Discussiones Mathematicae
General Algebra and Applications 41 (2021) 381–394
https://doi.org/10.7151/dmgaa.1368

ON SHEFFER STROKE UP-ALGEBRAS

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Abstract

In this paper, we introduce Sheffer Stroke UP-algebra (in short, SUP-algebra) and study its properties. We demonstrate that the Cartesian product of two SUP-algebras is a SUP-algebra. After presenting SUP-subalgebras, we define SUP-homomorphisms between SUP-algebras.

Keywords: SUP-algebra, Sheffer stroke operation, SUP-homomorphism.

2010 Mathematics Subject Classification: 06F05, 03G2, 03G10.

1. Introduction

The logical algebras (or algebras of logic) such as BCK-algebras, BCI-algebras, BCH-algebras, KU-algebras, and others, have a significant place among many algebraic structures. In 2017, Iampan introduced UP-algebras, and studied relationships between KU-algebras and UP-algebras, and as well as the notions of ideals, subalgebras, a partially ordered relation, congruences, filters, homomorphisms on this logical algebra. In fact, he showed that an UP-algebra is a generalization of KU-algebra. Again, he proved that an UP-ideal of an UP-algebra is an UP-subalgebra of this algebra and that an arbitrary intersection of UP-ideals of an UP-algebra is a UP-ideal of this algebra. Moreover, he studied the quotient set of an UP-algebra by a congruence on it and some properties about this congruence relation. By introducing UP-homomorphisms between UP-algebras, he gave some their features in his article [5]. After about a year, Romano intensively studied the notions of UP-ideals, UP-filters, a congruence relation on UP-algebras and their various properties [11–13]. Besides, in 2018, he presented UP-algebras with apartness which is a new concept in his preprint [14]. Also, he introduced the concept of pseudo-UP algebras [15] and examined homomorphisms on these structures [16]. Because the notion of UP-algebras is new, we continue to study many other concepts and features about this new notion.

On the other side, Sheffer originally introduced the Sheffer stroke operation [17]. This operation raise many scientists' curiosity because every Boolean axiom or operation can be expressed by means of this binary operation [6]. It leads to reductions of axioms or formulas for many algebraic structures. So, many researchers wish to apply such reduction to several algebraic structures such as ortholattices [2], orthoimplication algebras [1], filters of strong Sheffer stroke non-associative MV-algebras [7], Sheffer stroke Hilbert algebras [8], (fuzzy) filters of Sheffer stroke BL-algebras [9] and Sheffer stroke BG-algebras [10]. These reductions are convenient to many studies in logic and related areas because a system containing only the Sheffer stroke operation is complete (completeness of a logical system).

We first introduce fundamental definitions and notions of Sheffer Stroke UP-algebras. We demonstrate that a Sheffer Stroke UP-algebra is a UP-algebra with $x \cdot y := (y|(x|x))|(y|(x|x))$, and then prove that a Cartesian product of two SUP-algebras is a SUP-algebra. After defining a SUP-subalgebra of SUP-algebra, we describe SUP-homomorphism between SUP-algebras.

2. Preliminaries

In this section, fundamental definitions and notions about Sheffer stroke operation and UP-algebras are provided.

Definition 1 [6]**.** Let $\mathcal{A} = \langle A, | \rangle$ be a groupoid. The operation $|$ is said to be a *Sheffer stroke operation* if it satisfies the following conditions:

$(S1)$ $x|y = y|x$,

$(S2)$ $(x|x)|(x|y) = x$,

$(S3)$ $x|((y|z)|(y|z)) = ((x|y)|(x|y))|z$,

$(S4)$ $(x|((x|x)|(y|y)))|(x|((x|x)|(y|y))) = x$.

Definition 2 [5]**.** An algebra $\mathcal{A} = \langle A; \cdot, 0\rangle$ of type $(2, 0)$ is called a UP-algebra if it satisfies the following conditions for all $x, y, z \in A$:

$(UP-1) : (y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0$,
$(UP-2) : 0 \cdot x = x$,
$(UP-3) : x \cdot 0 = 0$, and
$(UP-4) : x \cdot y = y \cdot x = 0$ implies $x = y$.

Definition 3 [5]**.** Let $\mathcal{A} = \langle A; \cdot, 0\rangle$ be a UP-algebra. Then the binary relation $\leq$ defined by $x \leq y$ if and only if $x \cdot y = 0$ is a partial order on A.

Definition 4 [5]**.** Let $\mathcal{A} = \langle A; \cdot, 0\rangle$ be a UP-algebra. A subset B of A is called a UP-subalgebra of A if the fixed element 0 of A is in B and $\langle B; \cdot, 0\rangle$ forms a UP-algebras. Clearly, A and $\{0\}$ are UP-subalgebras of A.

