

HYPER RL-IDEALS IN HYPER RESIDUATED LATTICES

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Abstract

In this paper, we introduce the notion of a (strong) hyper RL-ideal in hyper residuated lattices and give some properties and characterizations of them. Next, we characterize the (strong) hyper RL-ideals generated by a subset and give some characterizations of the lattice of these hyper RL-ideals. Particularly, we prove that this lattice is algebraic and compact elements are finitely generated hyper RL-ideals, and obtain some isomorphism theorems. Finally, we introduce the notion of nodal hyper RL-ideals in a hyper residuated lattice and investigate their properties. We prove that the set of nodal hyper RL-ideals is a complete Brouwerian lattice and under suitable operations is a Heyting algebra.

Keywords: residuated lattice, MV-algebra, BL-algebra, hyper residuated lattice, hyper ideal.

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1. INTRODUCTION AND PRELIMINARIES

Residuated lattices were introduced by Ward and Dilworth [10] as a bounded lattice L endowed with a residuated operation \star with the residual \rightarrow such that $(L; \star)$ forms a commutative monoid and also satisfies the adjoint property

$$a \star b \leq c \Leftrightarrow a \leq b \rightarrow c$$

The main examples of residuated lattices are MV-algebras [5] and BL-algebras [7].

The hyperstructure theory was introduced by Marty at the 8th Congress of Scandinavian Mathematicians. After that many researchers applied it to algebraic structures. It was Borzooei and their co-authors that applied the hyperstructures to algebraic logics and introduced hyper K-algebras [4]. After that

Ghorbani *et al.* introduced hyper MV-algebras [6]. Mittas *et al.* [8] introduced the notion of a hyperlattice and superlattice. A superlattice [8] is a partially ordered set $\langle A; \leq \rangle$ endowed with two binary hyperoperations \vee and \wedge satisfying the following properties: for all $a, b, c \in A$,

- $a \in (a \vee a) \cap (a \wedge a)$,
- $a \vee b = b \vee a$, $a \wedge b = b \wedge a$,
- $a \vee (b \vee c) = (a \vee b) \vee c$, $a \wedge (b \wedge c) = (a \wedge b) \wedge c$,
- $a \in (a \vee (a \wedge b)) \cap (a \wedge (a \vee b))$,
- $a \leq b$ implies $b \in a \vee b$ and $a \in a \wedge b$,
- if $a \in a \wedge b$ or $b \in a \vee b$, then $a \leq b$.

If (X, \leq) is a partially ordered set, the ordering \leq can be extended to a binary relation \ll in the power set of X as

$$A \ll B \Leftrightarrow (\exists a \in A)(\exists b \in B) a \leq b.$$

Zahiri *et al.* applied the hyperstructure theory to residuated lattices and introduced hyper residuated lattices as a generalization of residuated lattices and hyper MV-algebras as a structure $L = \langle L, \vee, \wedge, \odot, \rightarrow, 0, 1 \rangle$, where

- $\langle L, \vee, \wedge, 0, 1 \rangle$ is a bounded superlattice (with the induced order \leq)
- $\langle L, \odot, 1 \rangle$ is a commutative semihypergroup with 1 as the identity,
- the pair (\odot, \rightarrow) satisfies the condition $a \odot b \ll c$ if and only if $b \ll a \rightarrow c$

In a hyper residuated lattice an auxiliary hyperoperation ' \oplus ' is defined as $x \oplus y = \neg x \rightarrow y$, where $\neg x = x \rightarrow 0$.

Proposition 1.1. *In any hyper residuated lattice L , for any $a, b, c \in L$ and $A, B, C \subseteq L$, the following properties hold:*

- (1) $1 \ll A$ implies that $1 \in A$,
- (2) $a \leq b$ implies that $1 \in a \rightarrow b$. Particularly, $1 \in a \rightarrow 1$, $1 \in 0 \rightarrow a$ and $1 \in a \rightarrow a$,
- (3) if 1 is a scalar element, $a \in 1 \rightarrow a$ and if $1 \in a \rightarrow b$, then $a \leq b$,
- (4) $A \ll B \rightarrow C$ if and only if $A \odot B \ll C$ if and only if $B \ll A \rightarrow C$,
- (5) $A \subseteq B$ implies that $A \ll B$,
- (6) $0 \in a \odot 0$, $0 \in 0 \oplus 0$,
- (7) $a \ll \neg\neg a$,
- (8) $a \leq b$ implies that $b \rightarrow c \ll a \rightarrow c$,
- (9) If $A \cap B \neq \emptyset$, then $A \ll B$.

