Discussiones Mathematicae
General Algebra and Applications 41 (2021) 411–417
https://doi.org/10.7151/dmgaa.1375

REVISITING THE REPRESENTATION THEOREM OF FINITE DISTRIBUTIVE LATTICES WITH PRINCIPAL CONGRUENCES. A *PROOF-BY-PICTURE* APPROACH

G. Grätzer and H. Lakser

Department of Mathematics
University of Manitoba
Winnipeg, MB R3T 2N2, Canada

e-mail: gratzer@me.com
hlakser@gmail.com

Abstract

A classical result of R.P. Dilworth states that every finite distributive lattice D can be represented as the congruence lattice of a finite lattice L. A sharper form was published in G. Grätzer and E.T. Schmidt in 1962, adding the requirement that all congruences in L be principal. Another variant, published in 1998 by the authors and E.T. Schmidt, constructs a planar semimodular lattice L. In this paper, we merge these two results: we construct L as a planar semimodular lattice in which all congruences are principal. This paper relies on the techniques developed by the authors and E.T. Schmidt in the 1998 paper.

Keywords: principal congruence, finite distributive lattice.

2010 Mathematics Subject Classification: 06B10.

1. Introduction

Let us start with the classical result of Dilworth from 1942 (see the book [1] for background information).

Theorem 1. *Every finite distributive lattice D can be represented as the congruence lattice of a finite lattice L.*

A sharper form was published in Grätzer and Schmidt [10] (see also Theorem 8.5 in [5]). The new idea was the use of standard ideals, see Grätzer [2] and Grätzer and Schmidt [9].

Theorem 2. *Every finite distributive lattice D can be represented as the congruence lattice of a finite relatively complemented lattice L.*

All congruences are principal in a finite relatively complemented lattice L. So we obtain the following variant of Theorem 2.

Theorem 3. *Every finite distributive lattice D can be represented as the congruence lattice of a finite lattice L in which all congruences are principal.*

Grätzer, Lakser and Schmidt [8] proved another variant of Theorem 2.

Theorem 4. *Let D be a finite distributive lattice. Then there exists a planar semimodular lattice L with $\operatorname{Con} L$ isomorphic to D.*

In this note, we combine Theorem 3 and 4, using the techniques developed for Theorem 4.

Theorem 5. *Every finite distributive lattice D can be represented as the congruence lattice of a planar semimodular lattice L in which all congruences are principal.*

There are other aspects of these constructions discussed in the book [5], for instance, the size of L. The constructions in Theorems 1, 2, and 5 are "large" (exponential), in Theorem 4 they are small (cubic polynomial).

There are related results in Grätzer and Lakser [6] and [7].

Outline

For a formal proof of Theorem 5, we need the formal proof of Theorem 4, as presented in Grätzer, Lakser and Schmidt [8]. There are two obvious solutions: copy the formal proof from [8] (making the editor unhappy) or require that the reader be familiar with the paper [8] (making the reader unhappy). So we choose the middle ground, we present a *Proof-by-Picture* (as defined in [5]) of Theorem 4. We do this in Section 2 and complete the proof of Theorem 5 in Section 3.

Notation

We use the notation as in [5].

In particular, for the ordered sets P and Q, we can form the (ordinal) sum, $P + Q$ and the glued sum $P \dotplus Q$, as illustrated in Figure 1. Observe that the glued sum $P \dotplus Q$ requires that P has a unit and Q has a zero (which are identified).

Coloring of a finite lattice L attaches a join-irreducible congruence to an edge (covering interval) of L generating it, see Figures 2–4 for examples.

2. Proof-by-Picture of Theorem 4

We start constructing the planar semimodular lattice L of Theorem 5 for the distributive lattice D and the ordered set $P = J(D)$ of Figure 2, with the three lattices, the planar semimodular lattices N (for Nondistributive), S (for Square), and R (for Rectangle). We glue them together and add some covering M_3-s, to obtain L, as sketched in Figure 5.

In Steps 1–4, we assume that P has no *isolated elements*, that is, for every $x \in P$, there is a $y \in P$ with $x < y$ or $y < x$.