Definition 5 [5]**.** Let $\mathcal{A} = \langle A; \cdot_A, 0_A\rangle$ and $\mathcal{B} = \langle B; \cdot_B, 0_B\rangle$ be UP-algebras. A mapping f from A to B is called a UP-homomorphism if $f(x \cdot_A y) = f(x) \cdot_B f(y)$ for all $x, y \in A$. Clearly, $f(0_A) = 0_B$.

3. The Sheffer Stroke UP-Algebras

In this section, we define Sheffer Stroke UP-algebra and give some notions about it.

Definition 6. A Sheffer stroke UP-algebra (briefly, SUP-algebras) is a structure $\langle A, |\rangle$ of type (2) such that 0 is the fixed element in A and the following conditions are satisfied for all $x, y, z \in A$:

$(SUP-1)$ $(((z|(x|x))|(z|(x|x)))|(((y|(x|x))|(z|(y|y)))|$
$((y|(x|x))|(z|(y|y)))))|(((z|(x|x))|(z|(x|x)))$
$|(((y|(x|x))|(z|(y|y)))|((y|(x|x))|(z|(y|y))))) = 0$,
$(SUP-2)$ $x|x = x|(0|0)$, and
$(SUP-3)$ $(x|(y|y))|(x|(y|y)) = 0$ and $(y|(x|x))|(y|(x|x)) = 0$ imply $x = y$.

Example 1. Given a structure $\langle A, |\rangle$, where $A\{0, a, b, c\}$ with Cayley table as below:

$\vert$	0	a	b	c
0	a	a	a	a
a	a	0	c	b
b	a	c	c	a
c	a	b	a	b

It is easy to show that this structure is a SUP-algebra.

Lemma 2. *In a SUP-algebra $\langle A, |\rangle$, the following hold for all $x, y, z \in A$:*

(1) $(x|(x|x))|(x|(x|x)) = 0$.

(2) $(y|(x|x))|(y|(x|x)) = 0$ *and* $(z|(y|y))|(z|(y|y)) = 0$ *imply* $(z|(x|x))|(z|(x|x)) = 0$.

(3) $(y|(x|x))|(y|(x|x)) = 0$ *imply* $(((y|(z|z))|(y|(z|z)))|(x|(z|z)))|(((y|(z|z))|(y|(z|z)))|(x|(z|z))) = 0$.

(4) $(y|(x|x))|(y|(x|x)) = 0$ *imply* $(((z|(x|x))|(z|(x|x)))|(z|(y|y)))|(((z|(x|x))|(z|(x|x)))|(z|(y|y))) = 0$.

(5) $(((x|(y|y))|(x|(y|y)))|(x|x))|(((x|(y|y))|(x|(y|y)))|(x|x)) = 0$.

(6) $x|(x|(y|y)) = 0|0$ *if and only if* $x|x = x|(y|y)$.

(7) $(((y|(y|y))|(y|(y|y)))|(x|x))|(((y|(y|y))|(y|(y|y)))|(x|x)) = 0$.

(8) $x|0 = 0|0$.

Proof. (1) Substituting, simultaneously, 0 instead of x and y, and x instead of z in $(SUP-1)$, we conclude

$$\begin{aligned}
0 &= (((x|(0|0))|(x|(0|0)))|(((0|(0|0))|(x|(0|0)))| \\
&\quad ((0|(0|0))|(x|(0|0)))))|(((x|(0|0))|(x|(0|0)))| \\
&\quad (((0|(0|0))|(x|(0|0)))|((0|(0|0))|(x|(0|0))))) \\
&= (((x|x)|(x|x))|(((0|0)|(x|x))|((0|0)|(x|x))))| \\
&\quad (((x|x)|(x|x))|(((0|0)|(x|x))|((0|0)|(x|x)))) && (SUP-2) \\
&= (((x|(x|x))|(x|(x|x)))|(0|0))| \\
&\quad (((x|(x|x))|(x|(x|x)))|(0|0)) && ((S1)-(S3)) \\
&= (((x|(x|x))|(x|(x|x)))|((x|(x|x))|(x|(x|x))))| \\
&\quad (((x|(x|x))|(x|(x|x)))|((x|(x|x))|(x|(x|x)))) && (SUP-2) \\
&= (x|(x|x))|(x|(x|x)), && (S2)
\end{aligned}$$

for all $x \in A$.

