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STUDY OF ADDITIVELY REGULAR Γ-SEMIRINGS AND DERIVATIONS

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Abstract

In this paper, the notions of commutator and derivation in additively regular Γ-semirings with (A_2, Γ)-condition are introduced. We also characterize Jordan product for additively regular Γ-semiring and establish some results which investigate the relationship between commutators, derivations and inner derivations. In 1957, E.C. Posner has shown that if there exists a non-zero centralizing derivation in a prime ring R, then R is commutative. This result is extended in the frame work of derivations of prime additively regular Γ-semirings.

Keywords: semirings, Γ-semirings, additively regular Γ-semirings, derivations and commutators.

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1. Introduction

The concept of derivation is quite old and plays vital role in algebraic geometry and algebra. The algebraists in this direction have studied the concept of derivation in semirings, Γ-rings and Γ-semirings. It is pertinent to note here that the results which are true for rings motivated the researchers to generalize the analogous results for derivations in Γ-rings and Γ-semirings. The concept of derivation in a prime $\Gamma-$ring was first introduced by Yang [12] in 1991. Over the years, the researchers studied the concept of derivation in Γ-rings and other algebraic structures [2, 3, 6]. The algebraic structure additively regular Γ-semiring

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is a generalization of semirings [9, 10, 11], additively regular semirings [5], and Γ-rings.

There are some algebraic structures in which binary operation "multiplication" fails. For instance, let R be the set of all $m \times n$ matrices over a boolean semiring under usual addition and multiplication of matrices. One can easily examine that R is not closed under multiplication. This problem has attracted the attention of various mathematicians for a long period. Therefore, another algebraic structure Γ was introduced; for example, consider A is an additive semigroup consisting of all homomorphisms from a semiring R_1 to semiring R_2 and Γ is an additive semigroup consisting of all homomorphisms from R_2 to R_1. Here the product g_1hg_2 belongs to A for any arbitrary elements g_1, g_2 of A and h of Γ. So, A is closed under multiplication. The importance of aforementioned algebraic structure Γ motivated us to explore the structure of Γ-semirings.

Rao [7, 8] introduced the notion of Γ-semirings and additively inverse Γ-semirings. According to Rao, if R_Γ and Γ are additive commutative semigroups with identity elements 0_{R_Γ} and 0_Γ respectively, then R_Γ is said to be a Γ-semiring if there exists a map $R_\Gamma \times \Gamma \times R_\Gamma \longrightarrow R_\Gamma$, defined as $(x, \gamma, y) \longmapsto x\gamma y$ such that $x\alpha(y + z) = x\alpha y + x\alpha z$; $(x + y)\alpha z = x\alpha z + y\alpha z$; $x(\alpha + \beta)y = x\alpha y + x\beta y$; $(x\alpha y)\beta z = x\alpha(y\beta z)$; $x\gamma 0_{R_\Gamma} = 0_{R_\Gamma}\gamma x = 0_{R_\Gamma}$ and $x\gamma 0_\Gamma = 0_\Gamma\gamma x = 0_\Gamma$ $\forall\ x, y, z \in R_\Gamma$, $\alpha, \beta, \gamma \in \Gamma$. Further, a Γ-semiring R_Γ is said to be additively regular if for each element $x \in R_\Gamma$ there exists an element $x' \in R_\Gamma$ such that $x = x + x' + x$. If in addition the element x' is unique and $x' = x' + x + x'$, then R_Γ is called an additively inverse Γ-semiring. Such an element x' is called pseudo inverse of x. Consider $M = \{0, 1, 2, \dots, 50\}$ and $R_\Gamma = \mathbb{Z} \times M = \{(a, r) : a \in \mathbb{Z},\ r \in M\}$. We define binary operations of addition $\oplus$ and multiplication $\odot$ by $(a, r) \oplus (b, s) = (a + b, \max(r, s))$ and $(a, r) \odot (b, s) = (ab, \min(r, s))$ for all $(a, r), (b, s) \in R_\Gamma$. Take $\Gamma = \{(0, m) : m \in M\}$ with same binary operations defined as above. One can easily check that R_Γ and Γ are additive commutative semigroups. Moreover, define map $R_\Gamma \times \Gamma \times R_\Gamma \longrightarrow R_\Gamma$ by $(a, r) \odot (0, m) \odot (b, s) = (0, \min(r, m, s))$. Then R_Γ is a Γ-semiring. Further, if we define the pseudo inverse of an element (a, r) of R_Γ by $(a, r)' = (-a, r)$. Then R_Γ is an additively inverse Γ-semiring. Throughout this article, additively inverse Γ-semiring along with 1 has been intensively explored and represented as "additively regular Γ-semiring" which will persuade the readers in its accuracy and truthfulness.

In present paper, we introduce and characterize the concept of derivations for additively regular Γ-semirings with (A_2, Γ)-condition. Here (A_2, Γ)-condition means that the sum of an element x of R_Γ and its pseudo inverse $x' \in R_\Gamma$ lies in the centre of R_Γ. For example, let $B = \{0, 1\}$ and $\Gamma = \{a, b\}$, where $0, 1$ and a, b are additively idempotent elements of R_Γ and Γ, respectively. Further, addition in B is defined by $0 + 1 = 1 = 1 + 0$ and in Γ by $a + b = b = b + a$. Moreover, a map $B \times \Gamma \times B \longrightarrow B$ is defined as $0a0 = 0a1 = 1a0 = 0b0 = 0b1 = 1b0 = 0$ and

$1a1 = 1b1 = 1$. Then B is an additively regular Γ-semiring with (A_2, Γ)-condition. Throughout this paper, R_Γ will denote an additively regular Γ-semiring with (A_2, Γ)-condition. In continuation, the study of commutators for additively regular Γ-semirings is also initiated which is the generalization of the commutators of rings. In section 3, some fundamental identities for commutators of additively regular Γ-semiring with (A_2, Γ)-condition are proved which are the generalization of some fundamental results of commutators in ring theory. The last section of this paper deals with the study of derivations and inner derivations. Also, some results are proved which establish the relationships between commutators and derivations. Finally, we extend Posner's second theorem for prime additively regular Γ-semirings with (A_2, Γ)-condition.

2. ADDITIVELY REGULAR Γ-SEMIRING WITH (A_2, Γ)-CONDITION

In this section, we prove some basic results and examples of additively regular Γ-semirings with (A_2, Γ)-condition. First we define commutativity and primeness of additively regular Γ-semiring R_Γ.

Definition 2.1. An additively regular Γ-semiring R_Γ is said to be commutative if $x\gamma y = y\gamma x \ \forall \ x, y \in R_\Gamma, \gamma \in \Gamma$.

Definition 2.2. An additively regular Γ-semiring R_Γ is said to be prime if $x\Gamma R_\Gamma \Gamma y = 0$ implies that either $x = 0$ or $y = 0$.

Now, we give an example of an additively regular Γ-semiring which is both commutative as well as prime.

Example 2.3. Let $R_\Gamma = \{0, 1, u\}$ and $\Gamma = \{\alpha, \beta\}$. We define operations with the help of following tables:

$+$	0	1	u
0	0	1	u
1	1	1	u
u	u	u	u

$+$	α	β
α	α	β
β	β	β

α	0	1	u
0	0	0	0
1	0	1	u
u	0	u	u

β	0	1	u
0	0	0	0
1	0	1	u
u	0	u	u

Then R_Γ is an additively regular Γ-semiring with (A_2, Γ)-condition and $a' = a$ for all $a \in R_\Gamma$. From the tables, it is clear that additively regular Γ-semiring R_Γ is prime and commutative.

Note that every additively regular semiring S is an additively regular Γ-semiring with $\Gamma = S$.

Next two examples show that every additively regular Γ-semiring may not satisfy (A_2, Γ)-condition.

Example 2.4. Let R_Γ be the set of all 2×2 matrices over boolean semiring B, i.e., $M_{2\times2}(B)$ and $\Gamma = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in B \right\}$. Define a map $R_\Gamma \times \Gamma \times R_\Gamma \to R_\Gamma$ by $(x, \gamma, y) \longmapsto x\gamma y$ for all $x, y \in R_\Gamma, \gamma \in \Gamma$. We define pseudo inverse of an element of R_Γ as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$. Then R_Γ is an additively regular Γ-semiring which do not satisfy (A_2, Γ)-condition under the usual multiplication of matrices.