Step 1. *Constructing N.* Take the eight-element, planar, semimodular lattice S_8 of Figure 3. We take three copies, $S_8(a, b)$, $S_8(b, c)$, $S_8(d, c)$, one for every covering pair in $P = J(D)$. Let $E = \mathsf{C}_2 \times \mathsf{C}_3$. We glue these together (preserving the colors!) as in Figure 4. More precisely, we glue $S_8(b, c)$ to E, and glue $S_8(d, c)$ to the top left boundary of E. Then we glue D to this lattice twice and glue $S_8(a, b)$ to the top. We denote by N_1 and N_2 the lower right and the upper right boundaries of N, respectively.

Step 2. *Constructing S.* We form N_2^2. In every covering square of the main vertical diagonal, we add an element to make it an M_3, forming the lattice S, see Figure 4. We denote by S_1 and S_2 the lower left and lower right boundaries of S, respectively. This will make a copy of the colors b and c in S_2, making them available for the M_3 insertions in Step 4b.

Step 3. *Constructing R.* Let the chain C_1 be isomorphic to $N_1 \dotplus S_1$. We choose a chain C of length four and color the edges with $\{a, b, c, d\}$ (in any order). Define $R = C \times C_1$. We denote by R_1, R_2, and R_1' the lower right, lower left, and upper left boundaries of R, respectively.

Step 4. *Constructing L.*

Step 4a. *Gluing N, S, and R.* We glue N and S by identifying N_2 with S_2 (preserving colors!); we call this lattice L_1. Then we glue L_1 and R by identifying R_1' with the lower right boundary of L_1 (preserving colors!); let L_2 be the lattice we obtain.

Step 4b. *Adding M_3-s to L_2.* Every color x occurs in $N_1 \dotplus S_1 = R_1'$ as the color of an edge. If x is not a maximal element in P, then x occurs in N_1 as the color of an edge (maybe many times). If x is a maximal element in P, then x occurs in S_1 as the color of an edge (maybe many times), so x occurs in S_2 as the color of an edge, and therefore also in R_1'.

So in the grid R, we take a "covering row" and a "covering column" hitting R_1' and R_2 in edges of color x, see Figure 5. They determine a covering square to which we add an element to obtain an M_3. We do this for all covering squares given by a covering row and a covering column both colored by x, thereby identifying all the principal congruences determined by a prime interval colored by x.

We repeat this for every color x.

The $S_8(u, v)$ sublattices then determine the desired order on the join-irreducible congruences—see Figure 3.

Step 5. *Adding the tail.* If there are $k > 0$ isolated elements, we form $\mathsf{C}_{k-1} \dot{+} L$; the tail is C_{k-1}.

This completes the *Proof-by-Picture* of Theorem 4.

3. Proving Theorem 5

We have to modify the construction of the planar semimodular lattice L of Section 2 to make all congruences principal. In Step 3, we choose a chain C of length four. Observe that the proof of Theorem 4 remains valid as long as every color is represented as the coloring of C.

Now we change the definition of C. For every $x \in D$, define

$$r(x) = \{\, a \in J(D) \mid x \leq a \,\},$$

and let C_x be a chain of $|r(x)| + 1$ elements, colored by the elements of $r(x)$ (in any order). Let $0_x, 1_x$ denote the bounds of C_x. Let C be the glued sum of the chains C_x for $x \in D$ (in any order). This chain C obviously satisfies the condition that every color is represented as the color of an edge in C.

Therefore, the lattice L constructed in Section 2 satisfies the requirements of Theorem 4. We only have to observe that all congruences are principal.

Let $\boldsymbol{\alpha}$ be a congruence of L. Let x be an element of D that corresponds to $\boldsymbol{\alpha}$ under an isomorphism between $\operatorname{Con} L$ and D. Since C_x is colored by the set $r(x)$, we conclude that in L, we have

$$\operatorname{con}(0_x, 1_x) = \boldsymbol{\alpha},$$

completing the proof.

