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NEW KINDS OF PREFILTERS IN EQ-ALGEBRAS

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Abstract

In this paper, the notions of n-fold positive implicative prefilter and n-fold implicative prefilter in EQ-algebras are introduced and several properties, characterizations and equivalent conditions are provided. It is proved that the quotient EQ-algebra induced by an n-fold positive implicative prefilter is n-idempotent. Also, it is proved that in an n-idempotent EQ-algebra, any filter is an n-fold positive implicative filter. In the sequel, we investigate the relationships between these two types of prefilters. Finally, some characterizations of n-fold implicative prefilters in bounded EQ-algebras are given.

Keywords: (n-idempotent) EQ-algebra, n-fold positive implicative prefilter, n-fold implicative prefilter.

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1. Introduction

EQ-algebras were proposed by Novák and De Baets [9] to introduce a special algebra as the correspondence of truth values for higher-order fuzzy logic (or fuzzy type theory). Fuzzy type theory itself, is a generalization of classical type theory having only ‘equality’ as a connective. Another motivation arose from the equational style of proof in logic. EQ-algebras are a generalization of residuated lattices. It has three connectives meet ($\wedge$), product ($\odot$) and fuzzy equality ($\sim$).

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Unlike residuated lattices in which the implication operation is derived from the product, in EQ-algebras the implication operation is derived the fuzzy equality. Then in EQ-algebras, the implication operation and the product no longer strictly form an adjoint pair in general. So, it is natural to extend some notions of residuated lattices to EQ-algebras and study their properties.

The filter theory plays a fundamental role in studying algebras of logic. From logical point of view, various filters correspond to various sets of provable formulas. In residuated lattices [11], several types of filters have been introduced, say [6, 9] and some important results have been obtained. For EQ-algebras, the notions of prefilters (which coincide with filters in residuated lattices) were proposed and some of their properties were obtained [2]. Furthermore, the notions of implicative and positive implicative prefilters in EQ-algebras were studied in [2].

As a generalization of (positive) implicative filters, the notion of n-fold (positive) implicative filters in some classes of residuated lattices such as BL-algebras and MTL-algebras were proposed [5, 12]. This motivates us to introduce the notions of n-fold implicative and n-fold positive implicative prefilters in EQ-algebras and investigate the properties and characterized them as it was done in residuated lattices.

This paper is organized as follows. In Section 2, the basic definitions, properties and theorems of EQ-algebras are reviewed. In Section 3, the notion of n-fold positive implicative prefilters of an EQ-algebras are defined and characterized. In Section 4, the notion of n-fold implicative prefilters of an EQ-algebras are introduced and characterizations of them are presented. Finally, some results which are proved directly in [8] for positive implicative prefilters and implicative prefilters in EQ-algebras are obtained.

2. Preliminaries

In this section, we give some fundamental definitions and results from the literature. For more details, we refer to the references.

Definition 2.1 [2, 9]**.** An EQ-algebra is an algebra $(L, \wedge, \odot, \sim, 1)$ of type (2,2,2,0) satisfying the following axioms:

(E1) $(L, \wedge, 1)$ is a $\wedge$-semilattice with top element 1. We set $x \leq y$ if and only if $x \wedge y = x$,

(E2) $(L, \odot, 1)$ is a commutative monoid and $\odot$ is isotone with respect to $\leq$,

(E3) $x \sim x = 1$,

(E4) $((x \wedge y) \sim z) \odot (s \sim x) = z \sim (s \wedge y)$,

(E5) $(x \sim y) \odot (s \sim t) \leq (x \sim s) \sim (y \sim t)$, (E6) $(x \wedge y \wedge z) \sim x \leq (x \wedge y) \sim x$,

(E7) $x \odot y \leq x \sim y$.

In any EQ-algebra L, the auxiliary operation '$\to$' (implication) is defined as

$$x \to y = (x \wedge y) \sim x.$$

Also, if L is bounded with the bottom element 0, the unary operation $\neg$ is defined as $\neg x = x \sim 0$. In this case, it is obvious that $\neg x = x \sim 1$ and $\neg 0 = 1$.

Definition 2.2 [9]**.** An EQ-algebra L is said to be

(i) separated, if $x \sim y = 1$ implies $x = y$ for all $x, y \in L$,

(ii) residuated, if for all $x, y, z \in L$ it satisfies
(Res) $(x \odot y) \wedge z = x \odot y \;\Leftrightarrow\; x \wedge ((y \wedge z) \sim y) = x$.
Classically, (Res) can be written as $x \odot y \leq z \Leftrightarrow x \leq y \to z$.

(iii) idempotent, if $x \odot x = x$ for all $x \in L$.

Lemma 2.3 [2]**.** Let L be an EQ-algebra. The following properties hold for any $x, y, z \in L$.

(1) $x \sim y = y \sim x$, $x \sim y \leq x \to y$,

(2) $x \odot y \leq x \wedge y \leq x, y$. Particularly, for natural number n, $x^n \leq x$, where $x^n = x \odot \cdots \odot x$ (n times).

(3) $x \leq 1 \sim x = 1 \to x \leq y \to x$.

(4) $x \to y \leq (y \to z) \to (x \to z)$.

(5) $x \to y \leq (z \to x) \to (z \to y)$.

(6) If $x \leq y$, then $x \to y = 1$ and $x \sim y = y \to x$.

(7) If $x \leq y$, then $z \to x \leq z \to x$ and $y \to z \leq x \to z$.

Theorem 2.4 [9]**.** In any EQ-algebra L, the following are equivalent:

(1) L is residuated.

(2) L is separated and satisfies $(x \odot y) \to z = x \to (y \to z)$.

