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ON HOM-LEIBNIZ AND HOM-LIE-YAMAGUTI SUPERALGEBRAS

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Abstract

In this paper some characterizations of Hom-Leibniz superalgebras are given and some of their basic properties are found. These properties can be seen as a generalization of corresponding well-known properties of Hom-Leibniz algebras. Considering the Hom-Akivis superalgebra associated to a given Hom-Leibniz superalgebra, it is observed that the Hom-super Akivis identity leads to an additional property of Hom-Leibniz superalgebras, which in turn gives a necessary and sufficient condition for Hom-super Lie admissibility of Hom-Leibniz superalgebras. We also show that every (left) Hom-Leibniz superalgebra has a natural super Hom-Lie-Yamaguti structure.

Keywords: Hom-Leibniz superalgebras, Hom-Akivis superalgebras, Hom-Lie-Yamaguti superalgebras.

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1. Introduction

A (left) Leibniz superalgebra is a superalgebra $(L = L_0 \oplus L_1, *)$ satisfying the identity

(1) $$x * (y * z) = (x * y) * z + (-1)^{|x||y|} y * (x * z)$$

for all homogeneous elements x, y, z in L.

As Leibniz algebras introduced by Loday [15] (and so they are sometimes called Loday algebras) as a noncommutative analogue of Lie algebras, Leibniz superalgebras are super noncommutative analogue of Lie superalgebras. Indeed, if the operation "$*$" of a given Leibniz superalgebra $(L, *)$ is super skew-symmetric, then $(L, *)$ is Lie superalgebra.

One of the problems in the general theory of a given class of (binary or binary-ternary) nonassociative algebras is the study of relationships between that class of algebras and the one of Lie algebras. In the same rule, the search of relationships between a class of nonassociative algebras and the one of Leibniz algebras is of interest (at least for constructing concrete examples of the given class of nonassociative algebras). In this setting, the existence of a Lie-Yamaguti structure on any (left) Leibniz algebra pointed out in [14] is a good illustration. The counterpart of this construction in the Hom-algebra setting has been investigated in [7]. Indeed, the authors show that every multiplicative left Hom-Leibniz algebra has a natural Hom-Lie-Yamaguti structure.

The Hom-algebra structures arisen first in quasi-deformation of Lie algebras of vector fields. Discrete modifications of vector fields via twisted derivations lead to Hom-Lie and quasi-Hom-Lie structures in which the Jacobi condition is twisted [10]. A general study and construction of Hom-Lie algebras have been considered (see [13, 19, 26]). Many researchers investigated Hom-structures on different objects. Yau introduced Hom-Malcev algebras and gave the connections between Hom-alternative algebras and Hom-Malcev algebras [22]. For further more information on other Hom-type algebras, one may refer to, e.g., [4, 9, 16, 23, 24, 25]. Next, generalizations of Hom-type algebras are introduced and discussed in the framework of superalgebras (see, e.g [1, 3]. In particular, Hom-Leibniz, Hom-Akivis and Hom-Lie-Yamaguti superalgebras are respectively introduced in [5, 21] and [8].

In this paper, after considering some characterizations and basic properties of Hom-Leibniz superalgebras, we show that every (left) Hom-Leibniz superalgebra has a natural super Hom-Lie-Yamaguti structure. We observe that the even part of this investigation recovers to the works done in [7] and in [11].

The rest of this paper is organized as follows. In Section two, we recall basic definitions in Hom-superalgebras theory and useful results about Hom-associative superalgebras and Hom-Leibniz superalgebras. Next, super-versions of some well-known properties of (left) Hom-Leibniz algebras [11] are displayed. Considering the specific properties of the binary and ternary operations of the Hom-Akivis superalgebra associated to a given Hom-Leibniz superalgebra, we infer a characteristic property of Hom-Leibniz superalgebras. This property in turn allows to give a necessary and sufficient condition for the super Hom-Lie admissibility of these Hom-superalgebras. In the last Section, we prove the existence of a super

Hom-Lie-Yamaguti structure on any (multiplicative) left Hom-Leibniz superalgebra. All vector superspaces are assumed to be over a fixed ground field $\mathbb{K}$ of characteristic 0.

2. Preliminaries

In this section, we provide some preliminaries, basic definitions [1, 3, 5, 8, 20, 21] and relevant results that are for further use. More precisely, some constructions of Hom-Leibniz superalgebras are given and super-versions of some well-known properties of (left) Hom-Leibniz algebras [11] are displayed. Considering the specific properties of the binary and ternary operations of the Hom-Akivis superalgebra associated to a given Hom-Leibniz superalgebra, we infer a characteristic property of Hom-Leibniz superalgebras. This property in turn allows to give a necessary and sufficient condition for the Hom-super Lie admissibility of these Hom-superalgebras.

Definition 2.1.

(i) Let $f : (A, *, \alpha) \to (A', *', \alpha')$ be a linear map, where $A = A_0 \oplus A_1$ and $A' = A'_0 \oplus A'_1$ are $\mathbb{Z}_2$-graded vector spaces. The map f is called an even (resp. odd) map if $f(A_i) \subset A'_i$ (resp. $f(A_i) \subset A'_{i+1}$), for $i = 0, 1$.

(ii) A (multiplicative) *n-ary Hom-superalgebra* is a triple $(A, \{,\dots,\}, \alpha)$ in which $A = A_0 \oplus A_1$ is a $\mathbb{Z}_2$-graded vector space, $\{,\dots,\} : A^n \longrightarrow A$ is an n-linear map such that $\{A_{i_1}, A_{i_2}, \dots, A_{i_n}\} \subseteq A_{i_1+i_2+\cdots+i_n}$ and $\alpha : A \to A$ is an even linear map such that $\alpha \circ \{,\dots,\} = \{,\dots,\} \circ \alpha^{\otimes n}$ (multiplicativity).

Binary ($n = 2$) Hom-superalgebas are called simply Hom-superalgebras. In this paper, we will be interesting in binary, ternary ($n = 3$) Hom-superalgebras and binary-ternary Hom-superalgebras (i.e., Hom-superalgebras with binary and ternary operations). For convenience, throughout this paper we assume that all (binary, ternary and binary-ternary) Hom-superalgebras are multiplicative and for a given vector superspace A, the set of all homogeneous elements will be denoted by $\mathcal{H}(A)$.

