

ON SHEFFER STROKE UP-ALGEBRAS

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Abstract

In this paper, we introduce Sheffer Stroke UP-algebra (in short, SUP-algebra) and study its properties. We demonstrate that the Cartesian product of two SUP-algebras is a SUP-algebra. After presenting SUP-subalgebras, we define SUP-homomorphisms between SUP-algebras.

Keywords: SUP-algebra, Sheffer stroke operation, SUP-homomorphism.

2010 Mathematics Subject Classification: 06F05, 03G2, 03G10.

1. INTRODUCTION

The logical algebras (or algebras of logic) such as BCK-algebras, BCI-algebras, BCH-algebras, KU-algebras, and others, have a significant place among many algebraic structures. In 2017, Iampan introduced UP-algebras, and studied relationships between KU-algebras and UP-algebras, and as well as the notions of ideals, subalgebras, a partially ordered relation, congruences, filters, homomorphisms on this logical algebra. In fact, he showed that an UP-algebra is a generalization of KU-algebra. Again, he proved that an UP-ideal of an UP-algebra is

an UP-subalgebra of this algebra and that an arbitrary intersection of UP-ideals of an UP-algebra is a UP-ideal of this algebra. Moreover, he studied the quotient set of an UP-algebra by a congruence on it and some properties about this congruence relation. By introducing UP-homomorphisms between UP-algebras, he gave some their features in his article [5]. After about a year, Romano intensively studied the notions of UP-ideals, UP-filters, a congruence relation on UP-algebras and their various properties [11–13]. Besides, in 2018, he presented UP-algebras with apartness which is a new concept in his preprint [14]. Also, he introduced the concept of pseudo-UP algebras [15] and examined homomorphisms on these structures [16]. Because the notion of UP-algebras is new, we continue to study many other concepts and features about this new notion.

On the other side, Sheffer originally introduced the Sheffer stroke operation [17]. This operation raise many scientists' curiosity because every Boolean axiom or operation can be expressed by means of this binary operation [6]. It leads to reductions of axioms or formulas for many algebraic structures. So, many researchers wish to apply such reduction to several algebraic structures such as ortholattices [2], orthoimplication algebras [1], filters of strong Sheffer stroke non-associative MV-algebras [7], Sheffer stroke Hilbert algebras [8], (fuzzy) filters of Sheffer stroke BL-algebras [9] and Sheffer stroke BG-algebras [10]. These reductions are convenient to many studies in logic and related areas because a system containing only the Sheffer stroke operation is complete (completeness of a logical system).

We first introduce fundamental definitions and notions of Sheffer Stroke UP-algebras. We demonstrate that a Sheffer Stroke UP-algebra is a UP-algebra with $x \cdot y := (y|(x|x))|(y|(x|x))$, and then prove that a Cartesian product of two SUP-algebras is a SUP-algebra. After defining a SUP-subalgebra of SUP-algebra, we describe SUP-homomorphism between SUP-algebras.

2. PRELIMINARIES

In this section, fundamental definitions and notions about Sheffer stroke operation and UP-algebras are provided.

Definition 1 [6]. Let $\mathcal{A} = \langle A, | \rangle$ be a groupoid. The operation $|$ is said to be a *Sheffer stroke operation* if it satisfies the following conditions:

- (S1) $x|y = y|x$,
- (S2) $(x|x)|(x|y) = x$,
- (S3) $x|((y|z)|(y|z)) = ((x|y)|(x|y))|z$,
- (S4) $(x|((x|x)|(y|y))|(x|((x|x)|(y|y)))) = x$.

Definition 2 [5]. An algebra $\mathcal{A} = \langle A; \cdot, 0 \rangle$ of type $(2, 0)$ is called a UP-algebra if it satisfies the following conditions for all $x, y, z \in A$:

- $(UP - 1) : (y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0,$
- $(UP - 2) : 0 \cdot x = x,$
- $(UP - 3) : x \cdot 0 = 0,$ and
- $(UP - 4) : x \cdot y = y \cdot x = 0$ implies $x = y.$

Definition 3 [5]. Let $\mathcal{A} = \langle A; \cdot, 0 \rangle$ be a UP-algebra. Then the binary relation \leq defined by $x \leq y$ if and only if $x \cdot y = 0$ is a partial order on A .

