x \notin \beta(I)$. Then $(x)^{\circ} \vee I \neq L$. Then there exists a maximal ideal N of L such that $(x)^{\circ} \vee I \subseteq N$. That implies $(x)^{\circ} \subseteq N$, which is a contradiction to $(x)^{\circ} \nsubseteq M$, for all $M \in K(I)$. Therefore $x \in \beta(I)$ and hence $\bigcap_{M \in K(I)} \Omega(M) \subseteq \beta(I.)$ Thus $\beta(I) = \bigcap_{M \in K(I)} \Omega(M)$. ∎

Finally, a set of all equivalent conditions are established for the set of all strongly disjunctive ideals of an ADL to become a sublattice of the ideal lattice of an ADL.

Theorem 4.11. *Let L be an ADL with maximal elements. Then the following conditions are equivalent:*

1. *For any $M \in Max\ L$, $\Omega(M)$ is maximal.*
2. *For any ideals I, J of L, $I \vee J = L$ implies $\beta(I) \vee \beta(J) = L$.*
3. *For any ideals I, J of L, $\beta(I \vee J) = \beta(I) \vee \beta(J)$.*
4. *If $M, N \in Max\ L$ with $M \neq N$, then $\Omega(M) \vee \Omega(N) = L$.*
5. *For any $M \in Max\ L$, M is the unique member of $Max\ L$ such that $\Omega(M) \subseteq M$.*

Proof. $1 \Rightarrow 2$: Assume 1. Let I, J be any ideals of L with $I \vee J = L$. Now, we prove that $\beta(I) \vee \beta(J) = L$. Suppose $\beta(I) \vee \beta(J) \neq L$. Then there exists

$N \in Max\ L$ such that $\beta(I) \vee \beta(J) \subseteq N$. That implies $\beta(I) \subseteq N$ and $\beta(J) \subseteq N$. Since $\beta(I) \subseteq N$, we we get that $\bigcap_{N \in K(I)} \Omega(N) \subseteq N$. Since N is a prime ideal of L, there exists $N_i \in K(I)$ such that $\Omega(N_i) \subseteq N$. By our assumption, we get that $N_i \subseteq N$. Since $N_i \in K(I)$, we get that $I \subseteq N$. Since $\beta(J) \subseteq N$, we get that $J \subseteq N$. That implies $I \vee J \subseteq N$. Since $I \vee J = L$, we get that $L = N$, which is a contradiction. Therefore $\beta(I) \vee \beta(J) = L$.

$2 \Rightarrow 3$: Assume 2. Clearly, we have that $\beta(I) \vee \beta(J) \subseteq \beta(I \vee J)$. Let $x \in \beta(I \vee J)$. Then $(x)^\circ \vee (I \vee J) = L$. That implies $(x)^\circ \vee I \vee (x)^\circ \vee J = L$. By our assumption, we get that $\beta((x)^\circ \vee I) \vee \beta((x)^\circ \vee J) = L$. That implies $x \in \beta((x)^\circ \vee I) \vee \beta((x)^\circ \vee J)$. Then there exist elements $a \in \beta((x)^\circ \vee I)$ and $b \in \beta((x)^\circ \vee J)$ such that $x = a \vee b$. Since $a \in \beta((x)^\circ \vee I)$, we have that $(a)^\circ \vee (x)^\circ \vee I = L$. Since $(a)^\circ \vee (x)^\circ \subseteq (a \wedge x)^\circ$, we get that $(a \wedge x)^\circ \vee I = L$ and hence $a \wedge x \in \beta(I)$. Similarly, we that $b \wedge x \in \beta(J)$. Now $x = x \wedge x = (a \vee b) \wedge x = (a \wedge x) \vee (b \wedge x) \in \beta(I) \vee \beta(J)$. Therefore $x \in \beta(I) \vee \beta(J)$ and hence $\beta(I \vee J) \subseteq \beta(I) \vee \beta(J)$. Thus $\beta(I \vee J) = \beta(I) \vee \beta(J)$.

$3 \Rightarrow 4$: Assume 3. Let $M,\ N \in Max\ L$ with $M \neq N$. Then choose elements $x,\ y \in L$ such that $x \in M \backslash N$ and $y \in N \backslash M$. Since $x \notin N$, we get that $N \vee (x] = L$. Let m_1 be any maximal element of L such that $m_1 \in N \vee (x]$. Then there exists an element $n \in N$ such that $n \vee x = m_1$. Since $y \notin M$, we get that $M \vee (y] = L$. Let m_2 be any maximal element of L such that $m_2 \in M \vee (y]$. Then there exists an element $m \in M$ such that $m \vee y = m_2$. Clearly, we have that $(n \vee x] = (m_1] = L$ and $(m \vee y] = (m_2] = L$. That implies $((n \vee v) \vee (m \vee y)] = (n \vee x] \vee (m \vee y] = L$. That implies $\beta(((n \vee v) \vee (m \vee y)]) = \beta(L)$. That implies $\beta((n \vee x]) \vee \beta((m \vee y]) = L$. Since $x, m \in M$ and $y, n \in N$, we get that $m \vee x \in M$ and $n \vee y \in N$. That implies $(m \vee x] \subseteq M$ and $(n \vee y] \subseteq N$. That implies $M \in K((n \vee x])$ and $N \in K((m \vee y])$. That $\beta((n \vee x]) \subseteq \Omega(M)$ and $\beta((m \vee y]) \subseteq \Omega(N)$. Since $\beta((n \vee x]) \vee \beta((m \vee y]) = L$, we get that $\Omega(M) \vee \Omega(N) = L$.

$4 \Rightarrow 5$: Assume 4. Clearly we have $\Omega(M) \subseteq M$, for all $M \in Max\ L$. We prove that M is unique. Let M, N be any two maximal ideal of L with $\Omega(M) \subseteq M$ and $\Omega(N) \subseteq M$. We prove that $M = N$. Suppose $M \neq N$. By our assumption, we have that $\Omega(M) \vee \Omega(N) = L$ and hence $M = L$, which is a contradiction. Therefore $M = N$.

$5 \Rightarrow 1$: Assume 5. Let $M \in Max\ L$. Clearly, we have that $\Omega(M) \subseteq M$. We prove that $\Omega(M)$ is maximal. Suppose N be any maximal ideal of L with $\Omega(M) \subseteq N$. Since $\Omega(N) \subseteq N$ and $\Omega(M) \subseteq N$, we get that $M = N$. Therefore $\Omega(M)$ is maximal. ■

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Received 26 April 2020
Revised 20 November 2020
Accepted 23 December 2020