Example 2.5. Let R be a non commutative ring and S be an additively regular semiring. Then the set $K = \{(a, \alpha) : a \in R, \alpha \in S\}$ is a non commutative additively regular semiring with operations pointwise addition and pointwise multiplication. We define pseudo inverse of an element of K as $(a, \alpha)' = (-a, \alpha')$. Take $\Gamma = \{(0, \beta) : 0 \in R, \beta \in S\}$ with operations pointwise addition and pointwise multiplication. Then Γ is an additive commutative semigroup. Further, define a map $K \times \Gamma \times K \to K$ by $(x, \gamma, y) = x\gamma y$ for all $x, y \in K, \gamma \in \Gamma$. Then K is an additively regular Γ-semiring.

Note that K satisfies (A_2, Γ)-condition only if S is commutative.

Throughout this paper, we consider an assumption $(*)$ $x\alpha y\beta z = x\beta y\alpha z$ for all $x, y, z \in R_\Gamma$ and $\alpha, \beta \in \Gamma$.

Lemma 2.6 [(Theorem 12, [8])]. *Let R_Γ be an additively regular Γ-semiring and $a, b \in R_\Gamma$, $\gamma \in \Gamma$. Then we have the following:*

(i) $a'' = a$,

(ii) $(a + b)' = a' + b'$,

(iii) $(a\gamma b)' = a'\gamma b = a\gamma b'$,

(iv) $a'\gamma b' = (a'\gamma b)' = (a\gamma b)'' = a\gamma b$.

Definition 2.7. The centre of an additively regular Γ-semiring R_Γ is the set $Z(R_\Gamma) = \{x \in R_\Gamma : x\gamma y = y\gamma x \ \forall \ y \in R_\Gamma, \gamma \in \Gamma\}$.

Proposition 2.8. *The centre of an additively regular Γ-semiring R_Γ is again an additively regular Γ-semiring.*

Proof. Let R_Γ be an additively regular Γ-semiring and $Z(R_\Gamma)$ be its centre. The map $Z(R_\Gamma) \times \Gamma \times Z(R_\Gamma) \longrightarrow Z(R_\Gamma)$ defined by $(a, \alpha, b) \longmapsto a\alpha b \ \forall \ a, b \in Z(R_\Gamma)$, $\alpha \in \Gamma$ is well defined map. Clearly, $Z(R_\Gamma)$ is an additive commutative semigroup and satisfies all the properties of Γ-semiring and hence $Z(R_\Gamma)$ is a Γ-semiring. Further, let $a \in Z(R_\Gamma)$. Then $a\gamma x = x\gamma a \ \forall \ x \in R_\Gamma, \gamma \in \Gamma$ implies that $(a\gamma x)' = (x\gamma a)'$, i.e., $a'\gamma x = x\gamma a' \ \forall \ x \in R_\Gamma, \gamma \in \Gamma$ and hence $a' \in Z(R_\Gamma)$. This completes the proof. ■

Remark 2.9. Let R_Γ be an additively regular Γ-semiring and X be a non-empty set. If $Map(X, R_\Gamma)$ is the set of all mappings from X into R_Γ, then define '+' in $Map(X, R_\Gamma)$ as $(f+g)(x) = f(x) + g(x)\ \forall\ f, g \in Map(X, R_\Gamma)$ and $Map(X, R_\Gamma) \times \Gamma \times Map(X, R_\Gamma) \longrightarrow Map(X, R_\Gamma)$ as $(f, \gamma, g) \longmapsto f\gamma g$ where $f\gamma g : X \longrightarrow R_\Gamma$ is defined by $(f\gamma g)(x) = f\ (x)\gamma g(x)\ \forall f, g \in Map(X, R_\Gamma)$, $\gamma \in \Gamma, x \in X$. Then $Map(X, R_\Gamma)$ is a Γ-semiring. Define $f' : X \longrightarrow R_\Gamma$ by $f'(x) = (f(x))'$ for each $f \in Map(X, R_\Gamma)$. Then it can be easily checked that f' is pseudo inverse of f and $f' \in Map(X, R_\Gamma)$ for each $f \in Map(X, R_\Gamma)$. Thus, $Map(X, R_\Gamma)$ is an additively regular Γ-semiring.

The proofs of the next two propositions are quite easy so we omit the proofs.

Proposition 2.10. *If R_Γ is an additively regular Γ-semiring, then $R_\Gamma[x]$ the set of all polynomials over R_Γ is an additively regular Γ-semiring.*

Proposition 2.11. *Let R_{Γ_1} be an additively regular Γ_1-semiring and R_{Γ_2} be an additively regular Γ_2-semiring. Then $R_\Gamma = R_{\Gamma_1} \times R_{\Gamma_2} = \{(r, s) : r \in R_{\Gamma_1}, s \in R_{\Gamma_2}\}$ is an additively regular $\Gamma = \Gamma_1 \times \Gamma_2$-semiring.*

3. COMMUTATORS OF ADDITIVELY REGULAR Γ-SEMIRINGS

In this section, we introduce the concept of α-commutator for additively regular Γ-semirings and generalize some results of commutators of rings.

Definition 3.1. Let R_Γ be an additively regular Γ-semiring and α be a fixed element of Γ. We define α-commutator as a mapping $[,]_\alpha : R_\Gamma \times \Gamma \times R_\Gamma \to R_\Gamma$ by $[x, y]_\alpha = x\alpha y + (y\alpha x)' = x\alpha y + y'\alpha x = x\alpha y + y\alpha x'$ for all $x, y \in R_\Gamma$. Then $[x, y]_\alpha$ is called α- commutator of x, y.

For convenience, we denote $x + x'$ by $x_\circ$ for each $x \in R_\Gamma$. Then clearly $x_\circ + x_\circ = x_\circ = x'_\circ$; $x + x_\circ = x$ and $x' + x_\circ = x'$.

Lemma 3.2. *If R_Γ is an additively regular Γ-semiring, then $(x\gamma y)_\circ = x_\circ\gamma y = x\gamma y_\circ = x_\circ\gamma y_\circ = y_\circ\gamma x_\circ = (y\gamma x)_\circ\ \forall\ x, y \in R_\Gamma, \gamma \in \Gamma$.*

Proof. By using Lemma 2.6, we have $(x\gamma y)_\circ = x\gamma y + x'\gamma y = x_\circ\gamma y$. Similarly, $(x\gamma y)_\circ = x\gamma y_\circ$. Now, $x_\circ\gamma y_\circ = (x + x')\gamma\,(y + y') = x\gamma y + x\gamma y' + x'\gamma y + x'\gamma y' = x\gamma y + x\gamma y' + x'\gamma y + x\gamma y = x\gamma y + x'\gamma y = x_\circ\gamma y$. Similarly, $y\gamma x_\circ = y_\circ\gamma x = y_\circ\gamma x_\circ = (y\gamma x)_\circ$. By (A_2, Γ)-condition, we have $x_\circ = x + x' \in Z(R_\Gamma)$. Thus $x_\circ\gamma y = y\gamma x_\circ$. Hence $(x\gamma y)_\circ = x_\circ\gamma y = x\gamma y_\circ = x_\circ\gamma y_\circ = y_\circ\gamma x_\circ = (y\gamma x)_\circ$. ■

In the next Theorem, we generalize some basic commutator identities of rings for additively regular Γ-semirings.

Theorem 3.3. *Let R_Γ be an additively regular Γ-semiring. Then for all $x, y, z, x_1, x_2, y_1, y_2 \in R_\Gamma$ and $\alpha, \beta \in \Gamma$, the following identities hold:*

(i) $[x + y, z]_\alpha = [x, z]_\alpha + [y, z]_\alpha$.

(ii) $[x, y + z]_\alpha = [x, y]_\alpha + [x, z]_\alpha$.

(iii) $[x, 0_{R_\Gamma}]_\alpha = [0_{R_\Gamma}, x]_\alpha = 0_{R_\Gamma}$.

(iv) $[x_1 + x_2, y_1 + y_2]_\alpha = [x_1, y_1]_\alpha + [x_1, y_2]_\alpha + [x_2, y_1]_\alpha + [x_2, y_2]_\alpha$.

(v) $([x, y]_\alpha)' = [y, x]_\alpha = [x, y']_\alpha = [x', y]_\alpha$. *(Anti-commutativity)*

(vi) $[[x, y]_\alpha, z]_\beta = [x, y]_\alpha \beta z + z\beta[y, x]_\alpha$.

(vii) $[nx, y]_\alpha = n[x, y]_\alpha$, *for any positive integer* n.

Proof. One can easily prove the identities (i) to (iv) by using Definition 3.1.