(2) Let $(y|(x|x))|(y|(x|x)) = 0$ and $(z|(y|y))|(z|(y|y)) = 0$ for all $x, y, z \in A$. We obtain from $(SUP-1)-(SUP-2)$ and $(S2)$ that

$$\begin{aligned}
0 &= (((z|(x|x))|(z|(x|x)))|(((y|(x|x))|(z|(y|y)))|((y|(x|x))|(z|(y|y)))))| \\
&\quad (((z|(x|x))|(z|(x|x)))|(((y|(x|x))|(z|(y|y)))|((y|(x|x))|(z|(y|y))))) \\
&= (((z|(x|x))|(z|(x|x)))|(((((y|(x|x))|(y|(x|x)))|((y|(x|x))|(y|(x|x))))|
\end{aligned}$$

$$
\begin{aligned}
&(((z|(y|y))|(z|(y|y)))|((z|(y|y))|(z|(y|y)))))|((((y|(x|x))|(y|(x|x)))| \\
&((y|(x|x))|(y|(x|x))))|(((z|(y|y))|(z|(y|y)))|((z|(y|y))|(z|(y|y)))))))| \\
&(((z|(x|x))|(z|(x|x)))|(((((y|(x|x))|(y|(x|x)))|((y|(x|x))|(y|(x|x))))| \\
&(((z|(y|y))|(z|(y|y)))|((z|(y|y))|(z|(y|y)))))|((((y|(x|x))|(y|(x|x)))| \\
&((y|(x|x))|(y|(x|x))))|(((z|(y|y))|(z|(y|y)))|((z|(y|y))|(z|(y|y)))))))) \\
={}& (((z|(x|x))|(z|(x|x)))|(((0|0)|(0|0))|((0|0)|(0|0))))| \\
&(((z|(x|x))|(z|(x|x)))|(((0|0)|(0|0))|((0|0)|(0|0)))) \\
={}& (((z|(x|x))|(z|(x|x)))|(0|0))| \\
&(((z|(x|x))|(z|(x|x)))|(0|0)) \\
={}& (((z|(x|x))|(z|(x|x)))|((z|(x|x))|(z|(x|x))))| \\
&(((z|(x|x))|(z|(x|x)))|((z|(x|x))|(z|(x|x)))) \\
={}& (z|(x|x))|(z|(x|x)).
\end{aligned}
$$

(3) Let $(y|(x|x))|(y|(x|x)) = 0$ for all $x, y \in A$. Substituting, simultaneously, z instead of x, x instead of y and y instead of z in $(SUP-1)$, we get from $(SUP-2)$ and $(S2)$ that

$$
\begin{aligned}
0 ={}& (((y|(z|z))|(y|(z|z)))|(((x|(z|z))|(y|(x|x)))|((x|(z|z))|(y|(x|x)))))| \\
&(((y|(z|z))|(y|(z|z)))|(((x|(z|z))|(y|(x|x)))|((x|(z|z))|(y|(x|x))))) \\
={}& (((y|(z|z))|(y|(z|z)))|(((x|(z|z))|(((y|(x|x))|(y|(x|x)))|((y|(x|x))| \\
&(y|(x|x)))))|((x|(z|z))|(((y|(x|x))|(y|(x|x)))|((y|(x|x))|(y|(x|x)))))))| \\
&(((y|(z|z))|(y|(z|z)))|(((x|(z|z))|(((y|(x|x))|(y|(x|x)))|((y|(x|x))| \\
&(y|(x|x)))))|((x|(z|z))|(((y|(x|x))|(y|(x|x)))|((y|(x|x))|(y|(x|x))))))) \\
={}& (((y|(z|z))|(y|(z|z)))|(((x|(z|z))|(0|0))|((x|(z|z))|(0|0))))| \\
&(((y|(z|z))|(y|(z|z)))|(((x|(z|z))|(0|0))|((x|(z|z))|(0|0)))) \\
={}& (((y|(z|z))|(y|(z|z)))|(((x|(z|z))|(x|(z|z)))|((x|(z|z))|(x|(z|z)))))| \\
&(((y|(z|z))|(y|(z|z)))|(((x|(z|z))|(x|(z|z)))|((x|(z|z))|(x|(z|z))))) \\
={}& (((y|(z|z))|(y|(z|z)))|(x|(z|z)))| \\
&(((y|(z|z))|(y|(z|z)))|(x|(z|z))),
\end{aligned}
$$

for all $x, y, z \in A$.

(4) Let $(y|(x|x))|(y|(x|x)) = 0$ for all $x, y \in A$. We obtain from $(SUP-1)-(SUP-2)$ and $(S1)-(S2)$ that

$$
\begin{aligned}
0 ={}& (((z|(x|x))|(z|(x|x)))|(((y|(x|x))|(z|(y|y)))|((y|(x|x))|(z|(y|y)))))| \\
&(((z|(x|x))|(z|(x|x)))|(((y|(x|x))|(z|(y|y)))|((y|(x|x))|(z|(y|y)))))
\end{aligned}
$$