Definition 4 [5]. Let $\mathcal{A} = \langle A; \cdot, 0 \rangle$ be a UP-algebra. A subset B of A is called a UP-subalgebra of A if the fixed element 0 of A is in B and $\langle B; \cdot, 0 \rangle$ forms a UP-algebra. Clearly, A and $\{0\}$ are UP-subalgebras of A .

Definition 5 [5]. Let $\mathcal{A} = \langle A; \cdot_A, 0_A \rangle$ and $\mathcal{B} = \langle B; \cdot_B, 0_B \rangle$ be UP-algebras. A mapping f from A to B is called a UP-homomorphism if $f(x \cdot_A y) = f(x) \cdot_B f(y)$ for all $x, y \in A$. Clearly, $f(0_A) = 0_B$.

3. THE SHEFFER STROKE UP-ALGEBRAS

In this section, we define Sheffer Stroke UP-algebra and give some notions about it.

Definition 6. A Sheffer stroke UP-algebra (briefly, SUP-algebras) is a structure $\langle A, | \rangle$ of type (2) such that 0 is the fixed element in A and the following conditions are satisfied for all $x, y, z \in A$:

- $(SUP - 1) (((z|(x|x))|(z|(x|x)))|(((y|(x|x))|(z|(y|y))|((y|(x|x))|(z|(y|y))))|(((z|(x|x))|(z|(x|x))|(((y|(x|x))|(z|(y|y))|((y|(x|x))|(z|(y|y))))))) = 0,$
- $(SUP - 2) x|x = x|(0|0),$ and
- $(SUP - 3) (x|(y|y))|(x|(y|y)) = 0$ and $(y|(x|x))|(y|(x|x)) = 0$ imply $x = y.$

Example 1. Given a structure $\langle A, | \rangle$, where $A = \{0, a, b, c\}$ with Cayley table as below:

	0	a	b	c
0	a	a	a	a
a	a	0	c	b
b	a	c	c	a
c	a	b	a	b

It is easy to show that this structure is a SUP-algebra.

Lemma 2. *In a SUP-algebra $\langle A, | \rangle$, the following hold for all $x, y, z \in A$:*

- (1) $(x|(x|x))|(x|(x|x)) = 0$.
- (2) $(y|(x|x))|(y|(x|x)) = 0$ and $(z|(y|y))|(z|(y|y)) = 0$ imply $(z|(x|x))|(z|(x|x)) = 0$.
- (3) $(y|(x|x))|(y|(x|x)) = 0$ imply $((y|(z|z))|(y|(z|z))|(x|(z|z))|(((y|(z|z))|(y|(z|z))|(x|(z|z)))) = 0$.
- (4) $(y|(x|x))|(y|(x|x)) = 0$ imply $((z|(x|x))|(z|(x|x))|(z|(y|y))|(((z|(x|x))|(z|(x|x))|(z|(y|y)))) = 0$.
- (5) $((x|(y|y))|(x|(y|y))|(x|x))|(((x|(y|y))|(x|(y|y))|(x|x))) = 0$.
- (6) $x|(x|(y|y)) = 0|0$ if and only if $x|x = x|(y|y)$.
- (7) $((y|(y|y))|(y|(y|y))|(x|x))|(((y|(y|y))|(y|(y|y))|(x|x))) = 0$.
- (8) $x|0 = 0|0$.

Proof. (1) Substituting, simultaneously, 0 instead of x and y , and x instead of z in $(SUP - 1)$, we conclude

$$\begin{aligned}
 0 &= (((x|(0|0))|(x|(0|0))|(((0|(0|0))|(x|(0|0))| \\
 &\quad ((0|(0|0))|(x|(0|0))))|(((x|(0|0))|(x|(0|0))| \\
 &\quad (((0|(0|0))|(x|(0|0))|((0|(0|0))|(x|(0|0)))))) \\
 &= (((x|x)|(x|x))|(((0|0)|(x|x))|((0|0)|(x|x))))| \\
 &\quad (((x|x)|(x|x))|(((0|0)|(x|x))|((0|0)|(x|x)))) \quad (SUP - 2) \\
 &= (((x|(x|x))|(x|(x|x))|(0|0))| \\
 &\quad (((x|(x|x))|(x|(x|x))|(0|0))) \quad ((S1) - (S3)) \\
 &= (((x|(x|x))|(x|(x|x))|((x|(x|x))|(x|(x|x))))| \\
 &\quad (((x|(x|x))|(x|(x|x))|((x|(x|x))|(x|(x|x)))) \quad (SUP - 2) \\
 &= (x|(x|x))|(x|(x|x)), \quad (S2)
 \end{aligned}$$

for all $x \in A$.