Definition 2.2. Let $(A, *, \alpha)$ be a Hom-superalgebra.

(i) The Hom-associator of A [1] is the trilinear map $as_\alpha : A \times A \times A \to A$ defined as

$$as_\alpha(x, y, z) = (x * y) * \alpha(z) - \alpha(x) * (y * z),\ \forall\, x, y, z \in \mathcal{H}(A).$$

(ii) A Hom-associative superalgebra [3] is a multiplicative Hom-superalgebra $(A, *, \alpha)$ such that

$$as_\alpha(x, y, z) = 0,\ \ \forall\ x, y, z \in \mathcal{H}(A)\ \ \text{(Hom-associativity)}.$$

Remark 2.3. If $\alpha = Id$ (the identity map) in $(A, *, \alpha)$, then its Hom-associator is just the usual associator of the superalgebra $(A, *)$. In the definition of Hom-superalgebras, the Hom-associativity is not assumed, i.e., $as(x, y, z) \neq 0$ in general. In this case $(A, *, \alpha)$ is said non-Hom-associative [12] or Hom-nonassociative [23] superalgebra and in [17] $(A, *, \alpha)$ is said to be a nonassociative Hom-superalgebra. This matches the generalization of associative superalgebras by the nonassociative ones.

Definition 2.4.

(i) A (left) Hom-Leibniz superalgebra [21] is a multiplicative Hom-superalgebra $(L, *, \alpha)$ satisfying the (left) super Hom-Leibniz identity

$$\alpha(x) * (y * z) = (x * y) * \alpha(z) + (-1)^{|x||y|}\alpha(y) * (x * z) \quad \forall x, y, z \in \mathcal{H}(L). \tag{2}$$

(ii) A Hom-Lie superalgebra [3] is a multiplicative Hom-superalgebra $(A, [,], \alpha)$ such that the operation $[,]$ is super skew-symmetic and the super Hom-Jacobi identity

$$[[x, y, \alpha(z)] + (-1)^{|x|(|y|+|z|)}[[y, z), \alpha(x)] \\ + (-1)^{|z|(|x|+|y|)}[[z, x], \alpha(y)] \quad \forall x, y, z \in \mathcal{H}(A)$$

holds.

Remark 2.5. The dual superidentity of (2) is given by

$$(x * y) * \alpha(z) = \alpha(x) * (y * z) + (-1)^{|y||z|}(x * z) * \alpha(y) \ \forall x, y, z \in \mathcal{H}(A). \tag{3}$$

which define a (right) Hom-Leibniz superalgebra. Actually, if the operation " $*$ " verifies (2), then the operation " $\cdot$ " defined by $x \cdot y = (-1)^{|x||y|}y * x$ for all homogeneous elements, verifies also (3) and then, from a (left) Hom-Leibniz superalgebra, one can deduce the (right) Hom-Leibniz superalgebra and vice-versa. Hence, in this note we consider only left Hom-Leibniz superalgebras that we call Hom-Leibniz superalgebras for short. For $\alpha = Id$ in $(L, *, \alpha))$ (resp. $(A, [,], \alpha)$), any Hom-Leibniz superalgebra (resp. Hom-Lie superalgebra) reduces to a classical Leibniz superalgebra $(L, *)$ (resp. Lie superalgebra $(A, [,])$). As for Leibniz superalgebras, if the operation "$*$" of a given Hom-Leibniz superalgebra $(L, *, \alpha)$ is super skew-symmetric, then $(L, *, \alpha)$ is a Hom-Lie superalgebra.

The following examples will be used. The first is obtained from [21] and the others are constructed using the first, from examples of Leibniz superalgebras [18].

Example 2.6. (i) Let $(L, *)$ be a Leibniz superalgebra and α be an even linear map of L such that $\alpha \circ * = * \circ \alpha^{\otimes 2}$. Then $(L, *_\alpha = \alpha \circ *, \alpha)$ is a Hom-Leibniz superalgebra [21].

(ii) Let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be a 4-dimensional superspace, where $L_{\bar{0}}$ is generated by $\{e_0\}$ and $L_{\bar{1}}$ is generated by $\{e_1, e_2, e_3\}$. Then $(L, *, \alpha)$ is a Hom-Leibniz superalgebra where the homomorphism $\alpha : L \to L$ is defined by $\alpha(e_0) = e_0, \alpha(e_1) = ae_1,\ \alpha(e_2) = be_1 + ae_2,\ \alpha(e_3) = ce_1 + be_2 + ce_2$ for any $a, b, c \in \mathbb{K}$ with the only non-zero products $e_1 * e_0 = ae_1,\ e_2 * e_0 = (a + b)e_1 + ae_2,\ e_3 * e_0 = (b + c)e_1 + (a + b)e_2 + ae_3$. It is not a Leibniz superalgebra if $a \neq 0$ since $e_1 * (e_0 * e_0) = 0 \neq a^2 e_1 = (e_1 * e_0) * e_0 - e_0 * (e_1 * e_0)$.

(iii) Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$ be a 4-dimensional superspace, where $A_{\bar{0}}$ is generated by $\{e_0\}$ and $A_{\bar{1}}$ is generated by $\{e_1, e_2, e_3\}$. Then $(A, *, \beta)$ is a Hom-Leibniz superalgebra where the homomorphism $\beta : A \to A$ is defined by $\beta(e_0) = e_0, \beta(e_1) = -e_1,\ \beta(e_2) = -e_2,\ \beta(e_3) = ae_1 + be_2 + ce_2$ for any $a, b, c \in \mathbb{K}$ with the only non-zero products $e_0 * e_1 = -e_1,\ e_2 * e_2 = e_0,\ e_2 * e_0 = -2e_1$.

We prove the following.

Proposition 2.7. *Let $(A, \cdot, \alpha)$ be a Hom-associative superalgebra. Consider an even linear map $D : A \longrightarrow A$ satisfying $\alpha \circ D = D \circ \alpha$ and*

$$D(x) \cdot D(y) = D(D(x) \cdot y) = D(x \cdot D(y))\ \forall x, y \in A. \tag{4}$$

Define an even bilinear map $*_D : A^{\times 2} \longrightarrow A$ such that

$$x *_D y := [D(x), y] = D(x) \cdot y - (-1)^{|x||y|} y \cdot D(x)\ \forall x, y \in \mathcal{H}(A).$$

Then $(A, *_D, \alpha)$ is a Hom-Leibniz superalgebra.