(v) By Lemma 2.6 and Definition 3.1, we have $([x, y]_\alpha)' = (x\alpha y + y'\alpha x)' = x'\alpha y + y\alpha x = [y, x]_\alpha$. Again, $([x, y]_\alpha)' = (x\alpha y + y'\alpha x)' = x\alpha y' + y'\alpha x' = [x, y']_\alpha$. Now, $[x', y]_\alpha = x'\alpha y + y\alpha x'' = x'\alpha y + y\alpha x = [y, x]_\alpha$.

(vi) Using Definition 3.1 and (v), we have $[[x, y]_\alpha, z]_\beta = [x, y]_\alpha\beta z + z\beta([x, y]_\alpha)' = [x, y]_\alpha\beta z + z\beta[y, x]_\alpha$.

(vii) By Lemma 2.6 and Definition 3.1, we have $[nx, y]_\alpha = nx\alpha y + y'\alpha nx = n\,(x\alpha y + y'\alpha x) = n[x, y]_\alpha$. ∎

Theorem 3.4. *Let R_Γ be an additively regular Γ-semiring. Then for all $x, y, z, u \in R_\Gamma$ and $\alpha, \beta, \gamma \in \Gamma$, the following identities are valid:*

(i) $[x, y\beta z]_\alpha = [x, y]_\alpha\beta z + y\beta[x, z]_\alpha$.

(ii) $[x\beta y, z]_\alpha = x\beta[y, z]_\alpha + [x, z]_\alpha\beta y$.

(iii) $[x\beta y, z\gamma u]_\alpha = x\beta[y, z]_\alpha\gamma u + [x, z]_\alpha\beta y\gamma u + z\gamma x\beta[y, u]_\alpha + z\gamma[x, u]_\alpha\beta y$.

Proof. (i) By assumption (*) and Definition 3.1, we have $[x, y\beta z]_\alpha = x\alpha y\beta z + y\beta z\alpha x' = x\alpha y\beta z + y\beta z\alpha\,(x' + x) + y\beta z\alpha x' = x\alpha y\beta z + y\beta\,(x' + x)\,\alpha z + y\beta z\alpha x' = x\alpha y\beta z + y\beta x'\alpha z + y\beta x\alpha z + y\beta z\alpha x' = x\alpha y\beta z + y\alpha x'\beta z + y\beta x\alpha z + y\beta z\alpha x' = [x, y]_\alpha\beta z + y\beta[x, z]_\alpha$.

Similarly we can prove (ii).

(iii) By using Definition 3.1, Lemma 2.6, Lemma 3.2 and assumption $(*)$ we have $[x\beta y, z\gamma u]_\alpha = x\beta y\alpha z\gamma u + z'\gamma u\alpha x\beta y = x\beta y\alpha z\gamma u + (z' + z + z')\gamma u\alpha x\beta y + z_\circ\gamma u\alpha x\beta y = x\beta y\alpha z\gamma u + z'\gamma u\alpha x\beta y + z_\circ\gamma u\alpha x\beta y + (z\gamma u)_\circ\alpha x\beta y = x\beta y\alpha z\gamma u + z'\gamma u\alpha x\beta y + (z\gamma u)_\circ\alpha x\beta y + x\beta(z\gamma u)_\circ\alpha y = x\beta y\alpha z\gamma u + z'\gamma u\alpha x\beta y + z\gamma u_\circ\alpha x\beta y + x\beta z\gamma u_\circ\alpha y = x\beta y\alpha z\gamma u + z'\gamma u\alpha x\beta y + z\gamma x\alpha u_\circ\beta y + x\beta z\alpha(y\gamma u)_\circ = x\beta y\alpha z\gamma u + z'\gamma u\alpha x\beta y + z\gamma x\alpha u\beta y + z\gamma x\alpha(u' + u_\circ)\beta y + x\beta z\alpha y_\circ\gamma u = x\beta[y, z]_\alpha\gamma u + z\gamma u\alpha x'\beta y + z\gamma x\alpha u\beta y + z\gamma x\beta u'\alpha y + z\gamma x\beta y\alpha u + z\gamma x\beta y\alpha u' + x\beta z\alpha y\gamma u = x\beta[y, z]_\alpha\gamma u + z\gamma[x, u]_\alpha\beta y + z\gamma x\beta[y, u]_\alpha + z\alpha x'\beta y\gamma u + x\alpha z\beta y\gamma u = x\beta[y, z]_\alpha\gamma u + [x, z]_\alpha\beta y\gamma u + z\gamma x\beta[y, u]_\alpha + z\gamma[x, u]_\alpha\beta y$. ∎

Note that by assumption $(*)$, we have $[x,y]_\alpha\beta z = [x,y]_\beta\alpha z$ and $x\alpha[y,z]_\beta = x\beta[y,z]_\alpha$ for all $x,y,z \in R_\Gamma, \alpha,\beta \in \Gamma$.

Now, we generalize the Jacobian identity of rings for additively regular Γ-semirings which might be useful to develop Lie type theory for additively regular Γ-semirings.

Theorem 3.5. *If R_Γ is an additively regular Γ-semiring, then $[x,[y,z]_\alpha]_\beta + [y,[z,x]_\alpha]_\beta = [[x,y]_\alpha, z]_\beta$ holds for all $x,y,z \in R_\Gamma, \alpha,\beta \in \Gamma$.*

Proof. Using Lemma 2.6, Lemma 3.2, Definition 3.1 and Theorem 3.3(v), we have $[x,[y,z]_\alpha]_\beta = ([[y,z]_\alpha, x]_\beta)' = [z,y]_\alpha\beta x + x\beta[y,z]_\alpha$ for all $x,y,z \in R_\Gamma, \alpha,\beta \in \Gamma$. Similarly $[y,[z,x]_\alpha]_\beta = [x,z]_\alpha\beta y + y\beta[z,x]_\alpha$. Therefore, $[x,[y,z]_\alpha]_\beta + [y,[z,x]_\alpha]_\beta$ $= [z,y]_\alpha\beta x + x\beta[y,z]_\alpha + [x,z]_\alpha\beta y + y\beta[z,x]_\alpha = z\alpha y\beta x + y'\alpha z\beta x + x\beta y\alpha z + x\beta z'\alpha y + x\alpha z\beta y + z'\alpha x\beta y + y\beta z\alpha x + y\beta x'\alpha z = (x\beta y\alpha z + y\beta x'\alpha z + z\alpha y\beta x + z'\alpha x\beta y) + (y'\alpha z\beta x + y\beta z\alpha x) + (x\beta z'\alpha y + x\alpha z\beta y) = (x\beta y\alpha z + y\beta x'\alpha z + z\alpha y\beta x + z\alpha x\beta y') + y_\circ\beta z\alpha x + x\beta z_\circ\alpha y = (x\beta y\alpha z + x\beta z_\circ\alpha y) + (z\alpha y\beta x + y_\circ\beta z\alpha x) + (y\beta x'\alpha z + z\alpha x\beta y') = (x\beta y\alpha z + x\beta y\alpha z_\circ) + (z\alpha y\beta x + z\alpha y_\circ\beta x) + (y\beta x'\alpha z + z\alpha x\beta y') = x\beta y\alpha z + z\alpha y\beta x + y\beta x'\alpha z + z\alpha x\beta y' = x\alpha y\beta z + y\alpha x'\beta z + z\beta y\alpha x + z\beta x\alpha y' = [[x,y]_\alpha, z]_\beta$. ■

Theorem 3.6. *If R_Γ is an additively regular Γ-semiring, then for all $x,y,z \in R_\Gamma$ and $\alpha,\beta,\gamma \in \Gamma$, the following identities hold:*

(i) $[x\alpha y, z]_\beta + [y\alpha z, x]_\beta + [z\alpha x, y]_\beta = [x,[y,z]_\alpha]_\beta + [y,[z,x]_\alpha]_\beta + [z,[x,y]_\alpha]_\beta$.

(ii) $[x\alpha y\beta z, u]_\gamma = x\alpha y\beta[z,u]_\gamma + x\alpha[y,u]_\gamma\beta z + [x,u]_\gamma\alpha y\beta z$.

(iii) $[x, y\beta z\gamma u]_\alpha = [x,y]_\alpha\beta z\gamma u + y\beta[x,z]_\alpha\gamma u + y\beta z\gamma[x,u]_\alpha$.

Proof. (i) By using Definition 3.1, Theorem 3.3(v) and Theorem 3.4(ii), $[x,[y,z]_\alpha]_\beta + [y,[z,x]_\alpha]_\beta + [z,[x,y]_\alpha]_\beta = x\beta[y,z]_\alpha + ([y,z]_\alpha)'\beta x + y\beta[z,x]_\alpha + ([z,x]_\alpha)'\beta y + z\beta[x,y]_\alpha + ([x,y]_\alpha)'\beta z = x\alpha[y,z]_\beta + [z,y]_\beta\alpha x + y\alpha[z,x]_\beta + [x,z]_\beta\alpha y + z\alpha[x,y]_\beta + [y,x]_\beta\alpha z = [x\alpha y, z]_\beta + [y\alpha z, x]_\beta + [z\alpha x, y]_\beta$.