(10) If B is a down-set, then $A \ll B$ implies that $A \cap B \neq \emptyset$.

We recall that a down-set in a partially ordered set $(P; \leq)$ is a subset A satisfying (HI), where (HI) is the following condition if $x \leq y$ and $y \in A$, then $x \in A$.

A closure operator on a set A is a mapping $C : 2^A \rightarrow 2^A$ with the following properties:

- (C1) $X \subseteq C(X)$,
- (C2) $C(C(X)) = C(X)$,
- (C3) $X \subseteq Y$ implies that $C(X) \subseteq C(Y)$.

If $X \subseteq A$ satisfies $C(X) = X$, X is said to be closed with respect to C and the set of these subsets are denoted by A_C . Closure operator C is said to be algebraic if $C(X) = \bigcup\{C(Y) : Y \subseteq X \text{ is finite}\}$. For more details on closure operators, we refer the reader to [9].

In the sequel, in this paper, L will denote a hyper residuated lattice and \mathcal{L} a complete lattice.

2. MAIN RESULTS

Definition 2.1. A down-set I of L is called a

- *strong hyper RL-ideal* if $x \oplus y \subseteq I$, for all $x, y \in I$,
- *hyper RL-ideal* if $x \oplus y \ll I$, for all $x, y \in I$.

By $\mathcal{SHI}(L)$ ($\mathcal{HI}(L)$) we mean the set of strong hyper RL-ideals (hyper RL-ideals) of L .

As a direct consequence of the definition and Proposition 1.1(5) we get that

Theorem 2.2. *Any strong hyper RL-ideal is a hyper RL-ideal.*

Example 2.3. Because of Proposition 1.1(6), it is obvious that in any hyper residuated lattice L , the singleton $\{0\}$ is a hyper RL-ideal but not necessary a strong hyper RL-ideal (see Example 2.8). Moreover, L itself is a (strong) hyper RL-ideal.

Example 2.4. Consider the hyper residuated lattice $(L; \vee, \wedge, \odot, \rightarrow, 0, 1)$, where $L = \{0, a, b, 1\}$ is a chain with the ordering $0 < a < b < 1$ and the hyperoperations \vee, \wedge, \odot and \rightarrow and the auxiliary hyperoperation \oplus are defined as in Tables 1-3 (see [3]). It is easy to verify that the singleton $\{0\}$ is a strong hyper RL-ideal and so a hyper RL-ideal of L . This example shows that proper hyper RL-ideals may not be exist, in general.

\vee	0	a	b	1	\wedge	0	a	b	1
0	L	$\{a, b, 1\}$	$\{b, 1\}$	$\{1\}$	0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a, b, 1\}$	$\{a, b, 1\}$	$\{b, 1\}$	$\{1\}$	a	$\{0\}$	$\{0, a\}$	$\{0, a\}$	$\{0, a\}$
b	$\{b, 1\}$	$\{b, 1\}$	$\{b, 1\}$	$\{1\}$	b	$\{0\}L$	$\{0, a\}$	$\{0, a, b\}$	$\{0, a, b\}$
1	$\{0, 1\}$	$\{1\}$	$\{1\}$	$\{1\}$	1	$\{0\}$	$\{0, a\}$	$\{0, a, b\}$	L

Table 1. Cayley tables of \vee and \wedge .

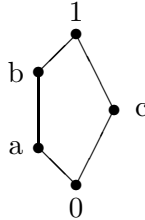
\odot	0	a	b	1	\rightarrow	0	a	b	1
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	0	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
a	$\{0\}$	$\{0, a\}$	$\{a\}$	$\{a\}$	a	$\{0, a\}$	$\{1\}$	$\{1\}$	$\{1\}$
b	$\{0\}$	$\{a\}$	$\{b\}$	$\{b\}$	b	$\{0\}$	$\{0, a\}$	$\{1\}$	$\{1\}$
1	$\{0\}$	$\{a\}$	$\{b\}$	$\{1\}$	1	$\{0\}$	$\{a\}$	$\{b\}$	$\{1\}$

Table 2. Cayley Tables of \oplus and \rightarrow .

\oplus	0	a	b	1
0	$\{0\}$	$\{a\}$	$\{b\}$	$\{1\}$
a	$\{0, a, 1\}$	$\{1\}$	$\{1\}$	$\{1\}$
b	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
1	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$

Table 3. Cayley table of \oplus of Example 2.4.