Proof. It suffices to prove (2). Pick $x, y, z \in \mathcal{H}(A)$, then

$$\alpha(x) *_D (y *_D z)$$

$$= [D(\alpha(x)), [D(y), z]] = [\alpha(D(x)), [D(y), z]]$$

by the condition $\alpha \circ D = D \circ \alpha$

$$= [\alpha(D(x)), D(y) \cdot z - (-1)^{|y||z|} z \cdot D(y)] \tag{5}$$

$$= [\alpha(D(x)), D(y) \cdot z] - (-1)^{|y||z|} [\alpha(D(x)), z \cdot D(y)]$$

$$= \alpha(D(x)) \cdot (D(y) \cdot z) - (-1)^{||(|y|+|z|)} (D(y) \cdot z) \cdot \alpha(D(x))$$

$$- (-1)^{|y||z|} \alpha(D(x)) \cdot (z \cdot D(y)) + (-1)^{|x|(|y|+|z|)+|y||z|} (z \cdot D(y)) \cdot \alpha(D(x))$$

$$
\begin{aligned}
&((x *_D y) * \alpha(z) + (-1)^{|x||y|}\alpha(y) *_D (x * z)\\
&= [D([D(x), y], \alpha(z)] + (-1)^{|x||y|}[D(\alpha(y)), [D(x), z]]\\
&= [D(D(x) \cdot y) - (-1)^{|x||y|}D(y \cdot D(x)), \alpha(z)]\\
&+ (-1)^{|x||y|}[\alpha(D(y)), D(x) \cdot z - (-1)^{|x||z|}z \cdot D(x)]\\
&= [D(x) \cdot D(y) - (-1)^{|x||y|}D(y) \cdot D(x), \alpha(z)]\\
&+ (-1)^{|x||y|}[\alpha(D(y)), D(x) \cdot z - (-1)^{|x||z|}z \cdot D(x)] \quad \text{(by (4))}\\
&= (D(x) \cdot D(y)) \cdot \alpha(z) - \underbrace{(-1)^{|z|(|x|+|y|)}\alpha(z) \cdot (D(x) \cdot D(y))}_{1}\\
&+ \underbrace{(-1)^{|x||y|}(D(y) \cdot D(x)) \cdot \alpha(z)}_{2}\\
&+ (-1)^{|x||y|+|z|(|x|+|y|)}\alpha(z) \cdot (D(y) \cdot D(x)) + \underbrace{(-1)^{|x||y|}\alpha(D(y)) \cdot (D(x) \cdot z)}_{2}\\
&- (-1)^{|y||z|}(D(x) \cdot z) \cdot \alpha(D(y)) - (-1)^{|x|(|y|+|z|)}\alpha(D(y)) \cdot (z \cdot D(x))\\
&+ \underbrace{(-1)^{|z|(|x|+|y|)}(z \cdot D(x)) \cdot \alpha(D(y))}_{1}\\
&= (D(x) \cdot D(y)) \cdot \alpha(z) + (-1)^{|y||z|+|x|(|y|+|z|)}\alpha(z) \cdot (D(y) \cdot D(x))\\
&- (-1)^{|y||z|}(D(x) \cdot z) \cdot \alpha(D(y)) - (-1)^{|x|(|y|+|z|)}\alpha(D(y)) \cdot (z \cdot D(x))\\
&\quad \text{(by Hom-associativity)}.
\end{aligned} \tag{6}
$$

The proof follows from (5), (6) and the Hom-associativity condition.

Let's recall the following notion which allows to construct a Hom-Leibniz superalgebra.

Definition 2.8 [20]**.** Let $(A, \cdot, \alpha)$ be a Hom-superalgebra and $\lambda \in \mathbb{K}$.

(i) A Rota-Baxter operator of weight λ of A is a linear map $P : A \longrightarrow A$ satisfies

$$P(x) \cdot P(y) = P(x \cdot P(y)) + P(P(x) \cdot y) + \lambda P(x \cdot y)\ \forall x,\ y \in \mathcal{H}(A). \tag{7}$$

(ii) A Rota-Baxter Hom-superalgebra of weight λ is a quadruple $(A, \cdot, \alpha, P)$, in which $(A, \cdot, \alpha)$ is a multiplicative Hom-superalgeba and $P : A \longrightarrow A$ is a Rota-Baxter operator of weight λ of A. If furthermore, $(A, \cdot, \alpha)$ is a multiplicative Hom-associative superalgebra, then $(A, \cdot, \alpha, P)$ is said a Rota-Baxter Hom-associative superalgebra of weight λ.

Now, we prove

Proposition 2.9. *Let $(A, \cdot, \alpha, P)$ be a Rota-Baxter Hom-associative superalgebra of weight 0 such that $\alpha \circ P = P \circ \alpha$. Define two even bilinear maps $\diamond, * : A^{\times 2} \longrightarrow A$ such that*

$$x \diamond y = x \cdot P(y) + P(x) \cdot y, \;\; x * y = (-1)^{|x||y|} y \diamond x - x \diamond y, \quad \forall x, y \in \mathcal{H}(A).$$

Then $(A, *, \alpha)$ is a Hom-Leibniz superalgebra.

Proof. The multiplicativity condition is obvious. Next, let's prove (2). Pick $x, y, z \in \mathcal{H}(A)$, then using the condition $\alpha \circ P = P \circ \alpha$ and (7) (with $\lambda = 0$), we have by a direct computation