- (2) Let $(y|(x|x))|(y|(x|x)) = 0$ and $(z|(y|y))|(z|(y|y)) = 0$ for all $x, y, z \in A$. We obtain from $(SUP - 1) - (SUP - 2)$ and $(S2)$ that

$$\begin{aligned}
 0 &= (((z|(x|x))|(z|(x|x))|(((y|(x|x))|(z|(y|y))|((y|(x|x))|(z|(y|y))))| \\
 &\quad (((z|(x|x))|(z|(x|x))|(((y|(x|x))|(z|(y|y))|((y|(x|x))|(z|(y|y)))))) \\
 &= (((z|(x|x))|(z|(x|x))|(((y|(x|x))|(y|(x|x))|((y|(x|x))|(y|(x|x))))|
 \end{aligned}$$

$$\begin{aligned}
 &(((z|(y|y))|(z|(y|y))|(z|(y|y))|(z|(y|y))))|(((y|(x|x))|(y|(x|x))| \\
 &((y|(x|x))|(y|(x|x))))|(((z|(y|y))|(z|(y|y))|(z|(y|y))|(z|(y|y))))| \\
 &(((z|(x|x))|(z|(x|x))|(((y|(x|x))|(y|(x|x))|(y|(x|x))|(y|(x|x))))| \\
 &(((z|(y|y))|(z|(y|y))|(z|(y|y))|(z|(y|y))))|(((y|(x|x))|(y|(x|x))| \\
 &((y|(x|x))|(y|(x|x))))|(((z|(y|y))|(z|(y|y))|(z|(y|y))|(z|(y|y)))))) \\
 = &(((z|(x|x))|(z|(x|x))|((0|0)|(0|0))|((0|0)|(0|0))))| \\
 &(((z|(x|x))|(z|(x|x))|((0|0)|(0|0))|((0|0)|(0|0)))) \\
 = &(((z|(x|x))|(z|(x|x))|(0|0))| \\
 &(((z|(x|x))|(z|(x|x))|(0|0)) \\
 = &(((z|(x|x))|(z|(x|x))|((z|(x|x))|(z|(x|x))))| \\
 &(((z|(x|x))|(z|(x|x))|((z|(x|x))|(z|(x|x)))) \\
 = &(z|(x|x))|(z|(x|x)).
 \end{aligned}$$

(3) Let $(y|(x|x))|(y|(x|x)) = 0$ for all $x, y \in A$. Substituting, simultaneously, z instead of x , x instead of y and y instead of z in $(SUP - 1)$, we get from $(SUP - 2)$ and $(S2)$ that

$$\begin{aligned}
 0 = &(((y|(z|z))|(y|(z|z))|(((x|(z|z))|(y|(x|x))|(x|(z|z))|(y|(x|x))))| \\
 &(((y|(z|z))|(y|(z|z))|(((x|(z|z))|(y|(x|x))|(x|(z|z))|(y|(x|x)))))) \\
 = &(((y|(z|z))|(y|(z|z))|(((x|(z|z))|(((y|(x|x))|(y|(x|x))|(y|(x|x))| \\
 &(y|(x|x))))|(((x|(z|z))|(((y|(x|x))|(y|(x|x))|(y|(x|x))|(y|(x|x))))))| \\
 &(((y|(z|z))|(y|(z|z))|(((x|(z|z))|(((y|(x|x))|(y|(x|x))|(y|(x|x))| \\
 &(y|(x|x))))|(((x|(z|z))|(((y|(x|x))|(y|(x|x))|(y|(x|x))|(y|(x|x)))))) \\
 = &(((y|(z|z))|(y|(z|z))|(((x|(z|z))|(0|0))|(x|(z|z))|(0|0))))| \\
 &(((y|(z|z))|(y|(z|z))|(((x|(z|z))|(0|0))|(x|(z|z))|(0|0)))) \\
 = &(((y|(z|z))|(y|(z|z))|(((x|(z|z))|(x|(z|z))|(x|(z|z))|(x|(z|z))))| \\
 &(((y|(z|z))|(y|(z|z))|(((x|(z|z))|(x|(z|z))|(x|(z|z))|(x|(z|z)))))) \\
 = &(((y|(z|z))|(y|(z|z))|(x|(z|z))| \\
 &(((y|(z|z))|(y|(z|z))|(x|(z|z))),
 \end{aligned}$$

for all $x, y, z \in A$.