(3) L is separated and satisfies $x \leq (x \to y) \to y$ for any $x, y \in L$.

Definition 2.5 [2, 8]**.** Let L be an EQ-algebra and F be a nonempty subset of L satisfying $1 \in F$.

(i) F is called a prefilter if it satisfies
(F2) $x, x \to y \in F$ imply $y \in F$.

(ii) F is called a filter if it satisfies (F2) and
(F3) if $x \to y \in F$, then $(x \odot z) \to (y \odot z) \in F$ for any $x, y, z \in L$.

(iii) F is called a positive implicative prefilter (resp. filter) if it satisfies (F2) (resp. (F2) and (F3)) and
(F4) $x \to (y \to z) \in F, x \to y \in F \Rightarrow x \to z \in F$, for all $x, y, z \in L$.

(iv) F is called an implicative prefilter (resp. filter) if it satisfies (F2) (resp. (F2) and (F3)) and
(F5) $z \to ((x \to y) \to x) \in F, z \in F \Rightarrow x \in F$, for all $x, y, z \in L$.

Remark 2.6. Obviously, any residuated EQ-algebra satisfies the property (WEP) $x \to (y \to z) = y \to (x \to z)$, while it may not be true in general. However, an EQ-algebra may contain those prefilters satisfying (WEP). Such a prefilters are said to satisfies the weak exchange principle. More general, EQ-algebra L is said to satisfies exchange principle if it satisfies (WEP) (see [8, Example 2.3].

Lemma 2.7 [2]**.** Any prefilter F of EQ-algebra L satisfies the following: for all $x, y, z \in L$,

(1) If $x \in F$ and $x \leq y$, then $y \in F$.

(2) If $x, x \sim y \in F$, then $y \in F$.

(3) If $x \sim y \in F$ and $y \sim z \in F$, then $x \sim z \in F$.

(4) If $x \to y \in F$ and $y \to z \in F$, then $x \to z \in F$.

In an EQ-algebra L, any prefilter F induces an equivalence relation $\equiv_F$ as follows:

$$(\forall x, y \in L) \;\; x \equiv_F y \Leftrightarrow x \sim y \in F.$$

If F is a filter of L, $\equiv_F$ is a congruence in L and so $\frac{L}{F}$, the set of all equivalence classes together with those operations induced from L forms an EQ-algebra, which is also separated (see [2]).

From now on, in this paper $L = (L, \wedge, \odot, \sim, 1)$ is an *EQ*-algebra, unless otherwise stated.

3. n-fold positive implicative prefilters

In this section, we introduce the notion of n-fold positive implicative prefilter and give some related results.

Definition 3.1. A prefilter (resp. filter) F of L is called an *n-fold positive implicative prefilter* (*filter*) if

(F5) $x^n \to (y \to z) \in F$, $x^n \to y \in F$ imply $x^n \to z \in F$, for all $x, y, z \in L$.

Example 3.2. Let $L = \{0, a, b, c, d, 1\}$ be a chain with the ordering $0 < a < b < c < d < 1$. Then L together with the operations $\odot$ and $\sim$ as shown in Tables 1 and 2 forms an EQ-algebra [9].

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	0	0	a	b
c	0	0	0	a	a	c
d	0	0	a	a	a	d
1	0	a	b	c	d	1

Table 1. Cayley table of $\odot$.

$\sim$	0	a	b	c	d	1
0	1	c	b	a	0	0
a	c	1	b	a	a	a
b	b	b	1	b	b	b
c	a	a	b	1	c	c
d	0	a	b	c	1	d
1	0	a	b	c	d	1

Table 2. Cayley table of $\sim$.

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	1	1	1	1
b	b	b	1	1	1	1
c	a	a	b	1	1	1
d	0	a	b	c	1	1
1	0	a	b	c	d	1

Table 3. Cayley table of $\to$.

One can see that $F = \{1, d\}$ is a 3-fold positive implicative prefilter of L, while it is not a 2-fold positive implicative prefilter, because $c^2 \to (a \to 0) = (c \odot c) \to c = a \to c = 1 \in F$ and $c^2 \to a = (c \odot c) \to a = a \to a = 1 \in F$, but $c^2 \to 0 = a \to 0 = c \notin F$.

Theorem 3.3. *Let F be an n-fold positive implicative prefilter of L. Then for any $x, y \in L$,*

$$(x^n \odot (x \to y)^n) \to y \in F. \tag{3.1}$$

Particularly, $(1 \to x)^n \to x \in F$.

Proof. Let F be an n-fold positive implicative prefilter of L and $x, y \in L$. Since $x^n \odot (x \to y)^n \leq x^n \leq x$ and $x^n \odot (x \to y)^n \leq (x \to y)^n \leq x \to y$, so by Lemma 2.3(6) we have

$$(x \odot (x \to y))^n \to x = (x^n \odot (x \to y)^n) \to x = 1 \in F$$

and

$$\begin{aligned}(x \odot (x \to y))^n \to (x \to y) &= (x^n \odot (x \to y)^n) \to (x \to y) \\ &= 1 \in F.\end{aligned}$$

Now, since F is an n-fold positive implicative prefilter we conclude that

$$(x^n \odot (x \to y)^n) \to y = (x \odot (x \to y))^n \to y \in F.$$

For the second part, it suffices in (3.1) we take $x = 1$ and $y = x$. ■

Theorem 3.4. *Let F be a prefilter of L. Then the following are equivalent:*
(1) *F is an n-fold positive implicative prefilter of L,*
(2) *$x^n \to (x^n \to y) \in F$ imply $x^n \to y \in F$, for any $x, y \in L$.*

Proof. (1)⇒(2) Let F be an n-fold positive implicative prefilter of L and $x^n \to (x^n \to y) \in F$, for $x, y \in L$. Since $x^n \to x^n = 1 \in F$ we have $x^n \to y \in F$.