$$\begin{aligned}
\alpha(x) * (y * z) = &\underbrace{(-1)^{|x|(|y|+|z|)+|y||z|}(z \cdot P(y)) \cdot \alpha(P(x))}_{1} \\
&+ \underbrace{(-1)^{|x|(|y|+|z|)+|y||z|}(P(z) \cdot y) \cdot \alpha(P(x))}_{2} \\
&+ \underbrace{(-1)^{|x|(|y|+|z|)+|y||z|}(P(z) \cdot P(y)) \cdot \alpha(x)}_{3} - \underbrace{(-1)^{|x|(|y|+|z|)}(y \cdot P(z)) \cdot \alpha(P(x))}_{13} \\
&- \underbrace{(-1)^{|x|(|y|+|z|)}(P(y) \cdot z) \cdot \alpha(P(x))}_{14} - \underbrace{(-1)^{|x|(|y|+|z|)}(P(y) \cdot P(z)) \cdot \alpha(x)}_{15} \\
&- \underbrace{(-1)^{|y||z|}\alpha(x) \cdot (P(z) \cdot P(y))}_{16} - \underbrace{(-1)^{|y||z|}\alpha(P(x)) \cdot (z \cdot P(y))}_{17} \\
&- \underbrace{(-1)^{|y||z|}\alpha(P(x)) \cdot (P(z) \cdot y)}_{18} + \underbrace{\alpha(x) \cdot (P(y) \cdot P(z))}_{10} + \underbrace{\alpha(P(x)) \cdot (y \cdot P(z))}_{11} \\
&+ \underbrace{\alpha(P(x)) \cdot (P(y) \cdot z)}_{12}
\end{aligned}$$

$$\begin{aligned}
(x * y) * \alpha(z) = &\underbrace{(-1)^{|z|(|x|+|y|)+|x||y|}\alpha(z) \cdot (P(y) \cdot P(x))}_{1} \\
&+ \underbrace{(-1)^{|z|(|x|+|y|)+|x||y|}\alpha(P(z)) \cdot (y \cdot P(x))}_{2} \\
&+ \underbrace{(-1)^{|z|(|x|+|y|)+|x||y|}\alpha(P(z)) \cdot (P(y) \cdot x)}_{3} - \underbrace{(-1)^{|z|(|x|+|y|)}\alpha(z) \cdot (P(x) \cdot P(y))}_{4} \\
&- \underbrace{(-1)^{|z|(|x|+|y|)}\alpha(P(z)) \cdot (x \cdot P(y))}_{5} - \underbrace{(-1)^{|z|(|x|+|y|)}\alpha(P(z)) \cdot (P(x) \cdot y)}_{6} \\
&- \underbrace{(-1)^{|x||y|}(y \cdot P(x)) \cdot \alpha(P(z))}_{7} - \underbrace{(-1)^{|x||y|}(P(y) \cdot x) \cdot \alpha(P(z))}_{8} \\
&- \underbrace{(-1)^{|x||y|}(P(y) \cdot P(x)) \cdot \alpha(z)}_{9} + \underbrace{(x \cdot P(y)) \cdot \alpha(P(z))}_{10} + \underbrace{(P(x) \cdot y) \cdot \alpha(P(z))}_{11} \\
&+ \underbrace{(P(x) \cdot P(y)) \cdot \alpha(z)}_{12}
\end{aligned}$$

$$
\begin{aligned}
&(-1)^{|x||y|}\alpha(x) * (y * z) = \underbrace{(-1)^{|z|(|x|+|y|)}(z \cdot P(x)) \cdot \alpha(P(y))}_{4}\\
&+ \underbrace{(-1)^{|z|(|x|+|y|)}(P(z) \cdot x) \cdot \alpha(P(y))}_{5}\\
&+ \underbrace{(-1)^{|z|(|x|+|y|)}(P(z) \cdot P(x)) \cdot \alpha(y)}_{6} - \underbrace{(-1)^{|y||z|}(x \cdot P(z)) \cdot \alpha(P(y))}_{16}\\
&- \underbrace{(-1)^{|y||z|}(P(x) \cdot z) \cdot \alpha(P(y))}_{17} - \underbrace{(-1)^{|y||z|}(P(x) \cdot P(z)) \cdot \alpha(y)}_{18}\\
&- \underbrace{(-1)^{|x|(|y|+|z|)}\alpha(y) \cdot (P(z) \cdot P(x))}_{13} - \underbrace{(-1)^{|x|(|y|+|z|)}\alpha(P(y)) \cdot (z \cdot P(x))}_{14}\\
&- \underbrace{(-1)^{|x|(|y|+|z|)}\alpha(P(y)) \cdot (P(z) \cdot x)}_{15} + \underbrace{(-1)^{|x||y|}\alpha(y) \cdot (P(x) \cdot P(z))}_{7}\\
&+ \underbrace{(-1)^{|x||y|}\alpha(P(y)) \cdot (x \cdot P(z))}_{8} + \underbrace{(-1)^{|x||y|}\alpha(P(y)) \cdot (P(x) \cdot z)}_{9}.
\end{aligned}
$$

Then, using the Hom-associativity condition, we obtain from the above expressions, the super Hom-Leibniz identity.

Definition 2.10 [21]**.** A Hom-dendriform superalgebra is a quadruple $(A, \prec, \succ, \alpha)$ consisting of a vector space A on which the operations $\prec, \succ A^{\times 2} \longrightarrow A$ are even bilinear maps and $\alpha : A \longrightarrow A$ is an even linear map satisfying

$$
\begin{aligned}
\alpha(x \prec y) &= \alpha(x) \prec \alpha(y)\\
\alpha(x \succ y) &= \alpha(x) \succ \alpha(y)\\
(x \prec y) \prec \alpha(z) &= \alpha(x) \prec (y \prec z) + \alpha(x) \prec (y \succ z)\\
(x \succ y) \prec \alpha(z) &= \alpha(x) \succ (y \prec z)\\
(x \prec y) \succ \alpha(z) + (x \succ y) \succ \alpha(z) &= \alpha(x) \succ (y \succ z)
\end{aligned}
$$

for all $x, y, z \in \mathcal{H}(A)$.

Proposition 2.11. *Let $(A, \prec, \succ, \alpha)$ be a Hom-dendriform superalgebra. Define two even bilinear maps $\diamond, [,] : A^{\times 2} \longrightarrow A$ such that*

$$x \diamond y = x \prec y + x \succ y, \ \ [x, y] = x \diamond y - (-1)^{|x||y|} y \diamond x, \quad \forall x, y \in \mathcal{H}(A).$$

Then $(A, [,], \alpha)$ is a Hom-Lie superalgebra.

Proof. The multiplicativity condition is obvious. Next, we know that $(A, \diamond, \alpha)$ is a Hom-associative superalgebra [21] and $(A, [,], \alpha)$ is a Hom-Lie superalgebra [3].