$$
\begin{aligned}
&= (((z|(x|x))|(z|(x|x)))|(((((y|(x|x))|(y|(x|x)))|((y|(x|x))|(y|(x|x))))| \\
&\quad (z|(y|y)))|((((y|(x|x))|(y|(x|x)))|((y|(x|x))|(y|(x|x))))|(z|(y|y)))))| \\
&\quad (((z|(x|x))|(z|(x|x)))|(((((y|(x|x))|(y|(x|x)))|((y|(x|x))|(y|(x|x))))| \\
&\quad (z|(y|y)))|((((y|(x|x))|(y|(x|x)))|((y|(x|x))|(y|(x|x))))|(z|(y|y))))) \\
&= (((z|(x|x))|(z|(x|x)))|(((0|0)|(z|(y|y)))|((0|0)|(z|(y|y)))))| \\
&\quad (((z|(x|x))|(z|(x|x)))|(((0|0)|(z|(y|y)))|((0|0)|(z|(y|y))))) \\
&= (((z|(x|x))|(z|(x|x)))|(((z|(y|y))|(0|0))|((z|(y|y))|(0|0))))| \\
&\quad (((z|(x|x))|(z|(x|x)))|(((z|(y|y))|(0|0))|((z|(y|y))|(0|0)))) \\
&= (((z|(x|x))|(z|(x|x)))|(((z|(y|y))|(z|(y|y)))|((z|(y|y))|(z|(y|y)))))| \\
&\quad (((z|(x|x))|(z|(x|x)))|(((z|(y|y))|(z|(y|y)))|((z|(y|y))|(z|(y|y))))) \\
&= (((z|(x|x))|(z|(x|x)))|(z|(y|y)))| \\
&\quad (((z|(x|x))|(z|(x|x)))|(z|(y|y))).
\end{aligned}
$$

(5) For all $x, y \in A$, we have from $(S1) - (S3)$, (1) and $(SUP-2)$ that

$$
\begin{aligned}
&(((x|(y|y))|(x|(y|y)))|(x|x))|(((x|(y|y))|(x|(y|y)))|(x|x)) \\
&= ((y|y)|((x|(x|x))|(x|(x|x))))|((y|y)|((x|(x|x))|(x|(x|x)))) \\
&= (0|(y|y))|(0|(y|y)) \\
&= (0|(y|(0|0)))|(0|(y|(0|0))) \\
&= (((0|0)|(0|0))|(y|(0|0)))|(((0|0)|(0|0))|(y|(0|0)))(S2) \\
&= 0.
\end{aligned}
$$

(6) Let $x|(x|(y|y)) = 0|0$ for all $x, y \in A$. Then we have

$$
\begin{aligned}
x|(y|y) &= x|(y|(0|0)) && (SUP-2) \\
&= x|(((y|y)|(y|y))|(0|0)) && (S2) \\
&= x|(((y|y)|(0|0))|(0|0)) && (SUP-2) \\
&= x|(((y|y)|(x|(x|(y|y))))|(x|(x|(y|y)))) && (hyp.) \\
&= x|((x|(y|y))|((x|(y|y))|(x|(y|y)))) && ((S3)\ and\ (S1)) \\
&= x|((((x|(y|y))|((x|(y|y))|(x|(y|y))))| \\
&\quad ((x|(y|y))|((x|(y|y))|(x|(y|y)))))| \\
&\quad (((x|(y|y))|((x|(y|y))|(x|(y|y))))| \\
&\quad ((x|(y|y))|((x|(y|y))|(x|(y|y)))))) && (S2) \\
&= x|(0|0) && (1) \\
&= x|x. && (SUP-2)
\end{aligned}
$$

Conversely, assume that $x|x = x|(y|y)$ for all $x, y \in A$. Then we conclude

$$\begin{aligned} x|(x|(y|y)) &= x|(x|x) && (hyp.) \\ &= ((x|(x|x))|(x|(x|x)))|((x|(x|x))|(x|(x|x))) && (S2) \\ &= 0|0. && (1) \end{aligned}$$

(7) For all $x, y \in A$, we obtain

$$\begin{aligned} &(((y|(y|y))|(y|(y|y)))|(x|x))|(((y|(y|y))|(y|(y|y)))|(x|x)) \\ &= (0|(x|x))|(0|(x|x)) && (1) \\ &= (0|(x|(0|0)))|(0|(x|(0|0))) && (SUP-2) \\ &= (((0|0)|(0|0))|((0|0)|x))|(((0|0)|(0|0))|((0|0)|x)) && ((S1)-(S2)) \\ &= 0. && (S2) \end{aligned}$$

(8) For all $x \in A$, we have

$$\begin{aligned} x|0 &= ((x|x)|(x|x))|((0|0)|(0|0)) && (S2) \\ &= ((x|x)|(0|0))|((0|0)|(0|0)) && (SUP-2) \\ &= 0|0. && ((S1) \text{ and } (S2)) \end{aligned}$$

■

Proposition 3. *Let $\langle A, |\rangle$ be a SUP-algebra. Then the binary relation $x \leq y$ if and only if $(y|(x|x))|(y|(x|x)) = 0$ is a partial order on A.*