(2)⇒(1) Let F be a prefilter of L, $x^n \to (y \to z) \in F$ and $x^n \to y \in F$, for $x, y, z \in L$. From Lemma 2.3(4) we get

$$x^n \to (y \to z) \leq ((y \to z) \to (x^n \to z)) \to (x^n \to (x^n \to z)),$$

whence

$$((y \to z) \to (x^n \to z)) \to (x^n \to (x^n \to z)) \in F. \tag{3.2}$$

Again by Lemma 2.3(4), we get $x^n \to y \leq (y \to z) \to (x^n \to z)$, whence

$$(y \to z) \to (x^n \to z) \in F. \tag{3.3}$$

Now, since F is a prefilter, by (3.2) and (3.3) we conclude that $x^n \to (x^n \to z) \in F$ and by (2), $x^n \to z \in F$. Therefore, F is an n-fold positive implicative prefilter of L. ■

Corollary 3.5. *If F is an n-fold positive implicative prefilter of L satisfying $x^n \sim (x^n \to y) \in F$, for any $x, y \in L$, then $x^n \to y \in F$.*

Proof. It follows from Lemma 2.3(1) and Theorem 3.4. ∎

Theorem 3.6. *Let F be a prefilter of L and satisfies the weak exchange principle. Then the following are equivalent:*

(1) *F is an n-fold positive implicative prefilter of L,*

(2) *If $x^n \to (y \to z) \in F$, then $(x^n \to y) \to (x^n \to z) \in F$, for any $x, y, z \in L$.*

Proof. (1)⇒(2) Let F be an n-fold positive implicative prefilter of L and $x^n \to (y \to z) \in F$, for $x, y, z \in L$. Then by Lemma 2.3(4),

$$(x^n \to y) \to y \leq (y \to z) \to ((x^n \to y) \to z)$$

and so by Lemma 2.3(4),

$$x^n \to ((x^n \to y) \to y) \leq x^n \to ((y \to z) \to ((x^n \to y) \to z)).$$

Since $(x^n \to y) \to (x^n \to y) = 1 \in F$ and F satisfies the weak exchange principle, we get that $x^n \to ((x^n \to y) \to y) = 1 \in F$ and so we have

$$x^n \to ((y \to z) \to ((x^n \to y) \to z)) \in F.$$

Now, since $x^n \to (y \to z) \in F$ and F is an n-fold positive implicative prefilter, we get that $x^n \to ((x^n \to y) \to z) \in F$ and so $(x^n \to y) \to (x^n \to z) \in F$.

(2)⇒(1) Suppose that $x^n \to (y \to z) \in F$ and $x^n \to y \in F$, for $x, y, z \in F$. By (2), $(x^n \to y) \to (x^n \to z) \in F$ and since F is a prefilter of L and $x^n \to y \in F$, we have $x^n \to z \in F$. Therefore, F is a n-fold positive implicative prefilter of L. ∎

Corollary 3.7. *Let L be a residuated EQ-algebra and F be a prefilter of L. Then the following are equivalent:*

(1) *F is an n-fold positive implicative prefilter,*

(2) *if $x^n \to (x^n \to y) \in F$, then $x^n \to y \in F$,*

(3) *if $x^n \to (y \to z) \in F$, then $(x^n \to y) \to (x^n \to z) \in F$ for any $x, y, z \in L$.*

Proof. The proof follows from Theorems 3.4 and 3.6 and Remark 2.6. ∎

Theorem 3.8. *Let $F \subseteq Q$ be two prefilters of L and L has exchange principle. If F is an n-fold positive implicative prefilter, then so is Q.*

Proof. Let $F \subseteq Q$ and F be an n-fold positive implicative prefilter of L. If $x^n \to (x^n \to y) \in Q$, for $x, y \in L$, then since L has exchange principle, we get that

$$\begin{aligned}
&x^n \to \big(x^n \to ((x^n \to (x^n \to y)) \to y)\big)\\
&= x^n \to \big((x^n \to (x^n \to y)) \to (x^n \to y)\big)\\
&= (x^n \to (x^n \to y)) \to (x^n \to (x^n \to y))\\
&= 1 \in F.
\end{aligned}$$

Hence, $x^n \to \big(x^n \to (x^n \to ((x^n \to y) \to y))\big) \in F$. Since F is an n-fold positive implicative prefilter of L, by Theorem 3.4, we get that

$$x^n \to ((x^n \to (x^n \to y)) \to y) \in F \subseteq Q.$$

And so we have

$$(x^n \to (x^n \to y)) \to (x^n \to y) \in F \subseteq Q.$$

Now, since $x^n \to (x^n \to y) \in Q$, we conclude that $x^n \to y \in Q$. Therefore, by Theorem 3.4, Q is an n-fold positive implicative prefilter of L. ■

Theorem 3.9. *Let L be an EQ-algebra satisfying*

$$x \odot y \to z \leq x \to (y \to z) \tag{3.4}$$

and F be an n-fold positive implicative prefilter of L. Then for any $x \in L$, $x^n \to x^{2n} \in F$.