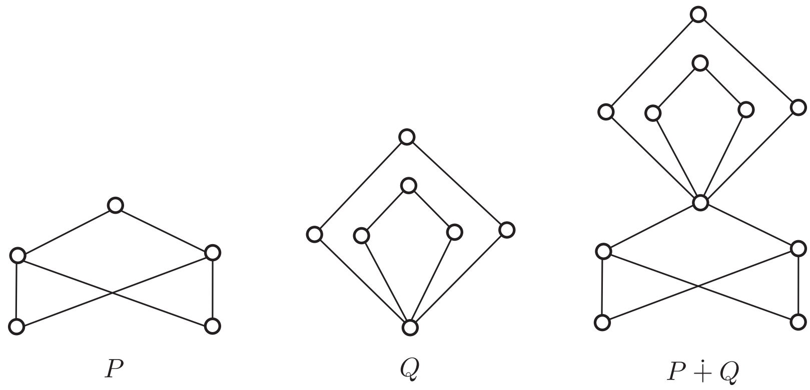


Figure 1. Glued sum of two ordered sets, P and Q.

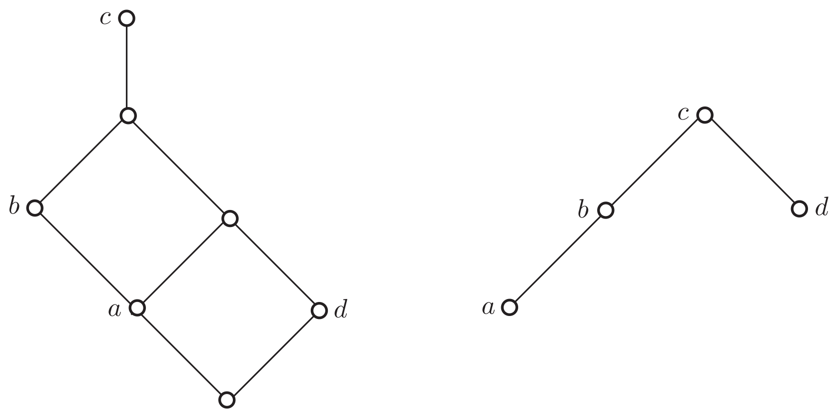


Figure 2. The lattice D to represent and the ordered set $P = J(D)$.

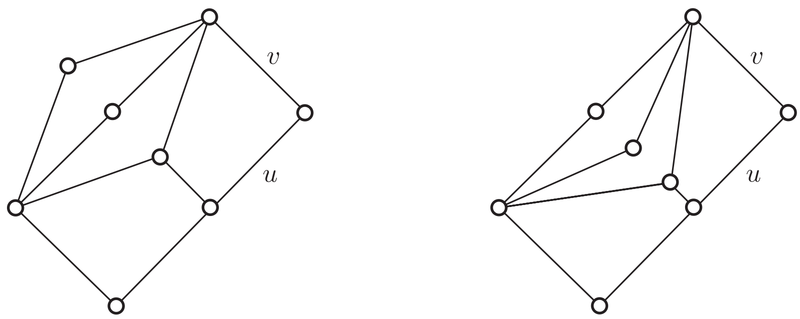
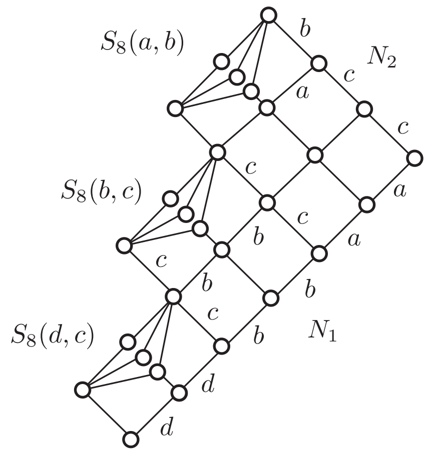


Figure 3. Two diagrams of the building block $S_8(u, v)$, $u \prec v$.



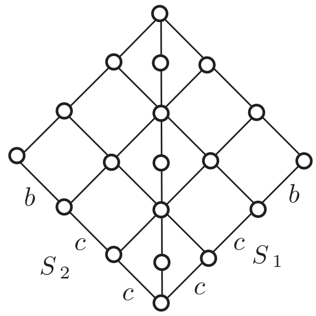


Figure 4. The lattices N and S.