One class of binary-ternary Hom-superalgebras that are of interest in our setting is the one of Hom-Akivis superalgebra. They are a $\mathbb{Z}_2$-graded generalization of Hom-Akivis algebras [12]. ∎

Definition 2.12. A Hom-Akivis superalgebra is a multiplicative binary-ternary Hom-superalgebra $(A = A_0 \oplus A_1, [,], \{,,\}, \alpha)$, such that

$$(8)\quad \begin{aligned} &[x,y] = -(-1)^{|x||y|}[y,x] \\ &[[x,y],\alpha(z)] + (-1)^{|x|(|y|+|z|)}[[y,z],\alpha(x)] + (-1)^{|z|(|x|+|y|)}[[z,x],\alpha(y)] \\ &= \{x,y,z\} + (-1)^{|x|(|y|+|z|)}\{y,z,x\} + (-1)^{|z|(|x|+|y|)}\{z,x,y\} \\ &+ -(-1)^{|x||y|}\{y,x,z\} - (-1)^{|y||z|}\{x,z,y\} - (-1)^{|z|(|x|+|y|)+|x||y|}\{z,y,x\} \end{aligned}$$

for all x, y, z in $\mathcal{H}(A)$. The identity (8) is called the Hom-super Akivis identity.

Observe that when $\alpha = Id$, the Hom-super Akivis identity (8) reduces to the usual super Akivis identity (see in [2]).

Now, we recall the following result which is useful.

Proposition 2.13 [5]**.** *Let $(A, *, \alpha)$ be a multiplicative Hom-superalgebra. Then $(A, [,], \{,,\}, \alpha)$ is a Hom-Akivis algebra where*

$$[x,y] = x * y - (-1)^{|x||y|} y * x \text{ and } \{x,y,z\} = as_\alpha(x,y,z)\ \forall x,y,z \in \mathcal{H}(A).$$

Corollary 2.14. *Let $(L, *, \alpha)$ be a Hom-Leibniz superalgebra. Consider on $(L, *, \alpha)$ the operations*

$$\begin{aligned} [x,y] &:= x * y - (-1)^{|x||y|} y * x \\ \{x,y,z\} &:= as_\alpha(x,y,z). \end{aligned}$$

Then $(L, [,], \{,,\})$ is a Hom-Akivis superalgebra.

Proof. The proof follows from Proposition 2.13 if observe that a Hom-Leibniz superalgebra is in particular a Hom-superalgebra. ■

Example 2.15. Consider the example of the Hom-Leibniz superalgebra in Example 2.6(ii). By Corollary 2.14, when we define the binary operation $[,]$ (super skew-symmetrization) and the ternary operation defined by $\{,,\} = as_\alpha$, we get a Hom-Akivis superalgebra $(L, [,], \{,,\}, \alpha)$ where nonzero products are $[e_0, e_1] = -ae_1 = -[e_1, e_0]$; $[e_0, e_2] = -(a+b)e_1 - ae_2 = -[e_2, e_0]$; $[e_0, e_3] = -(b+c)e_1 - (a+b)e_2 - ae_3 = -[e_3, e_0]$. Actually, $(L, [,], \alpha)$ is a Hom-Lie superalgebra.

From a given Hom-Leibniz superalgebra $(L, *, \alpha)$, in a term of Hom-assoiator, the identity (2) has the form

$$(9)\quad as_\alpha(x,y,z) = -(-1)^{|x||y|}\alpha(y) * (x * z)\ \forall\ x,y,z \in \mathcal{H}(L).$$

Thus the operations of the Hom-Akivis superalgebra associated with the Hom-Leibniz superalgebra $(L, *, \alpha)$ are the super skew-symmetrization and (9). Then the super Hom-Akivis identity (8) takes the form

$$
\begin{aligned}
&[[x, y], \alpha(z)] + (-1)^{|x|(|y|+|z|)}[[y, z], \alpha(x)] + (-1)^{|z|(|x|+|y|)}[[z, x], \alpha(y)] \\
&= as_\alpha(x, y, z) + (-1)^{|x|(|y|+|z|)} as_\alpha(y, z, x) + (-1)^{|z|(|x|+|y|)} as_\alpha(z, x, y) \\
&+ - (-1)^{|x||y|} as_\alpha(y, x, z) - (-1)^{|y||z|} as_\alpha(x, z, y) - (-1)^{|z|(|x|+|y|)+|x||y|} as_\alpha(z, y, x)
\end{aligned}
$$

that is, by (9),

$$
\begin{aligned}
&[[x, y], \alpha(z)] + (-1)^{|x|(|y|+|z|)}[[y, z], \alpha(x)] + (-1)^{|z|(|x|+|y|)}[[z, x], \alpha(y)] \\
&= (x * y) * \alpha(z) + (-1)^{|x|(|y|+|z|)}(y * z) * \alpha(x) \\
&+ (-1)^{|z|(|x|+|y|)}(z * x) * \alpha(y)
\end{aligned} \tag{10}
$$

for all $x, y, z \in \mathcal{H}(L)$. Hence, we get:

Proposition 2.16. *A Hom-Leibniz superalgebra $(L, *, \alpha)$ is Hom-super Lie admissible if and only if $(x * y) * \alpha(z) + (-1)^{|x|(|y|+|z|)}(y * z) * \alpha(x) + (-1)^{|z|(|x|+|y|)}$ $(z * x) * \alpha(y) = 0$, for all $x, y, z \in \mathcal{H}(L)$.*

Proof. The proof follows from Corollary 2.14 and (10). ■

Example 2.17. Consider the Hom-Leibniz superalgebra $(A, *, \beta)$ in Example 2.6(iii). We have,

$$(e_2 * e_2) * \beta(e_1) + (e_2 * e_1) * \beta(e_2) + (e_1 * e_2) * \beta(e_2) = e_1 \neq 0.$$

Then $(A, *, \beta)$ is not Hom-super Lie admissible.

Let's give the following useful properties of Hom-Leibniz superalgebras.

Proposition 2.18. *Let $(L, *, \alpha)$ be a Hom-Leibniz superalgebra. Then*

$$(x * y + (-1)^{|x||y|} y * x) * \alpha(z) = 0 \tag{11}$$

$$\alpha(x) * [y, z] = [x * y, \alpha(z)] + (-1)^{|x||y|}[\alpha(y), x * z] \tag{12}$$

for all $x, y, z \in \mathcal{H}(L)$.