Proof. Let F be an n-fold positive implicative prefilter filter of L. From (3.4) it follows that

$$x^n \odot x^n \to x^n \odot x^n \leq x^n \to (x^n \to (x^n \odot x^n))$$

and since

$$x^n \odot x^n \to x^n \odot x^n = x^{2n} \to x^{2n} = 1 \in F,$$

we get that $x^n \to (x^n \to (x^n \odot x^n)) \in F$. Since F is an n-fold positive implicative prefilter, by Theorem 3.4, we conclude that $x^n \to x^{2n} = x^n \to x^n \odot x^n \in F$. ■

Theorem 3.10. *Let F be a filter satisfying the weak exchange principle such that for any $x, y \in L$, $x^n \to x^{2n} \in F$ and $(x^n \odot (x^n \to y)) \to y \in F$. Then F is an n-fold positive implicative filter of L.*

Proof. Suppose that F is a filter satisfying the weak exchange principle, $x^n \to (y \to z) \in F$ and $x^n \to y \in F$, for $x, y, z \in L$. Then $y \to (x^n \to z) \in F$ and so by (F3) we get that $x^n \odot y \to x^n \odot (x^n \to z) \in F$ and $x^n \odot x^n \to x^n \odot y \in F$. By Lemma 2.7(4), $x^n \odot x^n \to x^n \odot (x^n \to z) \in F$ and since $x^n \to x^n \odot x^n \in F$, we get that $x^n \to (x^n \odot (x^n \to z)) \in F$. On the other hand, by hypothesis we have $(x^n \odot (x^n \to z)) \to z \in F$ and so by Lemma 2.7(4) we get that $x^n \to z \in F$. Therefore, F is an n-fold implicative filter of L. ■

As a result of Theorems 3.9 and 3.10, we provide the following theorem which is proved in [8] directly.

Theorem 3.11. *Let L be an EQ-algebra satisfying* (3.4) *and F be a filter of L. Then the following are equivalent:*

(1) *F is a positive implicative filter,*

(2) *for any $x, y \in L$, $x \to x^2 \in F$ and $(x \odot (x \to y)) \to y \in F$.*

Theorem 3.12. *Let F be a filter of residuated EQ-algebra L. Then F is an n-fold positive implicative filter of L if and only if $x^n \to x^{2n} \in F$, for any $x \in L$.*

Proof. Let F be an n-fold positive implicative filter of residuated EQ-algebra L. Then L satisfies in (3.4) and so by Theorem 3.9, $x^n \to x^{2n} \in F$, for any $x \in L$. Conversely, let $x^n \to (x^n \to y) \in F$, for $x, y \in L$. Then $x^n \odot x^n \to y = x^{2n} \to y \in F$, and so by Lemma 2.7(4), $x^n \to y \in F$. Therefore, F is an n-fold positive implicative filter of L. ■

Theorem 3.13. *In an EQ-algebra satisfying* (3.4), *any n-fold positive implicative prefilter is an $(n+1)$-fold positive implicative prefilter.*

Proof. Let L be an EQ-algebra satisfying (3.4) and F be an n-fold positive implicative prefilter of L. Also, assume that $x^{n+1} \to (y \to z) \in F$ and $x^{n+1} \to z \in F$, for $x, y, z \in L$. Since $x^{2n} = x^{n+n} \leq x^{n+1}$, by Lemma 2.3(7) we have $x^{n+1} \to (y \to z) \leq x^{n+n} \to (y \to z)$ and $x^{n+1} \to y \leq x^{n+n} \to y$. Hence

$$(x^2)^n \to (y \to z) = x^{n+n} \to (y \to z) \in F$$

and $(x^2)^n \to y = x^{n+n} \to y \in F$. Now, since F is an n-fold positive implicative prefilter of L, we get that $x^{2n} \to z = (x^2)^n \to z \in F$. From Theorem 3.9 we know that $x^n \to x^{2n} \in F$, whence by Lemma 2.7(4), $x^n \to z \in F$. Again from $x^{n+1} \leq x^n$ we get that $x^n \to z \leq x^{n+1} \to z$, whence $x^{n+1} \to z \in F$. Thus F is an $(n+1)$-fold positive implicative prefilter of L. ■

Theorem 3.14. *Let F be a filter of residuated EQ-algebra L. Then F is an n-fold positive implicative filter of L if and only if $x^{n+1} \to y \in F$ implies $x^n \to y \in F$, for any $x, y \in L$.*

Proof. Assume that F is an n-fold positive implicative filter of L and $x^{n+1} \to y \in F$, for $x, y \in L$. Then $x^n \to (x \to y) \in F$ and since $x^n \to x = 1 \in F$ and F is an n-fold positive implicative filter, we conclude that $x^n \to y \in F$. Conversely, Let $x^n \to (x^n \to y) \in F$ for $x, y \in L$. Then

$$x^{n+1} \to (x^{n-1} \to y) = (x^{n+1} \odot x^{n-1}) \to y = (x^n \odot x^n) \to y \in F$$

and so by hypothesis we get $x^n \to (x^{n-1} \to y) \in F$. Continuing this process we conclude that $x^{n+1} \to y \in F$, whence by hypothesis, it follows that $x^n \to y \in F$. Therefore, by Theorem 3.4, F is an n-fold positive implicative filter. ∎

Definition 3.15. Let L be an EQ-algebra. Then L is called an (n-*fold implicative*) n-idempotent EQ-algebra if $x^n = x^n \odot x^n$, for all $x \in L$.

Obviously, any idempotent EQ-algebra a 1-idempotent EQ-algebra.

Example 3.16. Let $L = \{0, a, b, 1\}$ be a chain with the ordering $0 < a < b < 1$. Then L together with the operations $\odot$ and $\sim$ as shown in Tables 4 and 5 forms an EQ-algebra. Routine calculations show that L an n-idempotent EQ-algebra, for any natural number n.

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

Table 4. Cayley table of $\odot$.

$\sim$	0	a	b	1
0	1	0	0	0
a	0	1	a	a
b	0	a	1	1
1	0	a	1	1

Table 5. Cayley table of $\sim$.