Example 2.5. Let $L = \{0, a, b, c, 1\}$ be a lattice whose Hasse diagram is below (Figure 1), and let $x \wedge y$ and $x \vee y$ be the set of all lower bounds and upper bounds (respectively) of $\{x, y\}$. Define the hyperoperations \odot and \rightarrow as in Table 4. Then L is a hyper residuated lattice (see [3]). It is not difficult to verify that $\{0\}$, $\{0, c\}$ and $\{0, a, b\}$ are strong hyper RL-ideals of L .

Figure 1. The Hasse diagram of L .

The next example shows that the converse of Theorem 2.2 does not hold, in general.

\odot	0	a	b	c	1	\rightarrow	0	a	b	c	1
0	{0}	{0}	{0}	{0}	{0}	0	{1}	{1}	{1}	{1}	{1}
a	{0}	{ a }	{ a }	{0}	{ a }	a	{ c }	{1}	{1}	{ c }	{1}
b	{0}	{ a }	{ a, b }	{0}	{ a, b }	b	{ b }	{ a, b, c }	{1}	{ c }	{1}
c	{0}	{0}	{0}	{ c }	{ c }	c	{ a, b }	{ a, b }	{ a, b }	{1}	{1}
1	{0}	{ a }	{ a, b }	{ c }	{1}	1	{0}	{1}	{ a, b }	{ c }	{1}

Table 4. Cayley Tables of \odot and \rightarrow .

Example 2.6. Consider the hyper residuated lattice $(L; \vee, \wedge, \odot, \rightarrow, 0, 1)$, where $L = \{0, a, b, 1\}$ is a chain with the ordering $0 < a < b < 1$ and the hyperoperations $\vee, \wedge, \odot, \rightarrow$ and \oplus are defined as the tables 5–6 (see [3]). Routine calculations show that the subsets $I = \{0, a\}$, $K = \{0, b\}$ and $J = \{0, a, b\}$ are hyper RL-ideals of L , which are not strong hyper RL-ideals because $a, 0 \in I \cap J$, while $a \oplus 0 = L$, which is not a subset of I and J . And $0 \in K$, while $0 \oplus 0 = \{0, 1\} \not\subseteq K$. Also, the singleton $\{0\}$ is a hyper RL-ideal, which is not a strong hyper RL-ideal. Indeed, L does not contain any proper strong hyper RL-ideals.

\vee	0	a	b	1	\wedge	0	a	b	1
0	L	{ $a, b, 1$ }	{ $b, 1$ }	{1}	0	{0}	{0}	{0}	{0}
a	{ $a, b, 1$ }	{ $a, b, 1$ }	{ $b, 1$ }	{1}	a	{0}	{0, a }	{0, a }	{0, a }
b	{ $b, 1$ }	{ $b, 1$ }	{ $b, 1$ }	{1}	b	{0}	{0, a }	{0, a, b }	{0, a, b }
1	{0, 1}	{1}	{1}	{1}	1	{0}	{0, a }	{0, a, b }	L

Table 5.

\rightarrow	0	a	b	1	\oplus	0	a	b	1
0	{1}	{1}	{1}	{1}	0	{0, 1}	{ a }	{ $b, 1$ }	{1}
a	{ $a, b, 1$ }	{ $a, 1$ }	{1}	{1}	a	L	{ $a, 1$ }	{ $b, 1$ }	{1}
b	{ $a, 1$ }	{ a }	{ $b, 1$ }	{1}	b	L	{ $a, 1$ }	{ $b, 1$ }	{1}
1	{0, 1}	{ a }	{ $b, 1$ }	{1}	1	{0, 1}	{ $a, 1$ }	{ $b, 1$ }	{1}

Table 6. Cayley table of \rightarrow and \oplus .