Lemma 4. *Let $\langle A, |\rangle$ be a SUP-algebra. Then*

1. $x \leq y$ *implies*
   (a) $y|(z|z) \leq x|(z|z)$,
   (b) $z|(x|x) \leq z|(y|y)$,
2. $x \leq y$ *if and only if* $y|y \leq x|x$,
3. $y|(x|x) \leq x$,
4. $y \leq (y|(x|x))|(y|(x|x))$,
5. $x \leq y$ *implies* $x \leq (y|(z|z))|(y|(z|z))$,
6. $z|(y|y) \leq z|(y|(x|x))$,
7. $((z|(y|y))|(z|(y|y)))|(x|x) \leq z|(y|(x|x))$ *and*
8. $x|((y|(z|z))|(y|(z|z))) \leq (x|(y|y))|((x|(z|z))|(x|(z|z)))$,

for all $x, y, z \in A$.

Proof. Let $\langle A, |\rangle$ be a SUP-algebra.

1. Let $x \leq y$. Then
   (a) it follows from Proposition 3, $(S1)$ and Lemma 2(3).
   (b) It is obtained from Proposition 3, $(S1)$ and Lemma 2(4).

2. $$\begin{aligned} x \leq y &\Leftrightarrow (y|(x|x))|(y|(x|x)) = 0 \\ &\Leftrightarrow ((x|x)|((y|y)|(y|y)))|((x|x)|((y|y) \\ &\quad |(y|y))) = (y|(x|x))|(y|(x|x)) = 0 \\ &\Leftrightarrow y|y \leq x|x \end{aligned}$$

   from Proposition 3, $(S1)$ and $(S2)$.

3. $$\begin{aligned} &(x|((y|(x|x))|(y|(x|x))))|(x|((y|(x|x))|(y|(x|x)))) \\ &= (((x|(x|x))|(x|(x|x)))|y)|(((x|(x|x))|(x|(x|x)))|y) \\ &= (0|y)|(0|y) \\ &= (0|0)|(0|0) \\ &= 0 \end{aligned}$$
   from $(S1)$–$(S3)$, Lemma 2 (1) and (8), it follows from Proposition 3 that $y|(x|x) \leq x$.

4. Since $y|(x|x) = (x|x)|((y|y)|(y|y)) \leq y|y$ from $(S1)$, $(S2)$ and (3), it is obtained from (2) and $(S2)$ that $y \leq (y|(x|x))|(y|(x|x))$.

5. Let $x \leq y$. Then $(y|(x|x))|(y|(x|x)) = 0$ by Proposition 3. Thus,
   $$\begin{aligned} &(((y|(z|z))|(y|(z|z)))|(x|x))|(((y|(z|z))|(y|(z|z)))|(x|x)) \\ &= ((z|z)|((y|(x|x))|(y|(x|x))))|((z|z)|((y|(x|x))|(y|(x|x)))) \\ &= ((z|z)|0)|((z|z)|0) \\ &= (0|0)|(0|0) \\ &= 0 \end{aligned}$$
   from $(S1)$–$(S3)$ and Lemma 2(8). Hence, $x \leq (y|(z|z))|(y|(z|z))$ from Proposition 3.

6. Since $y \leq (y|(x|x))|(y|(x|x))$ from (4), it follows from 1(b) and $(S2)$ that $z|(y|y) \leq z|(((y|(x|x))|(y|(x|x)))|((y|(x|x))|(y|(x|x)))) = z|(y|(x|x))$.

7. $$\begin{aligned} ((z|(y|y))|(z|(y|y)))|(x|x) &= (x|x)|((z|(y|y))|(z|(y|y))) \\ &\leq (x|x)|((z|(y|(x|x)))|(z|(y|(x|x)))) \\ &\leq z|(y|(x|x)) \end{aligned}$$

   from $(S1)$, 1(b), (3) and (6).

8. Since $x|(y|y) \leq y$ from (3), it is obtained from (S1), (S3) and 1(a) that $x|((y|(z|z))|(y|(z|z))) = y|((x|(z|z))|(x|(z|z))) \leq (x|(y|y))|((x|(z|z))|(x|(z|z)))$. ■

Theorem 5. *Let $\langle A, |\rangle$ be a SUP-algebra. If we define $x \cdot y := (y|(x|x))|(y|(x|x))$, then $\langle A; \cdot, 0\rangle$ is a UP-algebra.*

Proof. Let x, y, z be any elements in A.

$(UP-1)$: We have

$$\begin{aligned}
&(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) \\
&= (((z|(x|x))|(z|(x|x)))|(((y|(x|x))|(z|(y|y)))|((y|(x|x))|(z|(y|y)))))| \\
&\quad (((z|(x|x))|(z|(x|x)))|(((y|(x|x))|(z|(y|y)))|((y|(x|x))|(z|(y|y))))) \\
&= 0
\end{aligned}$$

from $(S2)$, $(S3)$ and $(SUP-1)$.