(ii) By Definition 3.1 and Lemma 3.2 we have $x\alpha y\beta[z,u]_\gamma + x\alpha[y,u]_\gamma\beta z + [x,u]_\gamma\alpha y\beta z = x\alpha y\beta z\gamma u + x\alpha y\beta u'\gamma z + x\alpha y\gamma u\beta z + x\alpha u'\gamma y\beta z + x\gamma u\alpha y\beta z + u'\gamma x\alpha y\beta z$ $= x\alpha y\beta z\gamma u + x\alpha y\beta u_\circ\gamma z + x\alpha u_\circ\gamma y\beta z + u'\gamma x\alpha y\beta z = x\alpha y\beta z\gamma u + x\alpha y\beta z\gamma u_\circ + u_\circ\alpha x\gamma y\beta z + u'\gamma x\alpha y\beta z = x\alpha y\beta z\gamma u + u'\gamma x\alpha y\beta z = [x\alpha y\beta z, u]_\gamma$.

Similarly we can prove (iii). ■

Theorem 3.7. *If R_Γ is an additively regular Γ-semiring, then for all $x, y, x_1, x_2, \ldots, x_n$, $y_1, y_2, \ldots, y_n \in R_\Gamma$ and $\alpha, \beta_1, \beta_2, \ldots, \beta_{n-1} \in \Gamma$, the following identities are valid:*

(i) $[x, y_1\beta_1 y_2\beta_2\cdots\beta_{n-1}y_n]_\alpha = [x,y_1]_\alpha\beta_1 y_2\beta_2\cdots\beta_{n-1}y_n + y_1\beta_1[x,y_2]_\alpha\beta_2\cdots\beta_{n-1}y_n + \cdots + y_1\beta_1 y_2\beta_2\cdots\beta_{n-1}[x,y_n]_\alpha$

(ii) $[x_1\beta_1 x_2\beta_2\cdots\beta_{n-1}x_n, y]_\alpha = x_1\beta_1 x_2\beta_2\cdots\beta_{n-1}[x_n,y]_\alpha + x_1\beta_1 x_2\beta_2\cdots\beta_{n-2}[x_{n-1},y]_\alpha\beta_{n-1}x_n + \cdots + [x_1,y]_\alpha\beta_1 x_2\beta_2\cdots\beta_{n-1}x_n$.

Proof. (i) We will prove the result by using induction on n, the result is already true for $n = 2, 3$ from Theorem 3.4(i) and Theorem 3.6(iii). Now assume that the result is true for $n = k - 1$, i.e., $[x, y_1\beta_1 y_2\beta_2 \cdots \beta_{k-2}y_{k-1}]_\alpha = [x, y_1]_\alpha\beta_1 y_2\beta_2 \cdots \beta_{k-2}y_{k-1} + y_1\beta_1[x, y_2]_\alpha\beta_2 \cdots \beta_{k-2}y_{k-1} + \cdots + y_1\beta_1 y_2\beta_2 \cdots \beta_{k-2}[x, y_{k-1}]_\alpha$. For $n = k$, by using Theorem 3.4(i) and induction hypothesis, we have $[x, y_1\beta_1 y_2\beta_2 \cdots \beta_{k-1}y_k]_\alpha = [x, (y_1\beta_1 y_2\beta_2 \cdots \beta_{k-2}y_{k-1})\beta_{k-1}y_k]_\alpha = [x, y_1\beta_1 y_2\beta_2 \cdots \beta_{k-2}y_{k-1}]_\alpha\beta_{k-1}y_k + y_1\beta_1 y_2\beta_2 \cdots \beta_{k-2}y_{k-1}\beta_{k-1}[x, y_k]_\alpha = [x, y_1]_\alpha \beta_1 y_2\beta_2 \cdots \beta_{k-1}y_k + y_1\beta_1[x, y_2]_\alpha\beta_2 \cdots \beta_{k-1}y_k + \cdots + y_1\beta_1 y_2\beta_2 \cdots \beta_{k-1}[x, y_k]_\alpha$.

Similarly we can prove (ii). ∎

The following identities are generalizations of commutator identities of ring theory (c.f. [1, 4]).

Proposition 3.8. *Let R_Γ be an additively regular Γ-semiring. Then for all $x, y, z \in R_\Gamma, \alpha, \beta, \gamma \in \Gamma$, the following identities hold:*

(i) $[x\alpha y, z]_\beta + [y\alpha z, x]_\beta = [y, z\alpha x]_\beta$

(ii) $[x\alpha y\beta z, u]_\gamma + [y\alpha z\beta u, x]_\gamma + [z\alpha u\beta x, y]_\gamma = [z, u\alpha x\beta y]_\gamma$.

Proof. (i) By using Lemma 2.6, Definition 3.1 and Lemma 3.2, (i) becomes $[x\alpha y, z]_\beta + [y\alpha z, x]_\beta = x\alpha y\beta z + z'\beta x\alpha y + y\alpha z\beta x + x\beta y\alpha z' = z'\beta x\alpha y + y\alpha z\beta x + x\beta y\alpha z_\circ = z'\beta x\alpha y + y\alpha z\beta x + z_\circ\beta x\alpha y = y\alpha z\beta x + z'\beta x\alpha y = y\beta z\alpha x + z'\alpha x\beta y = [y, z\alpha x]_\beta$.

(ii) By Lemma 2.6, Definition 3.1 and Lemma 3.2, (ii) reduces to $[x\alpha y\beta z, u]_\gamma + [y\alpha z\beta u, x]_\gamma + [z\alpha u\beta x, y]_\gamma = x\alpha y\beta z\gamma u + u'\gamma x\alpha y\beta z + y\alpha z\beta u\gamma x + x\gamma y\alpha z\beta u' + z\alpha u\beta x\gamma y + y'\gamma z\alpha u\beta x = x\alpha y\beta z\gamma u_\circ + u'\gamma x\alpha y\beta z + z\alpha u\beta x\gamma y + y_\circ\alpha z\beta u\gamma x = u_\circ\gamma x\alpha y\beta z + u'\gamma x\alpha y\beta z + z\beta u\gamma x\alpha y_\circ + z\beta u\gamma x\alpha y = u'\gamma x\alpha y\beta z + z\beta u\gamma x\alpha y = u'\alpha x\beta y\gamma z + z\gamma u\alpha x\beta y = [z, u\alpha x\beta y]_\gamma$. ∎

Definition 3.9. Let α be a fixed element of Γ. Then we define α-Jordan product as $(x \circ y)_\alpha = x\alpha y + y\alpha x$ for all $x, y \in R_\Gamma$.

Proposition 3.10. *Let R_Γ be an additively regular Γ-semiring. Then for all $x, y, z \in R_\Gamma, \alpha, \beta \in \Gamma$, the following α-Jordan identities hold:*

(i) $(x \circ y)_\alpha = (y \circ x)_\alpha$

(ii) $((x + y) \circ z)_\alpha = (x \circ z)_\alpha + (y \circ z)_\alpha$

(iii) $[(x \circ y)_\alpha, z]_\beta + [(y \circ z)_\alpha, x]_\beta = [y, (z \circ x)_\alpha]_\beta$.

Proof. The proofs of identities (i) and (ii) are quite obvious.