Proposition 2.7. *The intersection of any nonempty family of strong hyper RL-ideals is again a strong hyper RL-ideal.*

Proof. Let $\{I_\alpha : \alpha \in \Lambda\}$ be a nonempty indexed family of strong hyper RL-ideals of hyper residuated lattice L and let $x, y \in L$. If $x \leq y$ and $y \in \bigcap_{\alpha \in \Lambda} I_\alpha$, so $y \in I_\alpha$, for each $\alpha \in \Lambda$, whence $x \in I_\alpha$, for each $\alpha \in \Lambda$. Hence $x \in \bigcap_{\alpha \in \Lambda} I_\alpha$, showing that $\bigcap_{\alpha \in \Lambda} I_\alpha$ is a down-set. Now, assume that $x, y \in \bigcap_{\alpha \in \Lambda} I_\alpha$. Then

$x, y \in I_\alpha$, for each $\alpha \in \Lambda$ and since I_α is a strong hyper RL-ideal of L , then $x \oplus y \subseteq I_\alpha$, for each $\alpha \in \Lambda$. This implies that $x \oplus y \subseteq \bigcap_{\alpha \in \Lambda} I_\alpha$. ■

Example 2.8. Consider the hyper residuated lattice L in which $L = \{x_i : i \in \mathbb{N}\} \cup \{0, 1\}$ is a lattice whose Hasse diagram is below (Figure 2) and the hyper-operations \vee, \wedge, \odot and \rightarrow are defined as follows (see [11]):

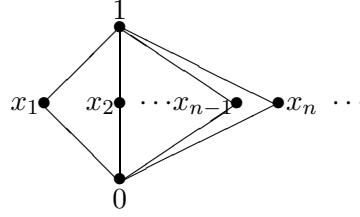


Figure 2. Hasse diagram of L .

$$a \vee b = \{c \in L : a \leq c, b \leq c\}, \quad a \wedge b = \{c \in L : c \leq a, c \leq b\}$$

$$a \odot b = a \wedge b \text{ and } a \rightarrow b = \begin{cases} \{1\} & : a \leq b \\ \{x_i : i \in \mathbb{N}\} & : a = 1, b \in L \setminus \{1\} \\ \{x_j : j \in \mathbb{N}, j \leq i\} \cup \{1\} & : a, b \in \{x_i : i \in \mathbb{N}\}, \\ & a = x_i, a \neq b \\ \{x_j : j \in \mathbb{N}, j \leq i\} \cup \{1\} & : a \in \{x_i : i \in \mathbb{N}\}, b = 0. \end{cases}$$

Routine calculations show that the sets $I_j = \{0, x_j\}$ and $J_k = \{0, x_1, \dots, x_k\}$ ($j \in \mathbb{N}$ and $j \in \mathbb{N} \setminus \{1\}$) are hyper RL-ideals of L , which are not strong hyper RL-ideals. Moreover, for all $r, s \in \mathbb{N}$ with $r \neq s$ we have $I_s \cap I_r = \{0, x_s\} \cap \{0, x_r\} = \{0\}$, which is not a hyper RL-ideal, because $0 \oplus 0 = \{x_i : i \in \mathbb{N}\} \not\subseteq I_r \cap I_s$. This shows that the intersection of hyper RL-ideals is not a hyper RL-ideal, in general.

Example 2.9. Consider the hyper residuated lattice L given in Example 2.5. As we see $I = \{0, c\}$ and $J = \{0, a, b\}$ are (strong) hyper RL-ideals of L , while $I \cup J = \{0, a, b, c\}$ is not a hyper RL-ideal (and so not a strong hyper RL-ideal), because $a \oplus c = \{1\} \not\subseteq I \cup J$.

Despite that the union of two (strong) hyper RL-ideal may not be a (strong) hyper RL-ideal, in general, but it is easily proved that

Proposition 2.10. *If C is a chain of (strong) hyper RL-ideals of L , $\bigcup C$ is again a (strong) hyper RL-ideal of L .*

In virtue of Proposition 2.7, any subset of L generates a strong hyper RL-ideal, which is the intersection of all strong hyper RL-ideals of L containing that subset. More precisely, if X is a subset of L , the intersection of all strong hyper

RL-ideals of L containing X is a strong hyper RL-ideal, denoted by $\langle X \rangle_s$. We observe that when $X = \emptyset$, $\langle X \rangle_s = L$. Furthermore, $\langle X \rangle_s$ satisfies the following properties:

- $\langle X \rangle_s = X$ if and only if X is a strong hyper RL-ideal of L .
- $X \subseteq Y$ implies that $\langle X \rangle_s \subseteq \langle Y \rangle_s$.
- $\langle \langle X \rangle_s \rangle_s = \langle X \rangle_s$.

Hence, we deduce that the mapping $X \mapsto \langle X \rangle_s$ is a closure operator on L , where the closed members of L are the strong hyper RL-ideals of L , i.e, the set $\mathcal{SHI}(L)$. Indeed, $\mathcal{SHI}(L)$ is a closed set system for L (see [9, Exercise §5.5]). Hence,

Theorem 2.11. $\mathcal{SHI}(L)$, as a poset with set-inclusion as the partial ordering, is a complete lattice.