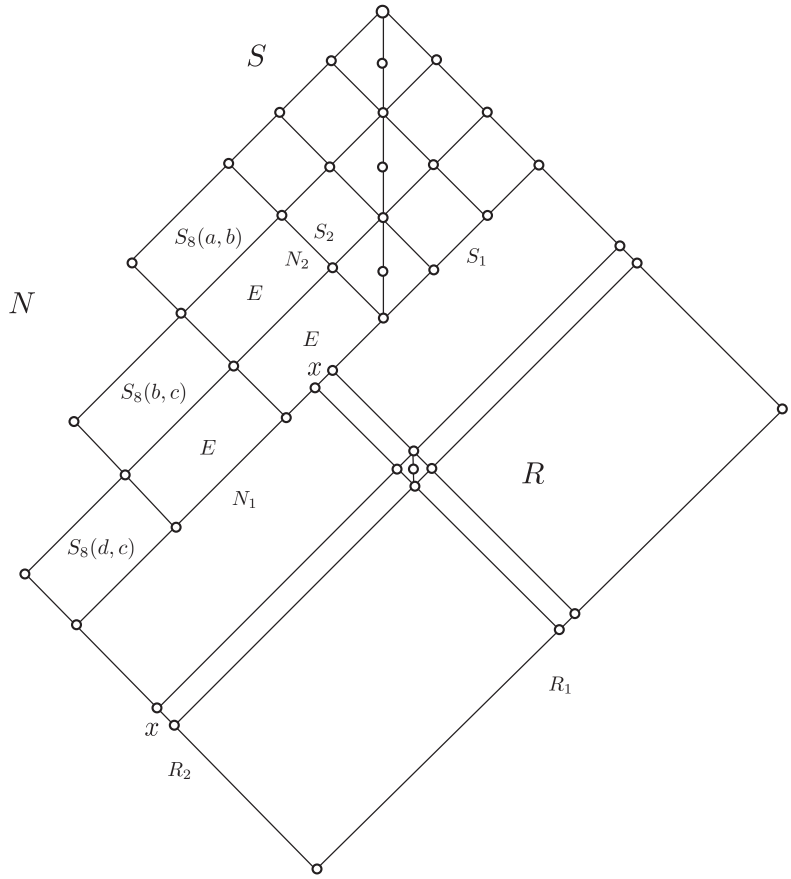


Figure 5. A sketch of L without the "tail".

References

[1] The Dilworth Theorems, Selected papers of Robert P. Dilworth. Edited by Kenneth P. Bogart, Ralph Freese, and Joseph P.S. Kung, Contemporary Mathematicians (Birkhäuser Boston, Inc., Boston, MA, 1990).
https://doi.org/10.1016/0001-8708(92)90026-h

[2] G. Grätzer, *Standard ideals*, Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **9** (1959) 81–97 (Hungarian).

[3] G. Grätzer, Lattice Theory: Foundation (Birkhäuser Verlag, Basel, 2011).

[4] G. Grätzer, *The order of principal congruences of a bounded lattice*, Algebra Univ. **70** (2013) 95–105.
https://doi.org/10.1007/s00012-013-0242-3

[5] G. Grätzer, The Congruences of a Finite Lattice, A "Proof-by-Picture" Approach, Second edition (Birkhäuser Verlag, Basel, 2016).
https://doi.org/10.1007/0-817

[6] G. Grätzer and H. Lakser, *Some preliminary results on the set of principal congruences of a finite lattice*, Algebra Univ. **79** (2018), paper no. 21.
https://doi.org/10.1007/s00012-018-0487-y

[7] G. Grätzer and H. Lakser, *Minimal representations of a finite distributive lattice by principal congruences of a lattice*, Acta Sci. Math. (Szeged) **85** (2019) 69–96.
https://doi.org/10.14232/actasm-017-060-9

[8] G. Grätzer, H. Lakser and E.T. Schmidt, *Congruence lattices of finite semimodular lattices*, Canad. Math. Bull. **41** (1998) 290–297.
https://doi.org/10.4153/cmb-1998-041-7

[9] G. Grätzer and E.T. Schmidt, *Standard ideals in lattices*, Acta Math. Acad. Sci. Hungar. **12** (1961) 17–86.

[10] G. Grätzer and E.T. Schmidt, *On congruence lattices of lattices*, Acta Math. Acad. Sci. Hungar. **13** (1962) 179–185.

Received 14 June 2020
Revised 31 October 2020
Accepted 31 October 2020