$(UP-2)$: It follows from $(SUP-2)$ and $(S2)$ that $0 \cdot x = (x|(0|0))|(x|(0|0)) = (x|x)|(x|x) = x$.

$(UP-3)$: We obtain $x \cdot 0 = (0|(x|x))|(0|(x|x)) = (0|0)|(0|0) = 0$ from Lemma 2(8) and $(S2)$.

$(UP-4)$: Let $x \cdot y = 0$ and $y \cdot x = 0$, i.e., $(y|(x|x))|(y|(x|x)) = 0$ and $(x|(y|y))|(x|(y|y)) = 0$. We get that $x = y$ from $(SUP-3)$. ■

Example 6. Given the SUP-algebra $\langle A, |\rangle$ in Example 1. Then UP-algebra $\langle A; \cdot, 0\rangle$ constructed from this SUP-algebra has the following Cayley table:

$\cdot$	0	a	b	c
0	0	a	b	c
a	0	0	0	0
b	0	c	0	c
c	0	b	b	0

Theorem 7. *Let $\langle A, |_A\rangle$ and $\langle B, |_B\rangle$ be SUP-algebras. Then, $\langle A \times B, |_{A\times B}\rangle$ is a SUP-algebra where the set $A \times B$ is the Cartesian product of A and B and the operation $|_{A\times B}$ on this set is defined by $(a_1, b_1)|_{A\times B}(a_2, b_2) := (a_1|_A a_2, b_1|_B b_2)$, and the fixed element is $0_{A\times B} := (0_A, 0_B)$.*

Proof. Let $(a_1, b_1), (a_2, b_2), (a_3, b_3)$ be arbitrary elements in $A \times B$.

$(SUP-1)$: Since $\langle A, |_A\rangle$ and $\langle B, |_B\rangle$ are SUP-algebras, we have
$((((a_3, b_3)|_{A\times B}((a_1, b_1)|_{A\times B}(a_1, b_1)))|_{A\times B}((a_3, b_3)|_{A\times B}((a_1, b_1)|_{A\times B}$
$(a_1, b_1))))|_{A\times B}((((a_2, b_2)|_{A\times B}((a_1, b_1)|_{A\times B}(a_1, b_1)))|_{A\times B}((a_3, b_3)|_{A\times B}$
$((a_2, b_2)|_{A\times B}(a_2, b_2))))|_{A\times B}(((a_2, b_2)|_{A\times B}((a_1, b_1)|_{A\times B}(a_1, b_1)))|_{A\times B}$
$((a_3, b_3)|_{A\times B}((a_2, b_2)|_{A\times B}(a_2, b_2))))))|_{A\times B}((((a_3, b_3)|_{A\times B}((a_1, b_1)|_{A\times B}$

$(a_1, b_1)))|_{A\times B}((a_3, b_3)|_{A\times B}((a_1, b_1)|_{A\times B}(a_1, b_1))))|_{A\times B}((((a_2, b_2)|_{A\times B}$
$((a_1, b_1)|_{A\times B}(a_1, b_1)))|_{A\times B}((a_3, b_3)|_{A\times B}((a_2, b_2)|_{A\times B}(a_2, b_2))))|_{A\times B}(((a_2,$
$b_2)|_{A\times B}((a_1, b_1)|_{A\times B}(a_1, b_1)))|_{A\times B}((a_3, b_3)|_{A\times B}((a_2, b_2)|_{A\times B}(a_2, b_2))))))$
$= ((((a_3|_A(a_1|_Aa_1))|_A(a_3|_A(a_1|_Aa_1)))|_A(((a_2|_A(a_1|_Aa_1))|_A$
$(a_3|_A(a_2|_Aa_2)))|_A((a_2|_A(a_1|_Aa_1))|_A(a_3|_A(a_2|_Aa_2)))))|_A$
$(((a_3|_A(a_1|_Aa_1))|_A(a_3|_A(a_1|_Aa_1)))|_A(((a_2|_A(a_1|_Aa_1))|_A$
$(a_3|_A(a_2|_Aa_2)))|_A((a_2|_A(a_1|_Aa_1))|_A(a_3|_A(a_2|_Aa_2))))),$
$(((b_3|_B(b_1|_Bb_1))|_B(b_3|_B(b_1|_Bb_1)))|_B(((b_2|_B(b_1|_Bb_1))|_B$
$(b_3|_B(b_2|_Bb_2)))|_B((b_2|_B(b_1|_Bb_1))|_B(b_3|_B(b_2|_Bb_2)))))|_b$
$(((b_3|_B(b_1|_Bb_1))|_B(b_3|_B(b_1|_Bb_1)))|_B(((b_2|_B(b_1|_Bb_1))|_B$
$(b_3|_B(b_2|_Bb_2)))|_B((b_2|_B(b_1|_Bb_1))|_B(b_3|_B(b_2|_Bb_2))))))$
$= (0_A, 0_B)$
$= 0_{A\times B}.$