$\to$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	a	1	1
1	0	a	1	1

Table 6. Cayley table of $\to$.

Theorem 3.17. *Let L be a residuated EQ-algebra and F be a filter of L. Then the following are equivalent:*

(1) *F is an n-fold positive implicative filter,*

(2) $\frac{L}{F}$ *is a residuated* n*-idempotent* EQ*-algebra.*

Proof. (1)⇒(2) Let F be an n-fold positive implicative filter of residuated EQ-algebra L. Then $\frac{L}{F}$ is a residuated EQ-algebra. By Theorem 3.12, $x^n \to x^n \odot x^n \in F$, for any $x \in L$. Hence, $[x]_F^n \leq [x]_F^n \odot [x]_F^n$ and so $[x]_F^{2n} = [x]_F^n$ and so by Definition 3.15, $\frac{L}{F}$ is an n-idempotent EQ-algebra.

(2)⇒(1) Let $\frac{L}{F}$ be a residuated n-idempotent EQ-algebra. Then $[x]_F^n \odot [x]_F^n = [x]_F^n$ and for any $[x]_F \in \frac{L}{F}$. Hence $x^n \to x^n \odot x^n \in F$, for any $x \in L$. Therefore, by Theorem 3.12, F is an n-fold positive implicative filter. ■

Theorem 3.18. *Let* L *be a residuated* EQ*-algebra and* F *be a filter of* L*. Then the following are equivalent:*

(1) F *is an* n*-fold positive implicative filter,*

(2) *If* $(x^n \odot y) \to z \in F$*, then* $(x \wedge y)^n \to z \in F$*, for any* $x, y, z \in L$*,*

(3) *If* $x^n \to (x^n \to y) \in F$*, then* $x^n \to y \in F$*, for any* $x, y \in L$*,*

(4) *If* $x^n \to (y \to z) \in F$*, then* $(x^n \to y) \to (x^n \to z) \in F$*, for any* $x, y, z \in L$*,*

(5) *If* $x^{n+1} \to y \in F$*, then* $x^n \to y \in F$*, for any* $x, y \in L$*,*

(6) $x^n \to x^n \odot x^n \in F$*, for any* $x \in L$*,*

(7) $\frac{L}{F}$ *is a residuated* n*-idempotent* EQ*-algebra.*

Proof. (1)⇒(2) Let F be an n-fold positive implicative filter and $(x^n \odot y) \to z \in F$, for $x, y \in L$. Then

$$(x^n \odot y) \to z = x^n \to (y \to z) = y \to (x^n \to z) \in F.$$

By Lemma 2.3(7) we have

$$\begin{aligned} y \to (x^n \to z) &\leq x \wedge y \to (x^n \to z) = x^n \to (x \wedge y \to z) \\ &\leq (x \wedge y)^n \to ((x \wedge y) \to z). \end{aligned}$$

Hence, $(x \wedge y)^n \to ((x \wedge y) \to z) \in F$ and since $(x \wedge y)^n \to (x \wedge y) = 1 \in F$, we get that $(x \wedge y)^n \to z \in F$.

(2)⇒(1) Let $x^n \to (x^n \to y) \in F$, for $x, y \in L$. Then $x^n \odot x^n \to y \in F$ and so by (1), $(x^n \wedge x)^n \to z \in F$ and since $x^n \leq x$, we get that $x^n \to z \in F$. Therefore, F is an n-fold positive implicative filter of L.

By Theorems 3.4, 3.6, 3.12, 3.14 and 3.17, the parts (1), (3), (4), (5), (6) and (7) are equivalent. ■

Since any residuated lattice (and so any BL-algebra [7], MV-algebra [1], MTL-algebra [4], R_0-algebra [10]) is a residuated EQ-algebra, by Theorem 3.18, we have the following corollary.

Corollary 3.19. *Let F be a filter of residuated lattice L. Then the following are equivalent:*

(1) *F is an n-fold positive implicative filter,*

(2) *if $(x^n \odot y) \to z \in F$, then $(x \wedge y)^n \to z \in F$, for any $x, y, z \in L$,*

(3) *if $x^n \to (x^n \to y) \in F$, then $x^n \to y \in F$, for any $x, y \in L$,*

(4) *if $x^n \to (y \to z) \in F$, then $(x^n \to y) \to (x^n \to z) \in F$, for any $x, y, z \in L$,*

(5) *if $x^{n+1} \to y \in F$, then $x^n \to y \in F$, for any $x, y \in L$,*

(6) *$x^n \to x^n \odot x^n \in F$, for any $x, y \in L$,*

(7) *$\frac{L}{F}$ is an n-idempotent residuated lattice.*

From Theorems 3.8 and 3.18 and that any residuated EQ-algebra satisfies the weak exchange principle we get

Corollary 3.20. *Let L be residuated EQ-algebra. Then the following are equivalent:*

(1) *L is an n-idempotent EQ-algebra,*

(2) *Every filter of L is an n-fold positive implicative filter,*

(3) *$\{1\}$ is an n-fold positive implicative filter.*

4. n-fold implicative prefilters

In this section, we introduce the notion of n-fold implicative prefilter and we give some related results.

Definition 4.1. Let L be an EQ-algebra and F be a nonempty subset of L. Then F is called an n-fold implicative prefilter (resp. filter) if $1 \in F$ and it satisfies

(F6) $z \to ((x^n \to y) \to x) \in F$ and $z \in F$ imply $x \in F$, for any $x, y, z \in L$.

Example 4.2. Consider the following Cayley tables (Tables 7 and 8) showing an EQ-algebra structure on the chain $L = \{0, a, b, c, 1\}$ with the order $0 < a < b < c < 1$.