(iii) By Lemma 2.6, Definition 3.1, Lemma 3.2 and Definition 3.9, we have $[(x\circ y)_\alpha, z]_\beta + [(y\circ z)_\alpha, x]_\beta = (x\alpha y + y\alpha x)\beta z + z\beta(x\alpha y' + y'\alpha x) + (y\alpha z + z\alpha y)\beta x + x\beta(y'\alpha z + z\alpha y') = y\alpha z\beta x + y\alpha x\beta z + x\alpha y\beta z + x\alpha y'\beta z + z\beta y'\alpha x + z\beta y\alpha x + z\beta x\alpha y' + x\alpha z\beta y' = y\alpha z\beta x + y\alpha x\beta z + x\alpha y_\circ\beta z + z\beta y_\circ\alpha x + z\alpha x\beta y' + x\alpha z\beta y' =$

$y\beta z\alpha x+y_\circ\beta z\alpha x+y\beta x\alpha z+y_\circ\beta x\alpha z+(z\circ x)_\alpha\beta y' = y\beta(z\alpha x+x\alpha z)+(z\circ x)_\alpha\beta y' = [y,(z\circ x)_\alpha]_\beta$. ∎

Proposition 3.11. *Let R_Γ be an additively regular Γ-semiring. Then for all $x, y, z \in R_\Gamma$ and $\alpha, \beta \in \Gamma$, the α-Jordan identity $(x\circ[y,z]_\beta)_\alpha + ([x,z]_\beta\circ y)_\alpha = [(x\circ y)_\alpha, z]_\beta$ holds.*

Proof. By Lemma 2.6, Definition 3.1 and Definition 3.9, the left hand side reduces to $(x\circ[y,z]_\beta)_\alpha+([x,z]_\beta\circ y)_\alpha = (x\circ(y\beta z+z'\beta y))_\alpha+((x\beta z+z'\beta x)\circ y)_\alpha = x\alpha y\beta z+x\alpha z'\beta y+y\beta z\alpha x+z'\beta y\alpha x+x\beta z\alpha y+z'\beta x\alpha y+y\alpha x\beta z+y\alpha z'\beta = (x\alpha y+y\alpha x)\beta z+x\alpha(z+z')\beta y+y\beta(z+z')\alpha x+z'\beta y\alpha x+z'\beta x\alpha y = (\ x\circ y)_\alpha\beta z+z_\circ\beta x\alpha y+z_\circ\beta y\alpha x+z'\beta x\alpha y+z'\beta y\alpha x = (\ x\circ y)_\alpha\beta z+z'\beta(x\alpha y+y\alpha x) = [(x\circ y)_\alpha, z]_\beta$. ∎

4. Derivations of additively regular Γ-semiring

In this section, we introduce the concept of derivation and inner derivation in additively regular Γ-semiring. Also, we establish the relationships between commutators and derivations of additively regular Γ-semirings.

Definition 4.1. A map $d: R_\Gamma \to R_\Gamma$ is called a derivation of additively regular Γ-semiring R_Γ if d is additive and d satisfies $d(x\gamma y) = d(x)\gamma y + x\gamma d(y)$ for all $x, y \in R_\Gamma, \gamma \in \Gamma$.

Example 4.2. Let R be an additively regular Γ-semiring. Take $R_\Gamma = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in R \right\}$ and $\Gamma = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in R \right\}$. Define a map $R_\Gamma \times \Gamma \times R_\Gamma \to R_\Gamma$ by $(x,\gamma,y) \longmapsto x\gamma y\ \forall\ x, y \in R_\Gamma, \gamma \in \Gamma$. Then R_Γ is an additively regular Γ-semiring under the usual multiplication of matrices. Define $d: R_\Gamma \to R_\Gamma$ by $d\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$, then d is a derivation on R_Γ.

Example 4.3. Let R_Γ be an additively regular Γ-semiring. Then by Proposition 2.10, $R_\Gamma[x]$ is also an additively regular Γ-semiring. We define $d: R_\Gamma[x] \to R_\Gamma[x]$ by $d(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots) = a_1 + 2a_2x + 3a_3x^2 + \cdots$ for all $a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots \in R_\Gamma[x]$. Then d is a derivation on $R_\Gamma[x]$.

Definition 4.4. Let R_Γ be an additively regular Γ-semiring and a be a fixed element of R_Γ and α be a fixed element of Γ. Define $d: R_\Gamma \to R_\Gamma$ by $d(x) = [a,x]_\alpha$, for all $x \in R_\Gamma$. The function d so defined can be easily checked to be additive and $d(x\gamma y) = [a,x\gamma y]_\alpha = x\gamma[a,y]_\alpha + [a,x]_\alpha\gamma y = x\gamma d(y) + d(x)\gamma y$ for all $x, y \in R_\Gamma, \gamma \in \Gamma$. Thus, d is a derivation which is called inner derivation of R_Γ determined by a and α.

Remark 4.5. For our convenience, we denote $d([x,y]_\beta) = [a,[x,y]_\beta]_\alpha$ by $[x,y]^d_\beta$.

Proposition 4.6. *If R_Γ is an additively regular Γ-semiring and d is a derivation on R_Γ, then*

(i) $d(x') = (d(x))'$ $\forall$ $x \in R_\Gamma$.

(ii) $d' : R_\Gamma \to R_\Gamma$ *is also a derivation on* R_Γ.

Proposition 4.7. *If R_Γ is an additively regular Γ-semiring and d is a derivation on R_Γ, then $d(x\gamma y) = d(x'\gamma y')$, $\forall$ $x, y \in R_\Gamma, \gamma \in \Gamma$.*

Proof. Let $x, y \in R_\Gamma, \gamma \in \Gamma$. Then by using Lemma 2.6 and Proposition 4.6(i), we have $d(x'\gamma y') = d(x')\gamma y' + x'\gamma d(y') = (d(x))'\gamma y' + x'\gamma(d(y))' = d(x)\gamma y + x\gamma d(y) = d(x\gamma y)$. ∎

Theorem 4.8. *If R_Γ is an additively regular Γ-semiring, $a \in R_\Gamma, \alpha \in \Gamma$ and d is an inner derivation determined by a and α, i.e., $d(x) = [a,x]_\alpha$, for all $x \in R_\Gamma$, then $[x\beta y, z]^d_\gamma +$ $[y\beta z, x]^d_\gamma = [y, z\beta x]^d_\gamma$ for all $x, y, z \in R_\Gamma, \beta, \gamma \in \Gamma$.*

Proof. By Lemma 2.6, Definition 3.1 and Remark 4.5, the left hand side reduces to $[x\beta y, z]^d_\gamma +$ $[y\beta z, x]^d_\gamma = [a,[x\beta y, z]_\gamma]_\alpha + [a,[y\beta z, x]_\gamma]_\alpha = a\alpha(x\beta y\gamma z + z'\gamma x\beta y) + (x\beta y\gamma z + z'\gamma x\beta y)\alpha a' + a\alpha(y\beta z\gamma x + x'\gamma y\beta z) + (y\beta z\gamma x + x'\gamma y\beta z)\alpha a' = a\alpha(x+x')\beta y\gamma z + a\alpha z'\gamma x\beta y + (x+x')\beta y\gamma z\alpha a' + z'\gamma x\beta y\alpha a' + a\alpha y\beta z\gamma x + y\beta z\gamma x\alpha a' = a\alpha z'\gamma x\beta y + z'\gamma x\beta y\alpha a' + a\alpha y\gamma z\beta(x + x_\circ) + y\gamma z\beta(x + x_\circ)\alpha a' = a\alpha z'\beta x\gamma y + z'\beta x\gamma y\alpha a' + a\alpha y\gamma z\beta x + y\gamma z\beta x\alpha a' = [y, z\beta x]^d_\gamma$. ∎

Proposition 4.9. *Let R_Γ be an additively regular Γ-semiring, $a \in R_\Gamma, \alpha \in \Gamma$ and d be an inner derivation determined by a and α, i.e., $d(x) = [a,x]_\alpha$ for all $x \in R_\Gamma$. Then for all $x, y, z, u \in R_\Gamma, \beta, \gamma, \delta \in \Gamma$, the following identities are valid:*

(i) $[x\beta y\gamma z, u]^d_\delta = (x\beta y\gamma [z,u]_\delta)^d + (x\beta [y,u]_\delta \gamma z)^d + ([x,u]_\delta \beta y\gamma z)^d$.

(ii) $[x\beta y\gamma z, u]^d_\delta + [y\beta z\gamma u, x]^d_\delta + [z\beta u\gamma x, y]^d_\delta = [z, u\beta x\gamma y]^d_\delta$.