Theorem 2.12. There is a closure operator C on $\mathcal{SHI}(L)$ such that $\mathcal{SHI}(L)$ is order-isomorphic to the lattice of closed subsets of $\mathcal{SHI}(L)$ with respect to C .

Proof. We define the mapping C with $C(X) = \{J \in \mathcal{SHI}(L) : J \subseteq \bigvee X\}$. It is easy to verify that C is a closure operator on $\mathcal{SHI}(L)$ such that the set of closed subsets of $\mathcal{SHI}(L)$ are those sets $I^\downarrow = \{J \in \mathcal{SHI}(L) : J \subseteq I\}$, where $I \in \mathcal{SHI}(L)$. Routine investigations show that the mapping $\Phi : \mathcal{SHI}(L) \rightarrow \mathcal{SHI}(L)_C$ with $I \mapsto I^\downarrow$ is a bijection such that Φ and Φ^{-1} are isotone. To prove this, it suffices we observe that $I \subseteq J$ if and only if $I^\downarrow \subseteq J^\downarrow$. Hence Φ is the desired order-isomorphism. ■

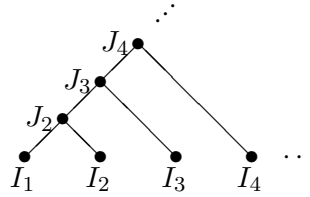


Figure 3. Hasse diagram of $\mathcal{HI}(L)$ in Example 2.8.

One important question is that how we can characterize the elements of generated (strong) hyper RL-ideal by a subset. For strong hyper RL-ideals we can see that for a subset X of L

$$(2.1) \quad \langle X \rangle_s \supseteq \{x \in L : x \ll a_1 \oplus a_2 \oplus \dots \oplus a_n; \text{ for some } n \in \mathbb{N}, a_1, a_2, \dots, a_n \in X\}.$$

As we saw, in Example 2.8, $\mathcal{HI}(L)$ is not closed with respect to the intersection, in general, whence $(\mathcal{HI}, \subseteq)$ is not a lattice. Figure 3 shows that the set of all

hyper RL-ideals of hyper residuated lattice L given in Example 2.8 is an upper semilattice. Actually, in this example, the supremum of any two elements is the union of them. Obviously, when the intersection is defined, $\mathcal{HI}(L)$ is a complete lattice, as well. Actually, in this case, $(\mathcal{SHI}(L), \subseteq)$ is a complete sublattice of the complete lattice $(\mathcal{HI}(L), \subseteq)$. Furthermore, we can characterize the hyper RL-ideal $\langle X \rangle$ generated by a subset X of L . Before we proceed, we give some preliminaries.

Definition 2.13. An element $a \in L$ is called a *scalar* with respect to the hyperoperation \oplus (or \oplus -scalar) if $|a \oplus b| = |b \oplus a| = 1$, where the symbol $|\cdot|$ denotes the cardinality of the set.

By $SC_\star(L)$, we mean the set of all scalar elements of L with respect the hyperoperation $\star \in \{\odot, \oplus\}$.

Example 2.4 shows that $SC_\odot(L)$ and $SC_\oplus(L)$ may not be equal, in general.

Example 2.14. Consider the hyper residuated lattice L given in Example 2.4. Routine calculations show that $SC_\oplus(L) = \{b, 1\}$ (see Table 3).

Lemma 2.15. *In any hyper residuated lattice, the following properties hold:*

- (1) $x \leq y$ implies that $\neg y \ll \neg x$
- (2) $x \leq y$ implies that $x \oplus z \ll y \oplus z$,
- (3) if \oplus is associative, then $SC_\oplus(L)$ is closed with respect to \oplus .

Proof. (1) Follows from Proposition 1.1(8).

(2) Let $x \leq y$, for $x, y \in L$. Then $\neg y \ll \neg x$, whence $a \leq b$, for some $a \in \neg y$ and some $b \in \neg x$. Hence $b \rightarrow z \ll a \rightarrow z \subseteq \neg y \rightarrow z = y \oplus z$ and since $b \rightarrow z \subseteq \neg x \rightarrow z = x \oplus z$, then $x \oplus z \ll y \oplus z$.

