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EQUIVALENT FORMS FOR A POSET TO BE MODULAR POSET

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Abstract

The notion of modular and distributive posets which generalize the corresponding notions from the lattice theory are introduced by J. Larmerova and J. Rachnek. Later some extended results of uniquely complemented lattice are derived to uniquely complemented posets. Now, in this paper, some equivalent conditions for a poset to be modular poset are given.

Keywords: poset, lattice, modular poset.

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1. Introduction

Modularity in lattice theory is an interesting topic. In [3], Larmerova and Rachunek introduced concepts of modular posets and distributive posets. They have proved that any distributive poset is a modular poset and some other results are proved. An example of modular poset which is not a distributive poset and not a lattice is given. An example of distributive poset which is not a lattice is given. Some results are derived on modular posets. In [5], Waphere and Joshi worked on uniquely complimented posets. Now, in this paper, some equivalent conditions of poset to be modular poset are given. Some properties of modular poset are derived. Dual of equivalent condition for a lattice to be modular lattice is quite natural in lattices. But dual of equivalent condition for a poset to be modular poset is not simple. Now we start with the following basic concepts.

2. Preliminaries

In this section, we begin with necessary definitions and terminologies in a poset P. Let $A \subseteq P$. The set $\mathcal{A}^u = \{x \in P/x \geq a$ for every $a \in A\}$ is called the upper cone of A. Dually,we have a concept of the lower cone $\mathcal{A}^l$ of A.$\mathcal{A}^{ul}$ shall mean $\{\mathcal{A}^u\}^l$ and $\mathcal{A}^{lu}$ shall mean $\{\mathcal{A}^l\}^u$. The lower cone $\{a\}^l$ is simply denoted by a^l and $\{a, b\}^l$ is it denoted by $(a, b)^l$. Similar notations are used for upper cones. Further, for $A, B \subseteq P$, $\{\ A \cup B\}^u$ is denoted by $\{A, B\}^u$ and for $x \in P$ the set $\{A \cup \{x\}\}^u$ is denoted by $\{\ A, x\}^u$. Similar notations are used for lower cones. Some properties of posets are given in the following

(i) $A^{lul} = A^l$, $A^{ulu} = A^u$.

(ii) $\{a^u\}^l = \{a\}^l = a^l$ and $\{a^l\}^u = \{a\}^u = a^u$.

(iii) $A \subseteq \mathcal{A}^{ul}$ and $A \subseteq \mathcal{A}^{lu}$.

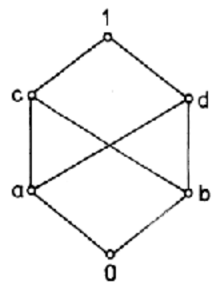
(iv) If $A \subseteq B$ then $B^l \subseteq A^l$ and $B^u \subseteq A^u$.

An element $y \in P$ is said to be a complement of $x \in P$, if $(x, y)^{ul} = (x, y)^{lu}$ $= P$. P is said to be complemented if each element of P has a complement in P and P is said to be uniquely complemented if each element $x \in P$ has a unique complement, denoted by $x^{'}$ in P. A distributive complemented poset is called a Boolean poset. Let P be a uniquely complemented poset and $A \subseteq P$. Denote $A^{'} = \{a^{'} : a \in A\}$. We say that P satisfies De Morgan's laws if $\{(x, y)^u\}^{'} = (x^{'}, y^{'})^l$ and $\{(x, y)^l\}^{'} = (x^{'}, y^{'})^u$ for all $x, y \in P$.

3. Modular Poset and some equivalent forms

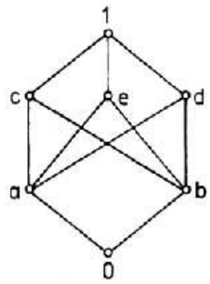
We begin this section with the definition of a Modular Poset.

Definition 3.1. A poset P is said to be Modular Poset if for all $a, b, c \in P$ with $a \leq c$ implies $(a, (b, c)^l)^{ul} = ((a, b)^u, c)^l$.



In the following, the examples of modular poset and distributive posets are given. The second diagram represents modular poset which is not a distributive poset and not a lattice where as the first diagram represents a distributive poset which is not a lattice.

In the above Hasse diagram of the poset is a distributive poset P^d but not a lattice because it satisfies distributive law $((x,y)^u, z)^l = ((x,z)^l, (y,z)^l)^{ul}$ for all $x, y, z \in P^d$. Now look the following Hasse diagram.



It is a modular poset but not a distributive poset because $((c,d)^u, e)^l = \{0, a, b, e\} \neq \{0, a, b, \} = ((c,e)^l, (d,e)^l)^{ul}$. In the following, some results on posets are proved which are used later.