(4) Let $(y|(x|x))|(y|(x|x)) = 0$ for all $x, y \in A$. We obtain from $(SUP - 1) - (SUP - 2)$ and $(S1) - (S2)$ that

$$\begin{aligned}
 0 = &(((z|(x|x))|(z|(x|x))|(((y|(x|x))|(z|(y|y))|(y|(x|x))|(z|(y|y))))| \\
 &(((z|(x|x))|(z|(x|x))|(((y|(x|x))|(z|(y|y))|(y|(x|x))|(z|(y|y))))))
 \end{aligned}$$

$$\begin{aligned}
&= (((z|(x|x))|(z|(x|x))))|(((y|(x|x))|(y|(x|x))|(y|(x|x))|(y|(x|x))))| \\
&\quad (z|(y|y))|(((y|(x|x))|(y|(x|x))|(y|(x|x))|(y|(x|x))))|(z|(y|y))))| \\
&\quad (((z|(x|x))|(z|(x|x))))|(((y|(x|x))|(y|(x|x))|(y|(x|x))|(y|(x|x))))| \\
&\quad (z|(y|y))|(((y|(x|x))|(y|(x|x))|(y|(x|x))|(y|(x|x))))|(z|(y|y)))) \\
&= (((z|(x|x))|(z|(x|x))))|(((0|0)|(z|(y|y))|(0|0)|(z|(y|y))))| \\
&\quad (((z|(x|x))|(z|(x|x))))|(((0|0)|(z|(y|y))|(0|0)|(z|(y|y)))) \\
&= (((z|(x|x))|(z|(x|x))))|(((z|(y|y))|(0|0))|(z|(y|y))|(0|0)))| \\
&\quad (((z|(x|x))|(z|(x|x))))|(((z|(y|y))|(0|0))|(z|(y|y))|(0|0))) \\
&= (((z|(x|x))|(z|(x|x))))|(((z|(y|y))|(z|(y|y))|(z|(y|y))|(z|(y|y))))| \\
&\quad (((z|(x|x))|(z|(x|x))))|(((z|(y|y))|(z|(y|y))|(z|(y|y))|(z|(y|y)))) \\
&= (((z|(x|x))|(z|(x|x))|(z|(y|y))))| \\
&\quad (((z|(x|x))|(z|(x|x))|(z|(y|y)))).
\end{aligned}$$

(5) For all $x, y \in A$, we have from (S1) – (S3), (1) and (SUP – 2) that

$$\begin{aligned}
&(((x|(y|y))|(x|(y|y))|(x|x))|(((x|(y|y))|(x|(y|y))|(x|x)) \\
&= (y|y)|((x|(x|x))|(x|(x|x))))|(y|y)|((x|(x|x))|(x|(x|x)))) \\
&= (0|(y|y))|(0|(y|y)) \\
&= (0|(y|(0|0))|(0|(y|(0|0)))) \\
&= (((0|0)|(0|0))|(y|(0|0))|(((0|0)|(0|0))|(y|(0|0))))(S2) \\
&= 0.
\end{aligned}$$

(6) Let $x|(x|(y|y)) = 0|0$ for all $x, y \in A$. Then we have

$$\begin{aligned}
x|(y|y) &= x|(y|(0|0)) && (SUP - 2) \\
&= x|(((y|y)|(y|y))|(0|0)) && (S2) \\
&= x|(((y|y)|(0|0))|(0|0)) && (SUP - 2) \\
&= x|(((y|y)|(x|(x|(y|y))))|(x|(x|(y|y)))) && (hyp.) \\
&= x|((x|(y|y))|((x|(y|y))|(x|(y|y)))) && ((S3) \text{ and } (S1)) \\
&= x|(((x|(y|y))|(x|(y|y))|(x|(y|y))))| \\
&\quad ((x|(y|y))|(x|(y|y))|(x|(y|y))))| \\
&\quad (((x|(y|y))|(x|(y|y))|(x|(y|y))))| \\
&\quad ((x|(y|y))|(x|(y|y))|(x|(y|y)))) && (S2) \\
&= x|(0|0) && (1) \\
&= x|x. && (SUP - 2)
\end{aligned}$$