Proof. (i) Taking right hand side of (i) and by using Lemma 2.6, Definition 3.1 and Remark 4.5, we get $(x\beta y\gamma [z,u]_\delta)^d + (x\beta [y,u]_\delta \gamma z)^d + ([x,u]_\delta \beta y\gamma z)^d = [a, x\beta y\gamma [z,u]_\delta]_\alpha + [a, x\beta [y,u]_\delta \gamma z]_\alpha + [a, [x,u]_\delta \beta y\gamma z]_\alpha = a\alpha x\beta y\gamma z\delta u + a\alpha x\beta y\gamma u\delta z' + x\beta y\gamma z\delta u\alpha a' + x\beta y\gamma u\delta z'\alpha a' + a\alpha x\beta y\delta u\gamma z + a\alpha x\beta u'\delta y\gamma z + x\beta y\delta u\gamma z\alpha a' + x\beta u'\delta y\gamma z\alpha a' + a\alpha x\delta u\beta y\gamma z + a\alpha u'\delta x\beta y\gamma z + x\delta u\beta y\gamma z\alpha a' + u'\delta x\beta y\gamma z\alpha a' = a\alpha x\beta y\gamma z\delta u + a\alpha u'\delta x\beta y\gamma z + x\beta y\gamma z\delta u\alpha a' + u'\delta x\beta y\gamma z\alpha a' + a\alpha x\beta y\gamma u\delta z_\circ + x\beta y\gamma u\delta z_\circ\alpha a' + a\alpha x\beta u_\circ\delta y\gamma z + x\beta u_\circ\delta y\gamma z\alpha a' = a\alpha x\beta y\gamma z\delta u + a\alpha u'\delta x\beta y\gamma z + x\beta y\gamma z\delta u\alpha a' + u'\delta x\beta y\gamma z\alpha a' + a\alpha x\beta y\gamma z_\circ\delta u + x\beta y\gamma z_\circ\delta u\alpha a' + a\alpha u_\circ\beta x\delta y\gamma z + u_\circ\beta x\delta y\gamma z\alpha a' = a\alpha x\beta y\gamma z\delta u + x\beta y\gamma z\delta u\alpha a' + a\alpha u'\delta x\beta y\gamma z + u'\delta x\beta y\gamma z\alpha a' = [a, [x\beta y\gamma z, u]_\delta]_\alpha = [x\beta y\gamma z, u]^d_\delta$.

(ii) Taking left hand side of (ii) and by using Lemma 2.6, Definition 3.1 and Remark 4.5, we have $[x\beta y\gamma z, u]^d_\delta + [y\beta z\gamma u, x]^d_\delta + [z\beta u\gamma x, y]^d_\delta = [a,\, [x\beta y\gamma z, u]_\delta]_\alpha +$

$[a,[y\beta z\gamma u,x]_\delta]_\alpha+[a,[z\beta u\gamma x,y]_\delta]_\alpha = a\alpha x\beta y\gamma z\delta u + a\alpha u\delta x'\beta y\gamma z + x\beta y\gamma z\delta u\alpha a' + u\delta x'\beta y\gamma z\alpha a' + a\alpha y\beta z\gamma u\delta x + a\alpha x'\delta y\beta z\gamma u + y\beta z\gamma u\delta x\alpha a' + x'\delta y\beta z\gamma u\alpha a' + a\alpha z\beta u\gamma x\delta y + a\alpha y'\delta z\beta u\gamma x + z\beta u\gamma x\delta y\alpha a' + y'\delta z\beta u\gamma x\alpha a' = a\alpha x_\circ\beta y\gamma z\delta u + a\alpha y_\circ\beta z\gamma u\delta x + a\alpha u\delta x'\beta y\gamma z + a\alpha z\beta u\gamma x\delta y + x_\circ\beta y\gamma z\delta u\alpha a' + y_\circ\beta z\gamma u\delta x\alpha a' + u\delta x'\beta y\gamma z\alpha a' + z\beta u\gamma x\delta y\alpha a' = a\alpha u\delta x_\circ\beta y\gamma z + a\alpha u\delta x'\beta y\gamma z + a\alpha z\gamma u\delta x\beta y_\circ + a\alpha z\gamma u\delta x\beta y + u\delta x_\circ\beta y\gamma z\alpha a' + u\delta x'\beta y\gamma z\alpha a' + z\gamma u\delta x\beta y_\circ\alpha a' + z\gamma u\delta x\beta y\alpha a' = a\alpha u\delta x'\beta y\gamma z + a\alpha z\gamma u\delta x\beta y + u\delta x'\beta y\gamma z\alpha a' + z\gamma u\delta x\beta y\alpha a' = a\alpha(u\beta x'\gamma y\delta z + z\delta u\beta x\gamma y) + (u\beta x'\gamma y\delta z + z\delta u\beta x\gamma y)\alpha a' = [z,u\beta x\gamma y]^d_\delta$. ∎

The next Theorem is the generalization of Jordan identity.

Theorem 4.10. *If R_Γ is an additively regular Γ-semiring, $a\in R_\Gamma,\alpha\in\Gamma$ and d is an inner derivation determined by a and α, then for all $x,y,z\in R_\Gamma,\beta,\gamma\in\Gamma$, the following identities hold:*

(i) $[(x\circ y)_\beta,z]^d_\gamma+[(y\circ z)_\beta,x]^d_\gamma=[y,(z\circ x)_\beta]^d_\gamma$.

(ii) $((x\circ[y,z]_\gamma)_\beta)^d+(([x,z]_\gamma\circ y)_\beta)^d=[(x\circ y)_\beta,z]^d_\gamma$.

Proof. (i) Using Lemma 2.6, Definition 3.1, Definition 3.9 and Remark 4.5, the left hand side of (i) becomes $[(x\circ y)_\beta,z]^d_\gamma+[(y\circ z)_\beta,x]^d_\gamma=[a,[(x\circ y)_\beta,z]_\gamma]_\alpha+[a,[(y\circ z)_\beta,x]_\gamma]_\alpha = a\alpha x\beta y\gamma z+a\alpha y\beta x\gamma z+a\alpha z'\gamma x\beta y+a\alpha z'\gamma y\beta x+x\beta y\gamma z\alpha a'+y\beta x\gamma z\alpha a'+z'\gamma x\beta y\alpha a'+z'\gamma y\beta x\alpha a'+a\alpha y\beta z\gamma x+a\alpha z\beta y\gamma x+a\alpha x'\gamma y\beta z+a\alpha x'\gamma z\beta y+y\beta z\gamma x\alpha a'+z\beta y\gamma x\alpha a'+x'\gamma y\beta z\alpha a'+x'\gamma z\beta y\alpha a'=a\alpha y\beta x\gamma z+a\alpha y\beta z\gamma x+a\alpha z'\gamma x\beta y+a\alpha x'\gamma z\beta y+y\beta z\gamma x\alpha a'+y\beta x\gamma z\alpha a'+z'\gamma x\beta y\alpha a'+x'\gamma z\beta y\alpha a'+a\alpha x_\circ\beta y\gamma z+a\alpha z_\circ\gamma y\beta x+x_\circ\beta y\gamma z\alpha a'+z_\circ\gamma y\beta x\alpha a' = a\alpha y\beta x\gamma z+a\alpha y\beta z\gamma x+a\alpha z'\gamma x\beta y+a\alpha x'\gamma z\beta y+y\beta z\gamma x\alpha a'+y\beta x\gamma z\alpha a'+z'\gamma x\beta y\alpha a'+x'\gamma z\beta y\alpha a'+a\alpha y\gamma z\beta x_\circ+a\alpha y\beta x\gamma z_\circ+y\gamma z\beta x_\circ\alpha a'+y\beta x\gamma z_\circ\alpha a' = a\alpha y\beta z\gamma x+a\alpha y\beta x\gamma z+a\alpha z'\gamma x\beta y+a\alpha x'\gamma z\beta y+y\beta z\gamma x\alpha a'+y\beta x\gamma z\alpha a'+z'\gamma x\beta y\alpha a'+x'\gamma z\beta y\alpha a' = a\alpha y\gamma z\beta x+a\alpha y\gamma x\beta z+a\alpha z\beta x\gamma y'+a\alpha x\beta z\gamma y'+y\gamma z\beta x\alpha a'+y\gamma x\beta z\alpha a'+z\beta x\gamma y'\alpha a'+x\beta z\gamma y'\alpha a'=[y,(z\circ x)_\beta]^d_\gamma$.