Proof. The identity (2) implies that

$$(x * y) * \alpha(z) = \alpha(x) * (y * z) - (-1)^{|x||y|}\alpha(y) * (x * z) \tag{13}$$

Then, interchanging x and y in the identity (13, we have

$$(y * x) * \alpha(z) = \alpha(y) * (x * z) - (-1)^{|x||y|}\alpha(x) * (y * z)$$

that is

$$(-1)^{|x||y|}(y * x) * \alpha(z) = (-1)^{|x||y|}\alpha(y) * (x * z) - \alpha(x) * (y * z) \tag{14}$$

for all $x, y, z \in \mathcal{H}(L)$. Then, adding memberwise (13) and (14), we come to the property (11).

Next we have

$$\begin{aligned}
&[x * y, \alpha(z)] + (-1)^{|x||y|}[\alpha(y), x * z]\\
&= (x * y) * \alpha(z) - (-1)^{|z|(|x|+|y|)}\alpha(z) * (x * y)\\
&+ (-1)^{|x||y|}\alpha(y) * (x * z) - (-1)^{|y||z|}(x * z) * \alpha(y)\\
&= \alpha(x) * (y * z) - (-1)^{|z|(|x|+|y|)}\alpha(z) * (x * y)\\
&- (-1)^{|y||z|}(x * z) * \alpha(y) \ \text{(by (2))}\\
&= \alpha(x) * (y * z) - (-1)^{|z|(|x|+|y|)}(z * x) * \alpha(y)\\
&+ - (-1)^{|z||y|}\alpha(x) * (z * y) - (-1)^{|y||z|}(x * z) * \alpha(y) \ \text{(by (2))}\\
&= \alpha(x) * (y * z) - (-1)^{|z||y|}\alpha(x) * (z * y) \ \text{(by (11))}\\
&= \alpha(x) * [y, z].
\end{aligned}$$

Therefore we get (12). ■

Remark 2.19. In Proposition 2.18, if x, y, z are in L_0, then one recovers the well-known properties of Hom-Leibniz algebras (see Proposition 3.1. in [11]).

Now, we consider Hom-Lie-Yamaguti superalgebras which are also useful. They are a twisted generalisation of Lie-Yamaguti superalgebras and a $\mathbb{Z}_2$-graded generalization of Hom-Lie-Yamaguti algebras [6].

Definition 2.20 [8]**.** A *Hom-Lie Yamaguti superalgebra* (Hom-LY superalgebra for short) is a binary-ternary superalgebra $(L, *, \{,,\}, \alpha)$ such that

$(SHLY1)$ $\alpha(x * y) = \alpha(x) * \alpha(y)$,

$(SHLY2)$ $\alpha(\{x, y, z\}) = \{\alpha(x,)\alpha(y), \alpha(z)\}$,

$(SHLY3)$ $x * y = -(-1)^{|x||y|}y * x$,

$(SHLY4)$ $\{x, y, z\} = -(-1)^{|x||y|}\{y, x, z\}$,

$(SHLY5)$ $\circlearrowleft_{(x,y,z)} (-1)^{|x||z|}[(x * y) * \alpha(z) + \{x, y, z\}] = 0$,

$(SHLY6)$ $\circlearrowleft_{(x,y,z)} (-1)^{|x||z|}[\{x * y, \alpha(z), \alpha(u)\}] = 0,$

$(SHLY7)$ $\{\alpha(x), \alpha(y), u * v\} = \{x, y, u\} * \alpha^2(v) + (-1)^{|u|(|x|+|y|)}\alpha^2(u) * \{x, y, v\},$

$(SHLY8)$ $\{\alpha^2(x), \alpha^2(y), \{u, v, w\}\} = \{\{x, y, u\}, \alpha^2(v), \alpha^2(w)\}$
$$+ (-1)^{|u|(|x|+|y|)}\{\alpha^2(u), \{x, y, v\}, \alpha^2(w)\}$$
$$+ (-1)^{(|x|+|y|)(|u|+|v|)}\{\alpha^2(u), \alpha^2(v), \{x, y, w\}\},$$

for all $u, v, w, x, y, z \in \mathcal{H}(L)$ and where $\circlearrowleft_{(x,y,z)}$ denotes the sum over cyclic permutation of x, y, z.

Note that the conditions $(SHLY1)$ and $(SHLY2)$ mean the multiplicativity of $(L, *, \{, , \}, \alpha)$.

3. Super Hom-Lie-Yamaguti structures on Hom-Leibniz superalgebras

In this section, we prove the existence of a super Hom-Lie-Yamaguti structure on any (multiplicative) left Hom-Leibniz superalgebra. This result generalizes the one in [7] whose Hom-analogue is in [14]. Note that our proof below essentially relies on some properties characterizing Hom-Leibniz superalgebras, obtained in Proposition 2.18. Specifically, we shall prove the following:

Theorem 3.1. *Every left Hom-Leibniz superalgebra has a natural Hom-Lie-Yamaguti superalgebra structure.*

Proof. In a left Hom-Leibniz superalgebra $(L, *, \alpha)$ consider the super skew-symmetrization
$$[x, y] := x * y - (-1)^{|x||y|} y * x$$
for all $x, y \in \mathcal{H}(L)$. In the following, consider the left translations $\Lambda_a b := a * b$ in $(L, *, \alpha)$. Then the identities (2) and (12) can be written respectively as

$$\Lambda_{\alpha(x)}(y * z) = (\Lambda_x y) * \alpha(z) + (-1)^{|x||y|}\alpha(y) * (\Lambda_x z), \tag{15}$$

$$\Lambda_{\alpha(x)}[y, z] = [\Lambda_x y, \alpha(z)] + (-1)^{|x||y|}[\alpha(y), \Lambda_x z] \tag{16}$$

In $(L, *, \alpha)$ consider the following ternary operation:

$$\{x, y, z\} := (-1)^{|x||y|} as_\alpha(y, x, z) - as_\alpha(x, y, z) \tag{17}$$

for all $x, y, z \in \mathcal{H}(L)$. Then (17), (2) and (9) imply

$$\{x, y, z\} = -(x * y) * \alpha(z). \tag{18}$$

Moreover, we have

$$\begin{aligned} [x, y] * \alpha(z) &= (x * y - (-1)^{|x||y|} y * x) * \alpha(z) \\ &= 2(x * y) * \alpha(z) \quad \text{(by (11))} \\ &= -2\{x, y, z\} \quad \text{(see (18))} \end{aligned} \tag{19}$$

so that

$$\{x, y, z\} = -\frac{1}{2}[x, y] * \alpha(z). \tag{20}$$

Thus (17), (18) and (20) are different expressions of the operation "$\{,,\}$" that are for use in what follows. Now we proceed to verify the validity on $(L, *, \alpha)$ of the set of identities $(SHLY1) - (SHLY8)$. The multiplicativity of $(L, *, \alpha)$ implies $(SHLY1)$ and $(SHLY2)$ while $(SHLY3)$ is the super skew-symmetrization and $(SHLY4)$ clearly follows from (17) (or (18)). Next, observe that $(HLY5)$ is just the Hom-super Akivis identity (8) for $(L, *, \alpha)$.