$(SUP-2)$: Since $\langle A, |_A\rangle$ is a SUP-algebra, we get

$$\begin{aligned} ((a_1, b_1)|_{A\times B}(a_1, b_1)) &= (a_1|_Aa_1, b_1|_Bb_1) \\ &= (a_1|_A(0_A|_A0_A), b_1|_B(0_B|_B0_B)) \\ &= (a_1, b_1)|_{A\times B}((0_A, 0_B)|_{A\times B}(0_A, 0_B)) \\ &= (a_1, b_1)|_{A\times B}(0_{A\times B}|_{A\times B}0_{A\times B}). \end{aligned}$$

$(SUP-3)$: Suppose

$((a_1, b_1)|_{A\times B}((a_2, b_2)|_{A\times B}(a_2, b_2)))|_{A\times B}$
$((a_1, b_1)|_{A\times B}((a_2, b_2)|_{A\times B}(a_2, b_2)))$
$= (0_A, 0_B)$
$= 0_{A\times B}$

and

$((a_2, b_2)|_{A\times B}((a_1, b_1)|_{A\times B}(a_1, b_1)))|_{A\times B}$
$((a_2, b_2)|_{A\times B}((a_1, b_1)|_{A\times B}(a_1, b_1)))$
$= (0_A, 0_B)$
$= 0_{A\times B}.$

Then, we have

$$\begin{aligned} 0_{A\times B} &= (0_A, 0_B) \\ &= ((a_1, b_1)|_{A\times B}((a_2, b_2)|_{A\times B}(a_2, b_2)))|_{A\times B} \\ &\quad ((a_1, b_1)|_{A\times B}((a_2, b_2)|_{A\times B}(a_2, b_2))) \\ &= ((a_1|_A(a_2|_Aa_2))|_A(a_1|_A(a_2|_Aa_2)), (b_1|_B(b_2|_Bb_2))|_B(b_1|_B(b_2|_Bb_2))) \end{aligned}$$

and

$$0_{A\times B} = (0_A, 0_B)$$

$$\begin{aligned} &= ((a_2,b_2)|_{A\times B}((a_1,b_1)|_{A\times B}(a_1,b_1)))|_{A\times B} \\ &\quad ((a_2,b_2)|_{A\times B}((a_1,b_1)|_{A\times B}(a_1,b_1))) \\ &= ((a_2|_A(a_1|_Aa_1))|_A(a_2|_A(a_1|_Aa_1)),(b_2|_B(b_1|_Bb_1))|_B(b_2|_B(b_1|_Bb_1))). \end{aligned}$$

Since $\langle A,|_A\rangle$ and $\langle B,|_B\rangle$ are SUP-algebras, we obtain $a_1 = a_2$ and $b_1 = b_2$. Thus, $(a_1,b_1) = (a_2,b_2)$. ■

Definition 7. Let $\mathcal{A} = \langle A,|\rangle$ be a SUP-algebra. A subset B of A is called a SUP-subalgebra of A if the fixed element 0 of A is in B and $\langle B,|\rangle$ forms a SUP-algebra. Clearly, A and $\{0\}$ are SUP-subalgebras of A.

Example 8. Consider the SUP-algebra A in Example 1. A subset $\{0,a\}$ of A is a SUP-subalgebra of A.

Now, we introduce a definition of homomorphisms on SUP-algebras and present a lemma about it.

Definition 8. Let $\langle A,|_A\rangle$ and $\langle B,|_B\rangle$ be SUP-algebras. A mapping $f : A \longrightarrow B$ is called a SUP-homomorphism if

$$f(x|_Ay) = f(x)|_Bf(y)$$

for all $x,y \in A$.

Lemma 9. *Let $\langle A,|_A\rangle$ and $\langle B,|_B\rangle$ be SUP-algebras, and let the mapping $f : A \longrightarrow B$ be a SUP-homomorphism. Then, $f(A)$ is a SUP-subalgebra of B.*

Proof. Let $\langle A,|_A\rangle$ and $\langle B,|_B\rangle$ be SUP-algebras, and let the mapping $f : A \longrightarrow B$ be a SUP-homomorphism. We show $0_B \in f(A)$. Since f is a SUP-homomorphism, we have

$$\begin{aligned} f(0_A) &= f((0_A|_A(0_A|_A0_A))|_A(0_A|_A(0_A|_A0_A))) \\ &= (f(0_A)|_B(f(0_A)|_Bf(0_A)))|_B(f(0_A)|_B(f(0_A)|_Bf(0_A))) \\ &= 0_B \end{aligned}$$

from Lemma 2(1). Thus, $0_B = f(0_A) \in f(A)$. For arbitrary elements $u,v,w \in f(A)$, there exist $x,y,z \in A$ such that $u = f(x)$, $v = f(y)$ and $z = f(w)$. Then