(ii) Using Lemma 2.6, Definition 3.1, Lemma 3.2, Definition 3.9 and Remark 4.5, we have $((x\circ[y,z]_\gamma)_\beta)^d+(([x,z]_\gamma\circ y)_\beta)^d = a\alpha x\beta y\gamma z+a\alpha x\beta z\gamma y'+a\alpha y\gamma z\beta x+a\alpha z\gamma y'\beta x+x\beta y\gamma z\alpha a'+x\beta z\gamma y'\alpha a'+y\gamma z\beta x\alpha a'+z\gamma y'\beta x\alpha a'+a\alpha x\gamma z\beta y+a\alpha z\gamma x'\beta y+a\alpha y\beta x\gamma z+a\alpha y\beta z\gamma x'+x\gamma z\beta y\alpha a'+z\gamma x'\beta y\alpha a'+y\beta x\gamma z\alpha a'+y\beta z\gamma x'\alpha a'= a\alpha x\beta y\gamma z+a\alpha z\gamma x'\beta y+a\alpha z\gamma y'\beta x+x\beta y\gamma z\alpha a'+a\alpha y\beta x\gamma z+z\gamma x'\beta y\alpha a'+y\beta x\gamma z\alpha a'+z\gamma y'\beta x\alpha a'+a\alpha y\gamma z\beta x_\circ+x\beta z\gamma y_\circ\alpha a'+y\gamma z\beta x_\circ\alpha a'+a\alpha x\beta z\gamma y_\circ = a\alpha x\beta y\gamma z+a\alpha z\gamma x'\beta y+a\alpha z\gamma y'\beta x+x\beta y\gamma z\alpha a'+a\alpha y\beta x\gamma z+z\gamma x'\beta y\alpha a'+y\beta x\gamma z\alpha a'+z\gamma y'\beta x\alpha a'+a\alpha x_\circ\beta y\gamma z+y_\circ\gamma x\beta z\alpha a'+x_\circ\beta y\gamma z\alpha a'+a\alpha y_\circ\gamma x\beta z = a\alpha x\beta y\gamma z+a\alpha y\beta x\gamma z+y\beta x\gamma z\alpha a'+x\beta y\gamma z\alpha a'+a\alpha z'\gamma x\beta y+a\alpha z'\gamma y\beta x+z'\gamma x\beta y\alpha a'+z'\gamma y\beta x\alpha a'=[(x\circ y)_\beta,z]^d_\gamma$. ∎

Next, we define a symmetric map.

Definition 4.11. Let R_Γ be an additively regular Γ-semiring. Then a mapping $B : R_\Gamma \times \Gamma \times R_\Gamma \to R_\Gamma$ is said to be symmetric, if $B(x, \gamma, y) = B(y, \gamma, x)$ for all $x, y \in R_\Gamma, \gamma \in \Gamma$.

Definition 4.12. A mapping $f : R_\Gamma \to R_\Gamma$ defined by $f(x) = B(x, \gamma, x)$ is called trace of B. Further, for an additively regular Γ-semiring R_Γ and derivation $d : R_\Gamma \to R_\Gamma$, we define a map $B_d : R_\Gamma \times \Gamma \times R_\Gamma \to R_\Gamma$ corresponding to derivation d as $B_d(x, \gamma, y) = [d(x), y]_\gamma + [d(y), x]_\gamma$ for all $x, y \in R_\Gamma, \gamma \in \Gamma$.

The following proposition shows that the mapping B_d is symmetric.

Proposition 4.13. *Let R_Γ be an additively regular Γ-semiring. Then following statements hold:*

(i) *If $d : R_\Gamma \to R_\Gamma$ is a derivation, then B_d is symmetric.*

(ii) *If f is trace of B_d, then $f(x+y) = f(x)+f(y)+2B_d(x, \gamma, y)$ for all $x, y \in R_\Gamma$.*

Proof. (i) By Definition 4.11 and Definition 4.12, we have $B_d(x, \gamma, y) = [d(x), y]_\gamma + [d(y), x]_\gamma = [d(y), x]_\gamma + [d(x), y]_\gamma = B_d(y, \gamma, x)$ for all $x, y \in R_\Gamma, \gamma \in \Gamma$. This shows that B_d is symmetric.

(ii) As $d : R_\Gamma \to R_\Gamma$ is a derivation, hence by Definition 4.12, we have $f(x+y) = B_d(x+y, \gamma, x+y) = [d(x+y), x+y]_\gamma + [d(x+y), x+y]_\gamma = [d(x), x]_\gamma + [d(x), x]_\gamma + [d(y), y]_\gamma + [d(y), y]_\gamma + 2([d(x), y]_\gamma + [d(y), x]_\gamma) = B_d(x, \gamma, x) + B_d(y, \gamma, y) + 2B_d(x, \gamma, y) = f(x) + f(y) + 2B_d(x, \gamma, y)$. ■

Proposition 4.14. *Let R_Γ be an additively regular Γ-semiring and d be a derivation of R_Γ into itself. Then for all $x, y, z \in R_\Gamma, \beta, \gamma \in \Gamma$, we have $B_d(x, \gamma, z)\beta y + x\beta B_d(y, \gamma, z) = [z, d(x)]_\gamma \beta y' + x'\beta[z, d(y)]_\gamma + [d(z), x\beta y]_\gamma$.*

Proof. By Lemma 2.6, Definition 4.12, Theorem 3.4(i) and Theorem 3.3(v), we have $B_d(x, \gamma, z)\beta y + x\beta B_d(y, \gamma, z) = [d(x), z]_\gamma \beta y + [d(z), x]_\gamma \beta y + x\beta[d(y), z]_\gamma + x\beta[d(z), y]_\gamma = [d(x), z']_\gamma \beta y' + x'\beta[d(y), z]'_\gamma + [d(z), x]_\gamma \beta y + x\beta[d(z), y]_\gamma = [z, d(x)]_\gamma \beta y' + x'\beta[z, d(y)]_\gamma + [d(z), x\beta y]_\gamma$. ■

Theorem 4.15. *Let R_Γ be an additively regular Γ-semiring and d be a derivation of R_Γ into itself. Then $B_d(x\beta y, \gamma, z) = B_d(x, \gamma, z)\beta y + x\beta B_d(y, \gamma, z) + d(x)\beta[y, z]_\gamma + [x, z]_\gamma \beta d(y)$ $\forall$ $x, y, z \in R_\Gamma, \beta, \gamma \in \Gamma$.*

Proof. By Definition 4.12, Lemma 2.6, Theorem 3.4(i) and Theorem 3.3, we have $B_d(x\beta y, \gamma, z) = [d(x\beta y), z]_\gamma + [d(z), x\beta y]_\gamma = [d(x)\beta y, z]_\gamma + [x\beta d(y), z]_\gamma + [d(z), x\beta y]_\gamma = [z, d(x)\beta y']_\gamma + [z, x'\beta d(y)]_\gamma + [d(z), x\beta y]_\gamma = [z, d(x)]_\gamma \beta y' + x'\beta[z, d(y)]_\gamma + d(x)\beta[y, z]_\gamma + [x, z]_\gamma \beta d(y) + [d(z), x]_\gamma \beta y + x\beta[d(z), y]_\gamma = B_d(x, \gamma, z)\beta y + x\beta B_d(y, \gamma, z) + d(x)\beta[y, z]_\gamma + [x, z]_\gamma \beta d(y)$. ■

Proposition 4.16. *Let R_Γ be an additively regular Γ-semiring with characteristic 2 and d be a derivation of R_Γ into itself. Then d^2 is again a derivation on R_Γ.*

Proof. Let d be a derivation on R_Γ. Then clearly d^2 is additive and $d^2(x\gamma y) = d(d(x)\gamma y + x\gamma d(y)) = d(d(x))\gamma y + 2d(x)\gamma d(y) + x\gamma d(d(y)) = d^2(x)\gamma y + x\gamma d^2(y)$. ■

Lemma 4.17. *Let R_Γ be an additively regular Γ-semiring such that $[x,y]_\gamma = 0$ $\forall\ x,y \in R_\Gamma, \gamma \in \Gamma$. Then R_Γ is commutative.*

The proof of this lemma is quite easy so we omit the proof.

Definition 4.18. An additive mapping f of an additively regular Γ-semiring R_Γ is said to be centralizing if $[[f(x),x]_\alpha, y]_\beta = 0$ for all $x, y \in R_\Gamma, \alpha, \beta \in \Gamma$. Moreover, f is said to be commuting if $[f(x),x]_\alpha = 0$ for all $x \in R_\Gamma, \alpha \in \Gamma$.

Remark 4.19. Let f be a centralizing map on a prime additively regular Γ-semiring R_Γ. Then $[[f(x),x]_\alpha, y]_\beta = 0$ for all $x, y \in R_\Gamma, \alpha, \beta \in \Gamma$, that is, $[f(x),x]_\alpha\beta y + (y' + y)\beta[f(x),x]_\alpha = y\beta[f(x),x]_\alpha$ for all $x, y \in R_\Gamma, \alpha, \beta \in \Gamma$. So, (A_2,Γ)-condition implies that $[f(x),x]_\alpha$ belongs to the centre of R_Γ for all $x \in R_\Gamma, \alpha \in \Gamma$. Moreover, the definition of f forces $[[f(x),x]_\alpha, x]_\beta = 0$ for all $x \in R_\Gamma, \alpha, \beta \in \Gamma$, that is, for all $x \in R_\Gamma, \alpha, \beta \in \Gamma$ we have

$0 = [f(x),x]_\alpha\beta x + x'\beta[f(x),x]_\alpha = [f(x),x]_\alpha\beta(x + x')$, since $[f(x),x]_\alpha$ belongs to the centre of R_Γ.