Conversely, assume that $x|x = x|(y|y)$ for all $x, y \in A$. Then we conclude

$$\begin{aligned} x|(x|(y|y)) &= x|(x|x) && \text{(hyp.)} \\ &= ((x|(x|x))|(x|(x|x)))|((x|(x|x))|(x|(x|x))) && \text{(S2)} \\ &= 0|0. && \text{(1)} \end{aligned}$$

(7) For all $x, y \in A$, we obtain

$$\begin{aligned} &(((y|(y|y))|(y|(y|y)))|(x|x))|(((y|(y|y))|(y|(y|y)))|(x|x)) \\ &= (0|(x|x))|(0|(x|x)) && \text{(1)} \\ &= (0|(x|(0|0))|(0|(x|(0|0)))) && \text{(SUP - 2)} \\ &= (((0|0)|(0|0))|((0|0)|x))|(((0|0)|(0|0))|((0|0)|x)) && \text{((S1) - (S2))} \\ &= 0. && \text{(S2)} \end{aligned}$$

(8) For all $x \in A$, we have

$$\begin{aligned} x|0 &= ((x|x)|(x|x))|((0|0)|(0|0)) && \text{(S2)} \\ &= ((x|x)|(0|0))|((0|0)|(0|0)) && \text{(SUP - 2)} \\ &= 0|0. && \text{((S1) and (S2))} \end{aligned} \quad \blacksquare$$

Proposition 3. *Let $\langle A, | \rangle$ be a SUP-algebra. Then the binary relation $x \leq y$ if and only if $(y|(x|x))|(y|(x|x)) = 0$ is a partial order on A .*

Lemma 4. *Let $\langle A, | \rangle$ be a SUP-algebra. Then*

1. $x \leq y$ implies
 - (a) $y|(z|z) \leq x|(z|z)$,
 - (b) $z|(x|x) \leq z|(y|y)$,
2. $x \leq y$ if and only if $y|y \leq x|x$,
3. $y|(x|x) \leq x$,
4. $y \leq (y|(x|x))|(y|(x|x))$,
5. $x \leq y$ implies $x \leq (y|(z|z))|(y|(z|z))$,
6. $z|(y|y) \leq z|(y|(x|x))$,
7. $((z|(y|y))|(z|(y|y)))|(x|x) \leq z|(y|(x|x))$ and
8. $x|((y|(z|z))|(y|(z|z))) \leq (x|(y|y))|((x|(z|z))|(x|(z|z)))$,

for all $x, y, z \in A$.

Proof. Let $\langle A, | \rangle$ be a SUP-algebra.

1. Let $x \leq y$. Then

(a) it follows from Proposition 3, (S1) and Lemma 2(3).

(b) It is obtained from Proposition 3, (S1) and Lemma 2(4).

2.

$$\begin{aligned} x \leq y &\Leftrightarrow (y|(x|x))|(y|(x|x)) = 0 \\ &\Leftrightarrow ((x|x)|((y|y)|(y|y))|((x|x)|((y|y)| \\ &\quad |(y|y)))) = (y|(x|x))|(y|(x|x)) = 0 \\ &\Leftrightarrow y|y \leq x|x \end{aligned}$$

from Proposition 3, (S1) and (S2).

$$\begin{aligned} 3. \quad &(x|((y|(x|x))|(y|(x|x))))|(x|((y|(x|x))|(y|(x|x)))) \\ &= (((x|(x|x))|(x|(x|x)))|y)|(((x|(x|x))|(x|(x|x)))|y) \\ &= (0|y)|(0|y) \\ &= (0|0)|(0|0) \\ &= 0 \end{aligned}$$

from (S1)–(S3), Lemma 2 (1) and (8), it follows from Proposition 3 that $y|(x|x) \leq x$.

4. Since $y|(x|x) = (x|x)|((y|y)|(y|y)) \leq y|y$ from (S1), (S2) and (3), it is obtained from (2) and (S2) that $y \leq (y|(x|x))|(y|(x|x))$.