(3) Assume that the hyperoperation \oplus is associative and $a, b \in SC(L)$. Then, for any $x \in L$ we have $|b \oplus x| = 1$ and so $|(a \oplus b) \oplus x| = |a \oplus (b \oplus x)| = 1$. Similarly, it is proved that $|x \oplus (a \oplus b)| = 1$. Hence $a \oplus b$ is a scalar element. ■

Theorem 2.16. *Assume that the hyperoperation \oplus is associative, $X \subseteq SC_\oplus(L)$ and $\mathcal{HI}(L)$ is closed with respect to the intersection. Then the hyper RL-ideal of L generated by X is characterized as*

$$\langle X \rangle = \{x \in L : x \ll a_1 \oplus a_2 \oplus \cdots \oplus a_n, \text{ for some } n \in \mathbb{N}, a_1, \dots, a_n \in X\}.$$

Particularly, if $X = \{a\}$ is singleton, then $\langle a \rangle = \{x \in L : x \ll na, \text{ for some } n \in \mathbb{N}\}$.

Proof. Let $A = \{x \in L : (\cdots ((x \oplus a_1) \oplus a_2) \oplus \cdots) \oplus a_n = \{0\}, \text{ for some } n \in \mathbb{N}, a_1, \dots, a_n \in X\}$. Reflexivity of \leq implies that $X \subseteq A$. Also, it is obvious that

$0 \in A$. Now, let $x, y \in A$. Then there exist $n, m \in \mathbb{N}$ and $a_1, \dots, a_n, b_1, \dots, b_m \in X$ such that

$$(2.2) \quad x \ll (\cdots (a_1 \oplus a_2) \oplus \cdots) \oplus a_n$$

and

$$(2.3) \quad y \ll (\cdots (b_1 \oplus b_2) \oplus \cdots) \oplus b_m.$$

Let $a_1 \oplus a_2 \oplus \cdots \oplus a_n = \{u\}$ and $b_1 \oplus b_2 \oplus \cdots \oplus b_m = \{v\}$. (2.2) and (2.3) state that $x \leq u$ and $y \leq v$ and so $x \oplus y \ll u \oplus y \ll u \oplus v$, whence $x \oplus y \ll u \oplus v \subseteq ((\cdots (a_1 \oplus a_2) \oplus \cdots) \oplus a_n) \oplus ((b_1 \oplus b_2) \oplus \cdots) \oplus b_m = (\cdots (((a_1 \oplus a_2) \oplus \cdots) \oplus a_n) \oplus b_1) \oplus \cdots) \oplus b_m$, whence $x \oplus y \ll A$ showing that A is a hyper RL-ideal of L . Now, let B be a hyper RL-ideal of L containing X and $x \in A$. Then $x \ll a_1 \oplus a_2 \oplus \cdots \oplus a_n$ for some $n \in \mathbb{N}$ and $a_1, \dots, a_n \in X \subseteq B$. Hence $x \ll \{u\} =_{def} a_1 \oplus a_2 \oplus \cdots \oplus a_n \ll B$, whence $x \in B$. Therefore, $A = \langle X \rangle$. ■

Theorem 2.17. *Under the conditions given in Theorem 2.16, the mapping $X \mapsto \langle X \rangle$ is an algebraic closure operator on L .*

Proof. Let $X \subseteq L$ and $x \in \langle X \rangle$. Then $x \ll a_1 \oplus \cdots \oplus a_n$, for some $n \in \mathbb{N}$ and $a_1, \dots, a_n \in X$. If we set $Y = \{a_1, \dots, a_n\}$, so Y is a finite subset of X and $x \in \langle Y \rangle$. Hence $\langle X \rangle \subseteq \bigcup \{ \langle Y \rangle : Y \subseteq X \text{ is finite} \}$, completes the proof. ■

By Theorem 2.17 and [9, I. Theorem 5.5], we conclude that

Corollary 2.18. *Under the conditions given in Theorem 2.16, $(\mathcal{HI}(L), \subseteq)$ as an algebraic lattice whose compact elements are precisely those $\langle X \rangle$, where X is a finite subset of L .*

Theorem 2.19. *Under the conditions of Theorem 2.16, there exists a set A and an algebraic closure operator C on A such that $\mathcal{HI}(L)$ is order-isomorphic with $\mathcal{HI}(L)_C$.*