It is not difficult to verify that the set $F = \{a, b, c, 1\}$ is an n-fold implicative prefilter, for any natural number n.

Theorem 4.3. *In an EQ-algebra, any n-fold implicative prefilter is a prefilter.*

Proof. By Definition 4.1, $1 \in F$. Let $x \to y \in F$ and $x \in F$. Then by Lemma 2.3(3) and (7), $y \leq 1 \to y$ and $x \to y \leq x \to (1 \to y) = x \to ((y^n \to 1) \to y)$, whence $x \to ((y^n \to 1) \to y) \in F$ and since $x \in F$, we get that $y \in F$. Therefore, F is a prefilter. ∎

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	0	a
b	0	0	0	0	b
c	0	0	0	0	c
1	0	a	b	c	1

Table 7. Cayley table of $\odot$.

$\sim$	0	a	b	c	1
0	1	0	0	0	0
a	0	1	b	b	b
b	0	b	1	c	c
c	0	b	c	1	1
1	0	b	c	1	1

Table 8. Cayley table of $\sim$.

Theorem 4.4. *Let F be a prefilter of L. Then the following are equivalent:*

(1) *F is an n-fold implicative prefilter of L,*

(2) *$(x^n \to y) \to x \in F$ implies, $x \in F$ for any $x, y \in L$.*

Proof. (1)⇒(2) Suppose that F is an n-fold implicative prefilter and $(x^n \to y) \to x \in F$. Since by Lemma 2.3(3), $(x^n \to y) \to x \leq 1 \to ((x^n \to y) \to x)$ we get that $1 \to ((x^n \to y) \to x) \in F$ and since $1 \in F$, we conclude that $x \in F$.

(2)⇒(1) Suppose that F is a prefilter. Then $1 \in F$, if $z \to ((x^n \to y) \to y) \in F$ and $z \in F$, then $(x^n \to y) \to x \in F$ and by (2), we get that $x \in F$. Therefore, F is an n-fold implicative filter. ■

Theorem 4.5. *Let F be an n-fold implicative filter of residuated EQ-algebra L. Then F is an n-fold positive implicative filter of L.*

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	b	1	1	1
c	0	b	c	1	1
1	0	b	c	1	1

Table 9. Cayley table of $\to$.

Proof. Let F be an n-fold implicative filter of residuated EQ-algebra L such that $x^{n+1} \to y \in F$, for $x, y \in L$. Then

$$\begin{aligned}
&(x^{n+1} \to y)^n \to (x^n \to y) \\
&= ((x^{n+1} \to y)^{n-1} \odot (x^{n+1} \to y)) \to (x^n \to y) \\
&= (x^{n+1} \to y)^{n-1} \to ((x^{n+1} \to y) \to (x^n \to y)) \\
&= (x^{n+1} \to y)^{n-1} \to ((x^{n+1} \to y) \to (x^{n-1} \to (x \to y))) \\
&= (x^{n+1} \to y)^{n-1} \to (x^{n-1} \to ((x^{n+1} \to y) \to (x \to y))) \\
&= (x^{n+1} \to y)^{n-1} \to (x^{n-1} \to ((x \to (x^n \to y)) \to (x \to y))) \\
&\geq (x^{n+1} \to y)^{n-1} \to (x^{n-1} \to ((x^n \to y) \to y)) \\
&= (x^{n+1} \to y)^{n-1} \to ((x^n \to y) \to (x^{n-1} \to y)) \\
&= (x^n \to y) \to ((x^{n+1} \to y)^{n-1} \to (x^{n-1} \to y)).
\end{aligned}$$

Now, we prove that

$$(x^{n+1} \to y)^n \to (x^n \to y) \geq (x^n \to y)^2 \to ((x^{n+1} \to y)^{n-2} \to (x^{n-2} \to y)).$$

Since $x^{n+1} \to y = x \to (x^n \to y)$, so $x \odot (x^{n+1} \to y) \leq (x^n \to y)$ and so

$$\begin{aligned}
&(x^n \to y) \odot (x^{n+1} \to y)^{n-1} \odot x^{n-1} \\
&= (x^n \to y) \odot (x^{n+1} \to y)^{n-2} \odot (x^{n+1} \to y) \odot x \odot x^{n-2} \\
&= (x^n \to y) \odot (x^{n+1} \to y)^{n-2} \odot x^{n-2} \odot x \odot (x^{n+1} \to y) \\
&\leq (x^n \to y) \odot (x^{n+1} \to y)^{n-2} \odot x^{n-2} \odot (x^n \to y) \\
&= (x^n \to y)^2 \odot (x^{n+1} \to y)^{n-2} \odot x^{n-2}.
\end{aligned}$$

Hence, by Lemma 2.3(7),

$$\begin{aligned}
&((x^n \to y)^2 \odot (x^{n+1} \to y)^{n-2} \odot x^{n-2}) \to y \\
&\leq ((x^n \to y) \odot (x^{n+1} \to y)^{n-1} \odot x^{n-1}) \to y.
\end{aligned}$$

Now,

$$\begin{aligned}
&((x^n \to y)^2 \to ((x^{n+1} \to y)^{n-2} \to (x^{n-2} \to y)) \\
&\leq ((x^n \to y) \to ((x^{n+1} \to y)^{n-1} \to (x^{n-1} \to y)).
\end{aligned}$$

Since

$$\begin{aligned}
&((x^n \to y) \to ((x^{n+1} \to y)^{n-1} \to (x^{n-1} \to y)) \\
&\leq (x^{n+1} \to y)^n \to (x^n \to y).
\end{aligned}$$