5. Let $x \leq y$. Then $(y|(x|x))|(y|(x|x)) = 0$ by Proposition 3. Thus,

$$\begin{aligned} &(((y|(z|z))|(y|(z|z))|(x|x))|(((y|(z|z))|(y|(z|z))|(x|x)) \\ &= ((z|z)|((y|(x|x))|(y|(x|x))))|((z|z)|((y|(x|x))|(y|(x|x)))) \\ &= ((z|z)|0)|((z|z)|0) \\ &= (0|0)|(0|0) \\ &= 0 \end{aligned}$$

from (S1)–(S3) and Lemma 2(8). Hence, $x \leq (y|(z|z))|(y|(z|z))$ from Proposition 3.

6. Since $y \leq (y|(x|x))|(y|(x|x))$ from (4), it follows from 1(b) and (S2) that $z|(y|y) \leq z|(((y|(x|x))|(y|(x|x))))|((y|(x|x))|(y|(x|x)))) = z|(y|(x|x))$.

7.

$$\begin{aligned} ((z|(y|y))|(z|(y|y))|(x|x)) &= (x|x)|((z|(y|y))|(z|(y|y))) \\ &\leq (x|x)|((z|(y|(x|x))|(z|(y|(x|x)))) \\ &\leq z|(y|(x|x)) \end{aligned}$$

from (S1), 1(b), (3) and (6).

$$\begin{aligned}
& (a_1, b_1)|_{A \times B}((a_3, b_3)|_{A \times B}((a_1, b_1)|_{A \times B}(a_1, b_1)))|_{A \times B}(((a_2, b_2)|_{A \times B} \\
& ((a_1, b_1)|_{A \times B}(a_1, b_1))|_{A \times B}((a_3, b_3)|_{A \times B}((a_2, b_2)|_{A \times B}(a_2, b_2))))|_{A \times B}(((a_2, \\
& b_2)|_{A \times B}((a_1, b_1)|_{A \times B}(a_1, b_1))|_{A \times B}((a_3, b_3)|_{A \times B}((a_2, b_2)|_{A \times B}(a_2, b_2)))))) \\
& = (((a_3|_A(a_1|_A a_1))|_A(a_3|_A(a_1|_A a_1)))|_A(((a_2|_A(a_1|_A a_1))|_A \\
& (a_3|_A(a_2|_A a_2)))|_A((a_2|_A(a_1|_A a_1))|_A(a_3|_A(a_2|_A a_2))))|_A \\
& (((a_3|_A(a_1|_A a_1))|_A(a_3|_A(a_1|_A a_1)))|_A(((a_2|_A(a_1|_A a_1))|_A \\
& (a_3|_A(a_2|_A a_2)))|_A((a_2|_A(a_1|_A a_1))|_A(a_3|_A(a_2|_A a_2))))), \\
& (((b_3|_B(b_1|_B b_1))|_B(b_3|_B(b_1|_B b_1)))|_B(((b_2|_B(b_1|_B b_1))|_B \\
& (b_3|_B(b_2|_B b_2)))|_B((b_2|_B(b_1|_B b_1))|_B(b_3|_B(b_2|_B b_2))))|_B \\
& (((b_3|_B(b_1|_B b_1))|_B(b_3|_B(b_1|_B b_1)))|_B(((b_2|_B(b_1|_B b_1))|_B \\
& (b_3|_B(b_2|_B b_2)))|_B((b_2|_B(b_1|_B b_1))|_B(b_3|_B(b_2|_B b_2)))))) \\
& = (0_A, 0_B) \\
& = 0_{A \times B}.
\end{aligned}$$

(SUP – 2) : Since $\langle A, |_A \rangle$ is a SUP-algebra, we get

$$\begin{aligned}
((a_1, b_1)|_{A \times B}(a_1, b_1)) &= (a_1|_A a_1, b_1|_B b_1) \\
&= (a_1|_A(0_A|_A 0_A), b_1|_B(0_B|_B 0_B)) \\
&= (a_1, b_1)|_{A \times B}((0_A, 0_B)|_{A \times B}(0_A, 0_B)) \\
&= (a_1, b_1)|_{A \times B}(0_{A \times B}|_{A \times B} 0_{A \times B}).
\end{aligned}$$