Proof. Let $A = \{ \langle X \rangle \in \mathcal{HI}(L) : X \text{ is a finite subset of } A \}$. For $X \subseteq A$ we let $C(X) = \{ J \in A : J \subseteq \bigvee X \}$. It is easy to verify that C is a closure operator and that the mapping $J \mapsto J^\downarrow$ is the desired order-isomorphism. ■

Remark 2.20. We will try to illustrate Theorem 2.16. Consider the hyper residuated lattice $(L = [0, 1], \vee, \wedge, \odot, \rightarrow)$ in which $[0, 1]$ is the real unit interval and the hyperoperations are defined as follows (see [3]):

$$a \odot b = a \wedge b = \min\{a, b\} \quad a \vee b = b \vee a = \begin{cases} L & a = b \\ L \setminus \{a\} & a < b \\ L \setminus \{b\} & b < a \end{cases}$$

$$a \rightarrow b = \begin{cases} 1 & a \leq b \\ b & a > b. \end{cases}$$

It is easy to verify that the hyperoperation \oplus is associative and also $SC_{\oplus}(L) = L$. Now, for $X = \{a\}$ (with $a \neq 0$) we have

$$\begin{aligned} \langle X \rangle &= \{x \in L : x \ll na, n \in \mathbb{N}\} \\ &= \{x \in L : x \ll a\} \cup \{x \in L : x \ll a \oplus a\} \cup \dots \\ &= [0, a] \cup [0, 1] \cup \dots = [0, 1]. \end{aligned}$$

Obviously, for $a = 0$ we have $\langle 0 \rangle = \{0\}$. This result is expected, because the only hyper RL-ideals of L are $\{0\}$ and L . In fact down-sets in L are sub-intervals $[0, a]$, for $a \in [0, 1]$. Routine calculations show that the only sub-interval of $[0, 1]$ which is closed with respect to the hyperoperation \oplus is $[0, 1]$.

3. NODAL HYPER RL-IDEALS

We recall that a node in a poset is an element which is comparable with every elements. So, in a bounded poset with 0 and 1 as the least element and the greatest element, 0 and 1 are always nodes of P .

Definition 3.1. In a lattice L , an element a is said to be distributive if for all $x, y \in L$ we have $a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y)$.

Proposition 3.2 [1]. *In a lattice, every node is distributive.*

Definition 3.3. A *nodal hyper RL-ideal* (*nodal strong hyper RL-ideal*) is a node of $\mathcal{HI}(L)$ (resp. $\mathcal{SHI}(L)$).

By $\mathcal{NHI}(L)$ ($\mathcal{NSHI}(L)$), we mean the set of all nodal hyper RL-ideals (nodal strong hyper RL-ideals) of L .

Example 3.4. From Example 2.3, we know that the singleton $\{0\}$ is a nodal hyper RL-ideal in every hyper residuated lattice. Also, the hyper RL-ideal J in Example 2.6 is a nodal hyper RL-ideal. In Example 2.5, the singleton $\{0\}$ is a nodal strong hyper RL-ideal, while the strong hyper RL-ideal $\{0, c\}$ is not nodal. Moreover, the hyper RL-ideals I and K in the hyper residuated lattice L given in Example 2.6 are not nodal.

Proposition 3.5. *If strong hyper RL-ideal I of L is such that for every $x \in I$ and $y \in L \setminus I$, $x \ll ny$, for some $n \in \mathbb{N}$, then I is a nodal strong hyper RL-ideal of L . Particularly, if L is linearly ordered, every strong hyper RL-ideal is a nodal strong hyper RL-ideal.*

Proof. Assume that J is a strong hyper RL-ideal of L such that $J \not\subseteq I$ and $x \in I$. Then, for $y \in J \setminus I$ we have $x \ll ny$, and since $ny \subseteq J$, we conclude that $x \in J$. This shows that $I \subseteq J$.

Now, if L is linearly ordered and I is a strong hyper RL-ideal of L , for every $x \in I$ and $y \in L \setminus I$ we have $x < y$, because x is a node. Hence, I is a nodal strong hyper RL-ideal of L . ■

Similarly, for hyper RL-ideals we have the next proposition with a bit modification.