Consider now $\circlearrowleft_{(x,y,z)} (-1)^{|x||z|}\{[x, y], \alpha(z), \alpha(u)\}$. Then

$$\begin{aligned}
&\circlearrowleft_{(x,y,z)} (-1)^{|x||z|}\{[x, y], \alpha(z), \alpha(u)\} \\
&= \circlearrowleft_{(x,y,z)} (-1)^{|x||z|}(-[x, y] * \alpha(z)) * \alpha^2(u) \quad \text{(by (18))} \\
&= 2(\circlearrowleft_{(x,y,z)} (-1)^{|x||z|}\{x, y, z\}) * \alpha^2(u) \quad \text{(by (20))} \\
&= -2((-1)^{|x||z|}(x * y) * \alpha(z) + (-1)^{|x||y|}(y * z) * \alpha(x) + (-1)^{|y||z|}(z * x) * \alpha(y)) * \alpha^2(u) \\
&= -2((-1)^{|x||z|}\alpha(x) * (y * z) - (-1)^{|x|(|y|+|z|)}\alpha(y) * (x * z) + (-1)^{|x||y|}(y * z) * \alpha(x) \\
&+ (-1)^{|y||z|}(z * x) * \alpha(y)) * \alpha^2(u) \quad \text{(by (2))} \\
&= -2(-1)^{|x||z|}(\alpha(x) * (y * z) + (-1)^{|x|(|y|+|z|)}(y * z) * \alpha(x)) * \alpha^2(u) \\
&- 2(-(-1)^{|x|(|y|+|z|)}\alpha(y) * (x * z) + (-1)^{|y||z|}(z * x) * \alpha(y)) * \alpha^2(u) \\
&= -2(-(-1)^{|x|(|y|+|z|)}\alpha(y) * (x * z) + (-1)^{|y||z|}(z * x) * \alpha(y)) * \alpha^2(u) \quad \text{(by (11))} \\
&= -2(-(-1)^{|x|(|y|+|z|)}\alpha(y) * (x * z) - (-1)^{|z|(|x|+|y|)}(x * z) * \alpha(y)) * \alpha^2(u) \quad \text{(by (11))} \\
&= 2(-1)^{|x|(|y|+|z|)}(\alpha(y) * (x * z) + (-1)^{|y|(|x|+|z|)}(x * z) * \alpha(y)) * \alpha^2(u) \\
&= 0 \quad \text{(by (11))}
\end{aligned}$$

so that we get $(SHLY6)$. Next

$$\begin{aligned}
&\{\alpha(x), \alpha(y), [u,v]\}\\
&= -\alpha(x*y)*\alpha([u,v]) \quad \text{(by (18) and multiplicativity)}\\
&= \Lambda_{-\alpha(x*y)}[\alpha(u), \alpha(v)]\\
&= [\Lambda_{-x*y}\alpha(u), \alpha^2(v)] + (-1)^{|u|(|x|+|y|)}[\alpha^2(u), \Lambda_{-x*y}\alpha(v)] \quad \text{(by (16))}\\
&= [\{x,y,u\}, \alpha^2(v)] + (-1)^{|u|(|x|+|y|)}[\alpha^2(u), \{x,y,v\}] \quad \text{(by (18))}
\end{aligned}$$

which is $(SHLY7)$. Finaly, we compute

$$\begin{aligned}
&\{\{x,y,u\}, \alpha^2(v), \alpha^2(w)\} + (-1)^{|u|(|x|+|y|)}\{\alpha^2(u), \{x,y,v\}, \alpha^2(w)\}\\
&+(-1)^{(|u|+|v|)(|x|+|y|)}\{\alpha^2(u), \alpha^2(v), \{x,y,w\}\}\\
&= \{-\Lambda_{x*y}\alpha(u), \alpha^2(v), \alpha^2(w)\} + (-1)^{|u|(|x|+|y|)}\{\alpha^2(u), -\Lambda_{x*y}\alpha(v), \alpha^2(w)\}\\
&+(-1)^{(|u|+|v|)(|x|+|y|)}\{\alpha^2(u), \alpha^2(v), -\Lambda_{x*y}\alpha(w)\}\\
&= -((-\Lambda_{x*y}\alpha(u)) * \alpha^2(v)) * \alpha^3(w) - (-1)^{|u|(|x|+|y|)}(\alpha^2(u) * (-\Lambda_{x*y}\alpha(v))) * \alpha^3(w)\\
&-(-1)^{(|u|+|v|)(|x|+|y|)}(\alpha^2(u) * \alpha^2(v)) * \alpha(-\Lambda_{x*y}\alpha(w))\\
&= (\Lambda_{\alpha(x*y)}\alpha(u*v)) * \alpha^3(w) + (-1)^{(|u|+|v|)(|x|+|y|)}\alpha^2(u*v) * \Lambda_{\alpha(x*y)}\alpha^2(w)\\
&\quad \text{(by (15) and multiplicativity)}\\
&= \Lambda_{\alpha^2(x*y)}(\alpha(u*v) * \alpha^2(w)) \quad \text{(by (15))}\\
&= -\alpha^2(x*y) * (-\alpha(u*v) * \alpha^2(w))\\
&= -(\alpha^2(x) * \alpha^2(y)) * \alpha(-(u*v) * \alpha(w)) \quad \text{(by multiplicativity)}\\
&= \{\alpha^2(x), \alpha^2(y), \{u,v,w\}\} \quad \text{(by (18))}.
\end{aligned}$$

Therefore $(L, [,], \{,,\}, \alpha)$ is a Hom-LY superalgebra. This completes the proof. ■

Now, let's give an example of Hom-Lie-Yamaguti superalgebras using Theorem 3.1.