(SUP – 3) : Suppose

$$\begin{aligned}
& ((a_1, b_1)|_{A \times B}((a_2, b_2)|_{A \times B}(a_2, b_2)))|_{A \times B} \\
& ((a_1, b_1)|_{A \times B}((a_2, b_2)|_{A \times B}(a_2, b_2))) \\
& = (0_A, 0_B) \\
& = 0_{A \times B}
\end{aligned}$$

and

$$\begin{aligned}
& ((a_2, b_2)|_{A \times B}((a_1, b_1)|_{A \times B}(a_1, b_1)))|_{A \times B} \\
& ((a_2, b_2)|_{A \times B}((a_1, b_1)|_{A \times B}(a_1, b_1))) \\
& = (0_A, 0_B) \\
& = 0_{A \times B}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
0_{A \times B} &= (0_A, 0_B) \\
&= ((a_1, b_1)|_{A \times B}((a_2, b_2)|_{A \times B}(a_2, b_2)))|_{A \times B} \\
& \quad ((a_1, b_1)|_{A \times B}((a_2, b_2)|_{A \times B}(a_2, b_2))) \\
&= ((a_1|_A(a_2|_A a_2))|_A(a_1|_A(a_2|_A a_2)), (b_1|_B(b_2|_B b_2))|_B(b_1|_B(b_2|_B b_2)))
\end{aligned}$$

and

$$0_{A \times B} = (0_A, 0_B)$$

$$\begin{aligned}
 &= ((a_2, b_2)|_{A \times B}((a_1, b_1)|_{A \times B}(a_1, b_1)))|_{A \times B} \\
 &\quad ((a_2, b_2)|_{A \times B}((a_1, b_1)|_{A \times B}(a_1, b_1))) \\
 &= ((a_2|_A(a_1|_A a_1))|_A(a_2|_A(a_1|_A a_1)), (b_2|_B(b_1|_B b_1))|_B(b_2|_B(b_1|_B b_1))).
 \end{aligned}$$

Since $\langle A, |_A \rangle$ and $\langle B, |_B \rangle$ are SUP-algebras, we obtain $a_1 = a_2$ and $b_1 = b_2$. Thus, $(a_1, b_1) = (a_2, b_2)$. ■

Definition 7. Let $\mathcal{A} = \langle A, | \rangle$ be a SUP-algebra. A subset B of A is called a SUP-subalgebra of A if the fixed element 0 of A is in B and $\langle B, | \rangle$ forms a SUP-algebra. Clearly, A and $\{0\}$ are SUP-subalgebras of A .

Example 8. Consider the SUP-algebra A in Example 1. A subset $\{0, a\}$ of A is a SUP-subalgebra of A .

Now, we introduce a definition of homomorphisms on SUP-algebras and present a lemma about it.

Definition 8. Let $\langle A, |_A \rangle$ and $\langle B, |_B \rangle$ be SUP-algebras. A mapping $f : A \rightarrow B$ is called a SUP-homomorphism if

$$f(x|_A y) = f(x)|_B f(y)$$

for all $x, y \in A$.

Lemma 9. Let $\langle A, |_A \rangle$ and $\langle B, |_B \rangle$ be SUP-algebras, and let the mapping $f : A \rightarrow B$ be a SUP-homomorphism. Then, $f(A)$ is a SUP-subalgebra of B .

Proof. Let $\langle A, |_A \rangle$ and $\langle B, |_B \rangle$ be SUP-algebras, and let the mapping $f : A \rightarrow B$ be a SUP-homomorphism. We show $0_B \in f(A)$. Since f is a SUP-homomorphism, we have

$$\begin{aligned}
 f(0_A) &= f((0_A|_A(0_A|_A 0_A))|_A(0_A|_A(0_A|_A 0_A))) \\
 &= (f(0_A)|_B(f(0_A)|_B f(0_A)))|_B(f(0_A)|_B(f(0_A)|_B f(0_A))) \\
 &= 0_B
 \end{aligned}$$

from Lemma 2(1). Thus, $0_B = f(0_A) \in f(A)$. For arbitrary elements $u, v, w \in f(A)$, there exist $x, y, z \in A$ such that $u = f(x)$, $v = f(y)$ and $w = f(z)$. Then