Proposition 3.6. *Assume that I is a hyper RL-ideal of L such that $L \setminus I \subseteq SC_{\oplus}(L)$. If for every $x \in I$ and $y \in L \setminus I$ we have $x \ll ny$, for some $n \in \mathbb{N}$, then I is a nodal hyper RL-ideal. Particularly, if L is linearly ordered, every hyper RL-ideal is a nodal hyper RL-ideal.*

Proposition 3.7. *Under the conditions given in Theorem 2.16, if I is a nodal hyper RL-ideal L , then for any $x \in I$ and $y \in L \setminus I$ we have $x \ll ny$, for some $n \in \mathbb{N}$.*

Proof. Let I be a nodal hyper RL-ideal of L , and $x \in I$ and $y \in L \setminus I$. Then $\langle x \rangle \subseteq I \subseteq \langle y \rangle$, whence $x \in \langle y \rangle$. Hence, for some $n \in \mathbb{N}$ we have $x \ll ny$. ■

Proposition 3.8. *If X is a set of nodal elements of L , then $\langle X \rangle_s$ is a nodal strong hyper RL-ideal of L . Particularly, if x is a node, $\langle x \rangle_s$ is a nodal strong hyper RL-ideal of L . A similar result holds for hyper RL-ideals.*

Proof. Let I be a strong hyper RL-ideal of L . If $X \subseteq I$, clearly $\langle X \rangle_s \subseteq I$. Otherwise, there exists $x \in X$ such that $x \notin I$, whence $i < x$, for all $i \in I$. By (2.1) we get that $i \in \langle x \rangle_s \subseteq \langle X \rangle_s$, whence $I \subseteq \langle X \rangle_s$.

The proof for hyper RL-ideals follows from (2.1). ■

Remark 3.9. We observe that $\mathcal{NHI}(L)$ (resp. $\mathcal{NSHI}(L)$) if is nonempty) together with set-inclusion as the partial ordering is a bounded chain which is closed with respect to the intersection and the union. Hence, by Proposition 3.2 and the observations just before in Definition 2.13 we get the next theorem.

Theorem 3.10. *$(\mathcal{NHI}(L), \subseteq)$ is a complete Brouwerian sublattice of $(\mathcal{HI}(L), \subseteq)$, which is itself a chain. Similarly, if L contains strong hyper RL-ideal, $(\mathcal{NSHI}(L), \subseteq)$ is a complete Brouwerian sublattice of $(\mathcal{SHI}(L), \subseteq)$, which is itself a chain.*

It is well-known that Heyting lattices are exactly complete Brouwerian lattices (see [2]). Hence, we have

Corollary 3.11. *The structure $(\mathcal{NH}\mathcal{I}(L), \cap, \cup, \rightarrow, \{0\}, L)$ is a Heyting algebra, where the operation \rightarrow is defined as $I \rightarrow J = \cup \{K \in \mathcal{NH}\mathcal{I}(L) : I \cap J \subseteq K\}$. More closely, the set of nodal strong hyper RL-ideals of L , if is nonempty, together with the same operations forms a Heyting subalgebra of $(\mathcal{NH}\mathcal{I}(L), \cap, \cup, \rightarrow, \{0\}, L)$.*

Theorem 3.12. *In a hyper residuated lattice, any proper hyper RL-ideal is contained in a maximal hyper RL-ideal.*

Proof. Let I be a proper hyper RL-ideal of L and S be the collection of all hyper RL-ideals of L containing I . Obviously, $S \neq \emptyset$. Now, let C be a chain in S . Clearly, $\bigcup C$ is a proper hyper RL-ideal containing I and so it belongs to S . So, by Zorn's lemma, S have a maximal element such as M . Now, we shall show that M is a maximal hyper RL-ideal of L . Assume that J is a proper hyper RL-ideal of L such that $M \subseteq J$. Then $J \in S$ and so we have $J \subseteq M$, by maximality. Thus M is maximal hyper RL-ideal of L . ■

Corollary 3.13. *Every hyper residuated lattice contain a maximal hyper RL-ideal.*

4. CONCLUSIONS

Hyper residuated lattices are a generalization of residuated lattices to hyperstructures. Their study is interesting for algebraic and logical reasons, especially from ideal theory point of view. In this paper, we introduced some types of hyper ideals such as hyper RL-ideals, strong hyper RL-ideals and nodal (strong) hyper RL-ideals and investigated their properties and characterizations of them. Particularly, we studied the lattice structure of them and proved that the lattice of (strong) hyper RL-ideals is complete and the lattice of nodal (strong) hyper RL-ideals is a Heyting lattice.

There are still many topics to study such as some other types of hyper RL-ideals and their structure, category of these hyper RL-ideals, applying the fuzzy set theory to hyper residuated lattices and introducing types of fuzzy hyper RL-ideals.

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