Example 3.2. Consider the example of Hom-Leibniz superalgebra in Example 2.6(ii). By Theorem 3.1, when we define the binary operation $[,]$ (super skew-symmetrization) and the ternary operation defined by (18), we get a Hom-Lie-Yamaguti superalgebra $(L, [,], \{,,\}, \alpha)$ with nonzero products $[e_0, e_1] = -ae_1 = -[e_1, e_0]$; $[e_0, e_2] = -(a+b)e_1 - ae_2 = -[e_2, e_0]$; $[e_0, e_3] = -(b+c)e_1 - (a+b)e_2 - ae_3 = -[e_3, e_0]$; $\{e_1, e_0, e_0\} = -a^2e_1 = -\{e_1, e_0, e_0\}$, $\{e_2, e_0, e_0\} = -2a(a+b)e_1 -$

$a^2e_2 = -\{e_0, e_2, e_0\}$, $\{e_3, e_0, e_0\} = -(a^2+b^2+4ab+2ac)e_1 - (2a(a+b)e_2 - a^2e_3 = -\{e_0, e_3, e_0\}$.

References

[1] K. Abdaoui, F. Ammar and A. Makhlouf, *Hom-alternative, Hom-Malcev and Hom-Jordan superalgebras*, Bull. Malays. Math. Sci. Soc. **40** (2017) 439–472. https://doi.org/10.1007/s40840-016-0323-5

[2] H. Albuquerque A. P. Santana *Akivis superalgebras and speciality*, Contemporary Math. **483** (2009) 13–22.

[3] F. Ammar and A. Makhlouf, *Hom-Lie superalgebras and Hom-Lie admissible superalgebras*, J. Algebra **324** (2010) 1513–1528. https://doi.org/10.1016/j.jalgebra.2010.06.014

[4] H. Ataguema, A. Makhlouf and S.D. Silvestrov, *Generalization of n-ary Nambu algebras and beyond*, J. Math. Phys. **50** (2009) 083501. https://doi.org/10.1063/1.3167801

[5] D. Gaparayi and A.N. Issa, *Hom-Akivis superalgebras*, J. Alg. Comput. Appl. **6** (2017) 36–51.

[6] D. Gaparayi and A.N. Issa, *A Twisted Generalization of Lie-Yamaguti Algebras*, Int. J. Algebra **7** (2012) 339–352.

[7] D. Gaparayi and A.N. Issa, *Hom-Lie-Yamaguti structures on Hom-Leibniz algebras*, Extracta Math. **28** (2013) 1–12.

[8] D. Gaparayi, S. Attan and A.N. Issa, *Hom-Lie-Yamaguti superalgebras*, Korean J. Math. **27** (2019) 175–192. https://doi.org/10.11568/kjm.2019.27.1.175

[9] Y. Fregier and A. Gohr, *Unital algebras of Hom-associative type and surjective or injective twistings*, J. Gen. Lie Theo. and Appl. **3** (2009) 285–295.

[10] J.T Hartwig, D. Larsson and S.D. Silvestrov, *Deformations of Lie algebras using σ-derivations*, J. Algebra **295** (2006) 314–361. https://doi.org/10.1016/j.jalgebra.2005.07.036

[11] A.N. Issa, *Some characterizations of Hom-Leibniz algebras*, International Int. Elect. J. Alg. **14** (2013) 1–9.

[12] A.N. Issa, *Hom-Akivis algebras*, Comment. Math. Univ. Carolin. **52** (2011). http://dml.cz/dmlcz/142800.

[13] Q. Jin and X. Li, *Hom-Lie algebra structures on semi-simple Lie algebras*, J. Algebra **319** (2008) 1398–1408. https://doi.org/10.1016/j.jalgebra.2007.12.005

[14] M.K. Kinyon and A. Weinstein, *Leibniz algebras, Courant algebroids and multiplications on reductive homogeneous spaces*, Amer. J. Math. **123** (2001) 525–550. https://doi.org/10.1353/ajm.2001.0017

[15] J-L. Loday. *Une version non commutative des algèbres de Lie: les algèbres de Leibniz*, Enseign. Math. **39** (1993) 269–293.

[16] A. Makhlouf, *Hom-alternative algebras and Hom-Jordan algebras*, Int. Elect. J. Alg. **8** (2010) 177–190.

[17] A. Makhlouf, *Paradigm of nonassociative Hom-algebras and Hom-superalgebras*, in: Proceedings of Jordan structures in algebra and Analysis Meeting, 143–177, Editional Circulo Rojo (Almeria, 2010).

[18] Qi Nie and Zhen Li, *The Classification of* 4*-dimensional Leibniz Superalgebras*, Math. Aeterna **7** 2017 375–380.

[19] I.P. Shestakov, *Prime Mal'tsev superalgebras*, Mat. Sb **182** (1991) 1357–1366

[20] B. Sun, *The construction of Hom-Novikov superalgebras*, Math. Aeterna **6** (2016) 605–609.

[21] C. Wang, Q. Zhang and Z. Wei, *Hom-Leibniz superalgebras and Hom-Leibniz Poisson superalgebras*, Hacet. J. Math. Stat. **44** (2015) 1163–1179.

[22] D. Yau, *Hom-Malcev, Hom-alternative, and Hom-Jardan algebras*, Int. Elect. J. Alg. **11** (2012) 177–217.

[23] D. Yau, *Hom-algebras and Homology*, J. Lie Theory **19** (2009) 409–421.

[24] D. Yau, *Hom-Novikov algebras*, J. Phys. A **44** (2011) 085202.

[25] D. Yau, *On n-ary Hom-Nambu and Hom-Nambu-Lie algebras*, J. Geom. Phys. **62** (2012) 506–522.
https://doi.org/10.1016/j.geomphys.2011.11.006

[26] J.X. Yuan, L.P. Sun and W.D. Liu, *Multiplicative Hom-Lie superalgebra structures on infinite dimensional simple Lie superalgebras of vector fields*, J. Geom. Phys. **84** (2014) 1–7.
https://doi.org/10.1016/j.geomphys.2014.06.001

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