Hence, $[f(x),x]_\alpha\Gamma(R_\Gamma + R'_\Gamma) = (0)$ leading to $[f(x),x]_\alpha\Gamma R_\Gamma = (0)$ for all $x \in R_\Gamma, \alpha \in \Gamma$ as R'_Γ is contained in R_Γ. Therefore, $[f(x),x]_\alpha\Gamma R_\Gamma\Gamma 1 = (0)$ for all $x \in R_\Gamma, \alpha \in \Gamma$. By using primeness of R_Γ we can conclude that $[f(x),x]_\alpha = 0$ for all $x \in R_\Gamma, \alpha \in \Gamma$. Therefore, every centralizing mapping of a prime additively regular Γ-semiring R_Γ is also commuting.

Theorem 4.20. *Let d be a non-zero derivation of prime additively regular Γ-semiring R_Γ such that $d([x,y]_\gamma) = 0$ for all $x, y \in R_\Gamma, \gamma \in \Gamma$. Then R_Γ is commutative.*

Proof. Let d be a derivation of R_Γ such that $d([x,y]_\gamma) = 0$ for all $x, y \in R_\Gamma$, $\gamma \in \Gamma$. Then by using Definitions 3.1 and 4.1, we are left with

(i) $$[d(x),y]_\gamma + [x,d(y)]_\gamma = 0 \text{ for all } x, y \in R_\Gamma,\ \gamma \in \Gamma.$$

By replacing y by $y\beta x$ in (i) and then using Definition 3.1, Theorem 3.4, Definition 4.1 and equation (i), we get $0 = [d(x), y\beta x]_\gamma + [x, d(y\beta x)]_\gamma = d\,(y)\,\beta[x,x]_\gamma + [x,y]_\gamma\beta d(x)$. So,

(ii) $$d\,(y)\,\beta[x,x]_\gamma + [x,y]_\gamma\beta d(x) = 0 \text{ for all } x, y \in R_\Gamma, \beta,\ \gamma \in \Gamma.$$

Replacing y by $r\alpha y$ in (ii) and then by using Theorem 3.4, Lemma 2.6, Lemma 3.2, equations (i) and (ii), we have $0 = d\,(r\alpha y)\,\beta[x,x]_\gamma + [x, r\alpha y]_\gamma\beta d(x) = [x,r]_\gamma\alpha y\beta d(x) + r\alpha[x,y]_\gamma\beta d(x) + d(r)\alpha y\beta[x,x]_\gamma + r\alpha d(y)\beta[x,x]_\gamma = (x\gamma r + r'\gamma x)$ $\alpha y\beta d(x) + r_\circ\gamma x\alpha y\beta d(x) + d(r)\alpha y\beta x\gamma x_\circ = [x,r]_\gamma\alpha y\beta d(x) + d(r)\gamma(x + x')\alpha y\beta x +$ $r\gamma x\alpha y\beta d(x) + r'\gamma x\alpha y\beta d(x) + r_\circ\gamma x\alpha y\beta d(x) = [x,r]_\gamma\alpha y\beta d(x) + d(r)\gamma(x + x_\circ)\alpha y\beta x +$

$d(r)\gamma x'\alpha y\beta x + r_\circ\gamma x\alpha y\beta d(x) + (d(x))'\gamma r\alpha y\beta x + d(x)\gamma r\alpha y\beta x = [x,r]_\gamma\alpha y\beta d(x) + (d(r)\gamma x + x'\gamma d(r))\alpha y\beta x + (x\gamma d(r) + d(r)\gamma x')\alpha y\beta x + (r\gamma d(x) + (d(x))'\gamma r)\alpha y\beta x + (r'\gamma d(x)+d(x)\gamma r)\alpha y\beta x = [x,r]_\gamma\alpha y\beta d(x)+([d(r),x]_\gamma+[r,d(x)]_\gamma)\alpha y\beta x+([x,d(r)]_\gamma + [d(x),r]_\gamma)\alpha y\beta x = [x,r]_\gamma\alpha y\beta d(x)$ for all $x,y,r \in R_\Gamma, \alpha,\beta,\gamma \in \Gamma$. Then by primeness of R_Γ, either $[x,r]_\gamma = 0$ for all $x,r \in R_\Gamma, \gamma \in \Gamma$ or $d(x) = 0$ for all $x \in R_\Gamma$. But as d is non-zero so we have $[x,r]_\gamma = 0$ for all $x,r \in R_\Gamma, \gamma \in \Gamma$. Thus by Lemma 4.17, R_Γ is commutative. ∎

Theorem 4.21. *Let d be a non-zero derivation of prime additively regular Γ-semiring R_Γ such that $[d(x),x]_\gamma = 0$ for all $x \in R_\Gamma, \gamma \in \Gamma$. Then R_Γ is commutative.*

Proof. As $0 = [d(x+y),x+y]_\gamma = [d(x),y]_\gamma + [d(y),x]_\gamma$. Hence we have

(i) $$[d(x),y]_\gamma + [d(y),x]_\gamma = 0 \text{ for all } x,y \in R_\Gamma,\ \gamma \in \Gamma.$$

By replacing y by $y\beta x$ in (i) and then using Theorem 3.4 and equation (i), we get $0 = [d(x),y\beta x]_\gamma + [d(y\beta x),x]_\gamma = d\,(y)\,\beta[x,x]_\gamma + [y,x]_\gamma\beta d(x)$. So,

(ii) $$d\,(y)\,\beta[x,x]_\gamma + [y,x]_\gamma\beta d(x) = 0 \text{ for all } x,y \in R_\Gamma, \beta,\gamma \in \Gamma.$$

Replacing y by $r\alpha y$ in (ii) and then by using Theorem 3.4, Lemma 2.6, Lemma 3.2, equations (i) and (ii), we get $0 = [r\alpha y,x]_\gamma\beta d(x) + d\,(r\alpha y)\,\beta[x,x]_\gamma = [r,x]_\gamma\alpha y\beta d(x)+d(r)\alpha y\beta x\gamma x_\circ = [r,x]_\gamma\alpha y\beta d(x)+x_\circ\gamma r\alpha y\beta d(x)+d(r)\gamma(x+x')\alpha y\beta x = [r,x]_\gamma\alpha y\beta d(x) + (d(r)\gamma x + d(r)\gamma x_\circ)\alpha y\beta x + d(r)\gamma x'\alpha y\beta x + x\gamma r\alpha y\beta d(x) + x_\circ\gamma r\alpha y\beta d(x) + x'\gamma r\alpha y\beta d(x) = [r,x]_\gamma\alpha y\beta d(x) + [d(r),x]_\gamma\alpha y\beta x + x\gamma d(r)\alpha y\beta x + d(r)\gamma x'\alpha y\beta x + x_\circ\gamma r\alpha y\beta d(x) + r_\circ\gamma x\alpha y\beta d(x) = [r,x]_\gamma\alpha y\beta d(x) + [d(r),x]_\gamma\alpha y\beta x + [x,d(r)]_\gamma\alpha y\beta x + r_\circ\gamma d(x)\alpha y\beta x + d(x)\gamma r_\circ\alpha y\beta x = [r,x]_\gamma\alpha y\beta d(x) + ([d(r),x]_\gamma + [d(x),r]_\gamma)\alpha y\beta x + ([r,d(x)]_\gamma + [x,d(r)]_\gamma)\alpha y\beta x = [r,x]_\gamma\alpha y\beta d(x)$ for all $x,y.r \in R_\Gamma, \alpha,\beta,\gamma \in \Gamma$. Then by primeness of R_Γ, either $[r,x]_\gamma = 0$ for all $x,r \in R_\Gamma, \gamma \in \Gamma$ or $d(x) = 0$ for all $x \in R_\Gamma$. But as d is non-zero so we have $[r,x]_\gamma = 0$ for all $x,r \in R_\Gamma, \gamma \in \Gamma$. Thus R_Γ is commutative. ∎

The next result is a generalization of Posner's second theorem for additively regular Γ-semiring R_Γ.

Theorem 4.22. *Let R_Γ be a prime additively regular Γ-semiring. If there is a non-zero centralizing derivation of R_Γ, then R_Γ is commutative.*

Proof. Let R_Γ be a prime additively regular Γ-semiring and d be a non-zero centralizing derivation of R_Γ. Then by using Remark 4.19, we have $[d(x),x]_\gamma = 0$ for all $x \in R_\Gamma, \gamma \in \Gamma$. Thus from Theorem 4.21, R_Γ is commutative. ∎

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