We get that

$$((x^n \to y)^2 \odot (x^{n+1} \to y)^{n-2} \odot x^{n-2}) \to y$$
$$\leq (x^{n+1} \to y)^n \to (x^n \to y).$$

By repeating this process we conclude that

$$((x^n \to y)^n \odot (x^{n+1} \to y)^{n-n} \odot x^{n-n}) \to y$$
$$\leq (x^{n+1} \to y)^n \to (x^n \to y).$$

Hence,

$$((x^n \to y)^n \to y) \leq (x^{n+1} \to y)^n \to (x^n \to y)$$

and so

$$((x^n \to y)^n \to y) \to ((x^{n+1} \to y)^n \to (x^n \to y)) = 1 \in F,$$

whence

$$(x^{n+1} \to y)^n \to ((x^n \to y)^n \to y) \to (x^n \to y)) \in F$$

and since F is a filter and $(x^{n+1} \to y) \in F$, we get that $(x^{n+1} \to y)^n \in F$. Hence

$$((x^n \to y)^n \to y) \to (x^n \to y)) \in F.$$

Now, since F is an n-fold implicative filter of L, by Theorem 4.4, $x^n \to y \in F$ and so by Theorem 3.14, F is an n-fold positive implicative filter of L. ■

Theorem 4.6. *Let L be a bounded EQ-algebra and F a prefilter of L. Then F is an n-fold implicative prefilter if and only if $\neg x^n \to x \in F$ implies, $x \in F$.*

Proof. Let F be a prefilter. Then by Lemma 2.3(6), from $0 \leq y$ we have $\neg x^n = x^n \to 0 \leq x^n \to y$. Hence, $(x^n \to y) \to x \leq \neg x^n \to x$. Now, if $(x^n \to y) \to x \in F$, then $\neg x^n \to x \in F$ and by hypothesis $x \in F$. Therefore, F is an n-fold implicative filter. Conversely, if $\neg x^n \to x \in F$, then $(x^n \to 0) \to x \in F$ and since F is an n-fold implicative prefilter, by Theorem 4.4, it follows that $x \in F$. ■

Theorem 4.7. *Let F and G be two prefilters of L such that $F \subseteq G$. If F is an n-fold implicative prefilter with the weak exchange principle, then G is an n-fold implictive prefilter.*

Proof. Let F be an n-fold implicative prefilter of L and $u = (x^n \to y) \to x \in G$, for $x, y \in L$. Then by Lemma 2.3(3) and (7), $x \leq u \to x$ and so $x^n \leq (u \to x)^n$ and $(u \to x)^n \to y \leq x^n \to y$. Hence

$$u = (x^n \to y) \to x \leq ((u \to x)^n \to y) \to x$$

and so $u \to ((u \to x)^n \to y) \to x = 1 \in F$ and since F has weak exchange principle, we get that $((u \to x)^n \to y) \to (u \to x) = 1 \in F$. Now, since F is an n-fold implicative prefilter, by Theorem 4.4, we conclude $u \to x \in F \subseteq G$ and by $u \in G$, we get that $x \in G$. Therefore, G is an n-fold implicative prefilter of L. ■

Theorem 4.8. *Let F be a prefilter of L. If F is an n-fold implicative prefilter, then F is an $(n+1)$-fold implicative prefilter.*

Proof. Let F be an n-fold implicative prefilter of L and $(x^{n+1} \to y) \to x \in F$, for $x, y \in L$. From $x^{n+1} \leq x^n$, by Lemma 2.3(7) we get $(x^{n+1} \to y) \to x \leq (x^n \to y) \to x$, whence $(x^n \to y) \to x \in F$. Now, by Theorem 4.4 we get that $x \in F$, means that F is an $(n+1)$-fold implicative prefilter of L. ■

Theorem 4.9. Assume that L is bounded and let F be a prefilter satisfying the weak exchange principle. Then F is an n-fold implicative prefilter if and only if $x \to (\neg z^n \to y) \in F$ and $y \to z \in F$ imply $x \to z \in F$.

Proof. Suppose that F is an n-fold implicative prefilter and $x \to (\neg z^n \to y) \in F$ and $y \to z \in F$, for $x, y, z \in L$. By Lemma 2.3(4) we have

$$\neg z^n \to (x \to y) \leq ((x \to y) \to (x \to z)) \to (\neg z^n \to (x \to z))$$

and

$$y \to z \leq (x \to y) \to (x \to z),$$

whence from $\neg z^n \to (x \to y) = x \to (z^n \to y) \in F$ and $y \to z \in F$ it follows that

$$((x \to y) \to (x \to z)) \to (\neg z^n \to (x \to z)) \in F$$

and $(x \to y) \to (x \to z) \in F$ and so $\neg z^n \to (x \to z) \in F$. Now, since $z \leq x \to z$, we get that $\neg z^n \to (x \to z) \leq \neg(x \to z)^n \to (x \to z)$, whence $\neg(x \to z)^n \to (x \to z) \in F$. Hence, by Theorem 4.6, $x \to z \in F$.