- We get

$$\begin{aligned} &(((w|_B(u|_Bu))|_B(w|_B(u|_Bu)))|_B(((v|_B(u|_Bu))|_B(w|_B(v|_Bv)))|_B \\ &((v|_B(u|_Bu))|_B(w|_B(v|_Bv)))))|_B(((w|_B(u|_Bu))|_B(w|_B(u|_Bu)))|_B \\ &(((v|_B(u|_Bu))|_B(w|_B(v|_Bv)))|_B((v|_B(u|_Bu))|_B(w|_B(v|_Bv))))) \\ &= (((f(z)|_B(f(x)|_Bf(x)))|_B(f(z)|_B(f(x)|_Bf(x))))|_B(((f(y)|_B \\ &\quad (f(x)|_Bf(x)))|_B(f(z)|_B(f(y)|_Bf(y))))|_B((f(y)|_B(f(x)|_Bf(x))) \\ &\quad |_B(f(z)|_B(f(y)|_Bf(y))))))|_B(((f(z)|_B(f(x)|_Bf(x)))|_B(f(z)|_B \end{aligned}$$

$$
\begin{aligned}
&(f(x)|_B f(x))))|_B(((f(y)|_B(f(x)|_B f(x)))|_B(f(z)|_B(f(y)|_B \\
&f(y))))|_B((f(y)|_B(f(x)|_B f(x)))|_B(f(z)|_B(f(y)|_B f(y)))))) \\
= {} & f((((z|_A(x|_A x))|_A(z|_A(x|_A x)))|_A(((y|_A(x|_A x))|_A(z|_A(y|_A y)))|_A \\
&((y|_A(x|_A x))|_A(z|_A(y|_A y)))))|_A(((z|_A(x|_A x))|_A(z|_A(x|_A x)))|_A \\
&(((y|_A(x|_A x))|_A(z|_A(y|_A y)))|_A((y|_A(x|_A x))|_A(z|_A(y|_A y)))))) \\
= {} & f(0_A) \\
= {} & 0_B.
\end{aligned}
$$

- We have

$$
\begin{aligned}
u|_B u &= f(x)|_B f(x) \\
&= f(x|_A x) \\
&= f(x|_A(0_A|_A 0_A)) \\
&= f(x)|_B(f(0_A)|_B f(0_A)) \\
&= f(x)|_B(0_B|_B 0_B) \\
&= u|_B(0_B|_B 0_B).
\end{aligned}
$$

- Let $(u|_B(v|_B v))|_B(u|_B(v|_B v)) = 0_B$ and $(v|_B(u|_B u))|_B(v|_B(u|_B u)) = 0_B$. Then we obtain $(x|_A(y|_A y))|_A(x|_A(y|_A y)) = 0_A$ and $(y|_A(x|_A x))|_A(y|_A(x|_A x)) = 0_A$ because

$$
\begin{aligned}
0_B &= (u|_B(v|_B v))|_B(u|_B(v|_B v)) \\
&= (f(x)|_B(f(y)|_B f(y)))|_B(f(x)|_B(f(y)|_B f(y))) \\
&= f((x|_A(y|_A y))|_A(x|_A(y|_A y)))
\end{aligned}
$$

and

$$
\begin{aligned}
0_B &= (v|_B(u|_B u))|_B(v|_B(u|_B u)) \\
&= (f(y)|_B(f(x)|_B f(x)))|_B(f(y)|_B(f(x)|_B f(x))) \\
&= f((y|_A(x|_A x))|_A(y|_A(x|_A x))),
\end{aligned}
$$

where f is a SUP-homomorphism. Hence, we get $x = y$ since $\langle A, |_A\rangle$ is a SUP-algebra. Thereby, it follows $u = f(x) = f(y) = v$. Thus $f(A)$ is a SUP-subalgebra of B. ■

Remark 10. The class of all SUP-algebras forms a variety.

4. CONCLUSION

In the present paper, it has been given a SUP-algebra which is the Sheffer stroke reduction of UP-algebras, and examined a Cartesian product, a SUP-subalgebra, a SUP-homomorphism and many properties in SUP-algebras. After introducing Sheffer Stroke UP-algebras and giving their features, it is showed that a Sheffer Stroke UP-algebra is a UP-algebra with $x \cdot y := (y|(x|x))|(y|(x|x))$, and proved that a Cartesian product of two SUP-algebras is a SUP-algebra. By defining a SUP-subalgebra of SUP-algebra, it has been presented a SUP-homomorphism between SUP-algebras.

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Received 28 September 2020
Revised 19 October 2020
Accepted 19 October 2020