- We get

$$\begin{aligned}
 &(((w|_B(u|_B u))|_B(w|_B(u|_B u)))|_B(((v|_B(u|_B u))|_B(w|_B(v|_B v)))|_B \\
 &((v|_B(u|_B u))|_B(w|_B(v|_B v))))|_B(((w|_B(u|_B u))|_B(w|_B(u|_B u)))|_B \\
 &(((v|_B(u|_B u))|_B(w|_B(v|_B v)))|_B((v|_B(u|_B u))|_B(w|_B(v|_B v)))))) \\
 &= (((f(z)|_B(f(x)|_B f(x)))|_B(f(z)|_B(f(x)|_B f(x))))|_B(((f(y)|_B \\
 &(f(x)|_B f(x)))|_B(f(z)|_B(f(y)|_B f(y))))|_B((f(y)|_B(f(x)|_B f(x))) \\
 &|_B(f(z)|_B(f(y)|_B f(y))))))|_B(((f(z)|_B(f(x)|_B f(x)))|_B(f(z)|_B
 \end{aligned}$$

$$\begin{aligned}
& (f(x)|_B f(x))|_B(((f(y)|_B(f(x)|_B f(x)))|_B(f(z)|_B(f(y)|_B \\
& f(y))))|_B((f(y)|_B(f(x)|_B f(x)))|_B(f(z)|_B(f(y)|_B f(y)))))) \\
= & f((((z|_A(x|_A x))|_A(z|_A(x|_A x)))|_A(((y|_A(x|_A x))|_A(z|_A(y|_A y)))|_A \\
& ((y|_A(x|_A x))|_A(z|_A(y|_A y))))|_A(((z|_A(x|_A x))|_A(z|_A(x|_A x)))|_A \\
& (((y|_A(x|_A x))|_A(z|_A(y|_A y)))|_A((y|_A(x|_A x))|_A(z|_A(y|_A y)))))) \\
= & f(0_A) \\
= & 0_B.
\end{aligned}$$

- We have

$$\begin{aligned}
u|_B u &= f(x)|_B f(x) \\
&= f(x|_A x) \\
&= f(x|_A(0_A|_A 0_A)) \\
&= f(x)|_B(f(0_A)|_B f(0_A)) \\
&= f(x)|_B(0_B|_B 0_B) \\
&= u|_B(0_B|_B 0_B).
\end{aligned}$$

- Let $(u|_B(v|_B v))|_B(u|_B(v|_B v)) = 0_B$ and $(v|_B(u|_B u))|_B(v|_B(u|_B u)) = 0_B$. Then we obtain $(x|_A(y|_A y))|_A(x|_A(y|_A y)) = 0_A$ and $(y|_A(x|_A x))|_A(y|_A(x|_A x)) = 0_A$ because

$$\begin{aligned}
0_B &= (u|_B(v|_B v))|_B(u|_B(v|_B v)) \\
&= (f(x)|_B(f(y)|_B f(y)))|_B(f(x)|_B(f(y)|_B f(y))) \\
&= f((x|_A(y|_A y))|_A(x|_A(y|_A y)))
\end{aligned}$$

and

$$\begin{aligned}
0_B &= (v|_B(u|_B u))|_B(v|_B(u|_B u)) \\
&= (f(y)|_B(f(x)|_B f(x)))|_B(f(y)|_B(f(x)|_B f(x))) \\
&= f((y|_A(x|_A x))|_A(y|_A(x|_A x))),
\end{aligned}$$

where f is a SUP-homomorphism. Hence, we get $x = y$ since $\langle A, |_A \rangle$ is a SUP-algebra. Thereby, it follows $u = f(x) = f(y) = v$. Thus $f(A)$ is a SUP-subalgebra of B . ■

Remark 10. The class of all SUP-algebras forms a variety.

4. CONCLUSION

In the present paper, it has been given a SUP-algebra which is the Sheffer stroke reduction of UP-algebras, and examined a Cartesian product, a SUP-subalgebra,

a SUP-homomorphism and many properties in SUP-algebras. After introducing Sheffer Stroke UP-algebras and giving their features, it is showed that a Sheffer Stroke UP-algebra is a UP-algebra with $x \cdot y := (y|(x|x))|(y|(x|x))$, and proved that a Cartesian product of two SUP-algebras is a SUP-algebra. By defining a SUP-subalgebra of SUP-algebra, it has been presented a SUP-homomorphism between SUP-algebras.

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Received 28 September 2020

Revised 19 October 2020

Accepted 19 October 2020