Conversely, let $\neg x^n \to x \in F$, for $x \in L$. Then by Lemma 2.3(3), $\neg x^n \to x \leq 1 \to (\neg x^n \to x)$ and so $1 \to (\neg x^n \to x) \in F$ and since $x \to x = 1$, we get that $1 \to x \in F$ and since $1 \in F$, we have $x \in F$. Therefore, by Theorem 4.6, F is an n-fold implicative prefilter of L. ■

Theorem 4.10. *Let F be an n-fold implicative prefilter with the weak exchange principle. Then $(x^n \to y) \to y \in F$ implies $(y \to x) \to x \in F$, for any $x, y \in L$.*

Proof. Let F be an n-fold implicative prefilter of L and $(x^n \to y) \to y \in F$. Put $u = (y \to x) \to x$. By Lemma 2.3(3), $(x^n \to y) \to y \leq (y \to x) \to ((x^n \to y) \to x)$ and since F is a prefilter with the weak exchange principle, we get that $(y \to x) \to ((x^n \to y) \to x) \in F$ and so $(x^n \to y) \to ((y \to x) \to x) = (x^n \to y) \to u \in F$. By Lemma 2.3(2), $x \leq (y \to x) \to x = u$ and so $x^n \leq u^n$. Hence, by Lemma 2.3(6), $u^n \to y \leq x^n \to y$ and so $(x^n \to y) \to u \leq (u^n \to y) \to u$. Therefore, $(u^n \to y) \to u \in F$ and since F is an n-fold implicative prefilter, by Theorem 4.4, we get that $u \in F$ and so $(y \to x) \to x \in F$. ∎

Theorem 4.11. *Let F be an n-fold positive implicative prefilter of L. If $(x \to y)^n \to y \in F$ implies $(y \to x) \to x \in F$, for any $x, y \in L$, then F is an n-fold implicative prefilter of L.*

Proof. Let $(x^n \to y) \to x \in F$, for $x, y \in L$. Then by Lemma 2.3(4), $(x^n \to y) \to x \leq y \to x$ and $(x^n \to y) \to x \leq (x \to y) \to ((x^n \to y) \to y)$ and so $y \to x \in F$ and $(x \to y) \to ((x^n \to y) \to y) \in F$. Since $(x \to y)^n \leq (x \to y)$, by Lemma 2.3(6), we get that

$$(x \to y) \to ((x^n \to y) \to y) \leq (x \to y)^n \to ((x^n \to y) \to y)$$

Since F is a prefilter, we conclude that $(x \to y)^n \to ((x^n \to y) \to y) \in F$. Since $x^n \leq x$, we have $x \to y \leq x^n \to y$ and since $(x \to y)^n \leq x \to y$, we get that $(x \to y)^n \leq x^n \to y$ and so $(x \to y)^n \to (x^n \to y) = 1 \in F$. Now, since F is an n-fold positive implicative prefilter of L and $(x \to y)^n \to ((x^n \to y) \to y) \in F$, we get that $(x \to y)^n \to y \in F$ and so by hypothesis $(y \to x) \to x \in F$ and since $y \to x \in F$, we conclude that $x \in F$. Therefore, by Theorem 4.4, F is an n-fold implicative prefilter of L. ∎

As a result of Theorems 4.10 and 4.12, we provide the following theorem which is proved in [8] directly.

Theorem 4.12. *Let F be a positive implicative prefilter of L with the weak exchange principle. Then the following are equivalent:*

(1) *F is an implicative prefilter,*

(2) *$(x \to y) \to y \in F$ implies $(y \to x) \to x \in F$, for any $x, y \in L$.*

Theorem 4.13. *Let L be a bounded EQ-algebra and F be an n-fold positive implicative prefilter of L. If $\neg(\neg x)^n \in F$ implies $x \in F$, for any $x \in L$, then F is an n-fold implicative prefilter of L.*

Proof. Let $\neg x^n \to x \in F$, for $x \in L$. Then by Lemma 2.3(2), $\neg x^n \to x \leq \neg x \to (x^n \to 0)$. Since $(\neg x)^n \leq \neg x$, by Lemma 2.3(2), we get $\neg x \to (\neg x^n \to 0) \leq (\neg x)^n \to (\neg x^n \to 0)$. Hence, $\neg x^n \to x \leq (\neg x)^n \to (\neg x^n \to 0)$ and since F is a prefilter, we get that $(\neg x)^n \to (\neg x^n \to 0) \in F$. Now, since $x^n \leq x$, we have $\neg x \leq \neg x^n$ and by $(\neg x)^n \leq \neg x$ we conclude that $(\neg x)^n \leq \neg x^n$. Therefore, $(\neg x)^n \to \neg x^n = 1 \in F$ and since F is an n-fold positive implicative prefilter of L, we get that $\neg(\neg x)^n = (\neg x)^n \to 0 \in F$ and so by hypothesis we conclude that $x \in F$. Therefore, by Theorem 4.6, F is an n-fold implicative prefilter of L. ∎

Theorem 4.14. *Let L be a bounded EQ-algebra and F be an n-fold implicative prefilter of L. Then $\neg\neg x^n \in F$ implies $x \in F$.*

Proof. Suppose that F is n-fold implicative prefilter and $\neg\neg x^n \in F$, then by Lemma 2.3(7), $\neg\neg x^n = \neg x^n \to 0 \leq \neg x^n \to x$. Hence, $\neg x^n \to x \in F$ and so by Theorem 4.6, $x \in F$. ∎

As a result of Theorems 4.13 and 4.14, we provide the following theorem which is proved in [8] directly.

Theorem 4.15. *Let L be a bounded EQ-algebra and F be a positive implicative prefilter of L. Then F is an implicative prefilter if and only if $\neg\neg x \in F$ implies $x \in F$.*

5. Conclusions

The results of this paper are devoted to study two new classes of prefilters in EQ-algebras which so-called n-fold positive implicative prefilters and n-fold implicative prefilters. We investigated and characterized the properties and characterizations of these prefilters. In particular, the extension theorem for n-fold positive implicative and n-fold implicative prefilters are obtained. Also, we studied the relation between n-fold positive implicative prefilters and n-fold implicative prefilters in (residuated) EQ-algebras and we provided some results which are proved directly in [8] for positive implicative prefilters and implicative prefilters in EQ-algebras. In future work, we will introduce other types of n-fold prefilters in EQ-algebras and study the relation between them.

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