Lemma 3.2. *Let $(P, \leq)$ be poset then $(a,b)^u \subseteq (a, (b,c)^l)^u$ for every $a, b, c \in P$.*

Proof. Let $x \in (a,b)^u$. Then $x \geq a, x \geq b$. Let $t \in (b,c)^l$, this mean that $t \leq b, t \leq c$. Then we have $x \geq b \geq t$. Thus $x \geq a, x \geq t$ for all $t \in (b,c)^l$ which implies $x \in (a, (b,c)^l)^u$.Thus we have $(a,b)^u \subseteq (a, (b,c)^l)^u$. ∎

Lemma 3.3. *Let $(P, \leq)$ be a poset and a,b,c $\in P$, if $a \leq c$ then $((a,b)^u, c) \subseteq (a, (b,c)^l)^u$.*

Proof. Let $z \in ((a,b)^u, c)$. Then $z \in (a,b)^u \cup \{c\}$. This implies $z \in (a,b)^u$ or $z = c$. If $z = c$, then we have to show that $c \in (a, (b,c)^l)^u$. Suppose $t \in (b,c)^l$. Then $t \leq b, t \leq c$. Hence $c \in (a, (b,c)^l)^u$. By lemma 3.2 if z$\in$ $(a,b)^u$ then z$\in$ $(a, (b,c)^l)^u$.In either case z$\in$ $(a, (b,c)^l)^u$. Therefore $((a,b)^u, c) \subseteq (a, (b,c)^l)^u$. By a property (iv) in the preliminaries,we have $(a, (b,c)^l)^{ul} \subseteq ((a,b)^u, c)^l$. ∎

Corollary 3.4. *Let$(P, \leq)$ be a poset and a,b,c $\in P$ if $a \leq c$, then $(a, (b,c)^l)^{ul} \subseteq ((a,b)^u, c)^l$.*

Proof. By Lemma 3.3, for $a \leq c$ we have $((a,b)^u, c) \subseteq ((a, (b,c)^l))^u$. Now, By property(iv),we get $(a, (b,c)^l)^{ul} \subseteq ((a,b)^u, c)^l$. ∎

In this paper, we take P be a poset to mean that $(P, \leq)$ be a poset. It is also given an equivalent condition of a poset to be modular poset in following theorem.

Theorem 3.5. *Let P be a poset then P is a modular poset if and only if $a \leq c$ implies $(a, (b, c)^l)^{ul} \supseteq ((a, b)^u, c)^l$.*

Proof. Suppose P is modular poset. Let $a \leq c$, then we have $(a, (b, c)^l)^{ul} = ((a, b)^u, c)^l$. Hence $((a, b)^u, c)^l \subseteq (a, (b, c)^l)^{ul}$. Conversely assume that, $a \leq c$. Then $((a, b)^u, c)^l \subseteq (a, (b, c)^l)^{ul}$. Now, we have to prove that $(a, b)^u \subseteq (a, (b, c)^l)^u$. Let $z \in (a, b)^u$. Then $z \geq a$, $z \geq b$ and hence $z \geq a$, $z \geq b \geq (b, c)^l$. Therefore $z \geq a$, $z \geq (b, c)^l$. So that $z \in (a, (b, c)^l)^u$. Thus we have $(a, b)^u \subseteq (a, (b, c)^l)^u$. Now, prove that $((a, b)^u, c) \subseteq (a, (b, c)^l)^u$. Let $z \in ((a, b)^u, c)$. Then $z \in (a, b)^u$ or $z = c$. If $z = c$, then $c \in (a, (b, c)^l)^u$ since we have $a \leq c$. So that $c \geq a$, $c \geq t$, for any $t \in (b, c)^l$. If $t \in (b, c)^l$, then $t \leq b, t \leq c$ and hence $c \in (a, (b, c)^l)^u$. Therefore $((a, b)^u, c) \subseteq (a, (b, c)^l)^u$. By the poset property,in the prilimnaries we have $A \subseteq B$ implies $A^l \supseteq B^l$. Therefore $((a, b)^u, c)^l \supseteq (a, (b, c)^l)^{ul}$. Thus we have $(a, (b, c)^l)^{ul} = ((a, b)^u, c)^l$. Hence P is modular poset. ∎

Now, we define the the dual of a modular poset as follows. Let $P*$ be a poset then $P*$ is a said to be a dual modular poset if, $a \leq c$ implies $(a, (b, c)^u)^{lu} = ((a, b)^l, c)^u$ for all $a, b, c \in P*$.

Theorem 3.6. *Let $P*$ be a poset then $P*$ is a dual modular poset if and only if $[a \leq c$ implies $(a, (b, c)^u)^{lu} \subseteq ((a, b)^l, c)^u]$.*

Proof. Suppose $P*$ is dual modular poset. Let $a \leq c$. Then $(a, (b, c)^u)^{lu} = ((a, b)^l, c)^u$ and hence $((a, b)^l, c)^u \supseteq (a, (b, c)^u)^{lu}$. Conversely, assume that $a \leq c \Rightarrow (a, (b, c)^u)^{lu} \subseteq ((a, b)^l, c)^u$. First, we prove that $(a, b)^l \supseteq (a, (b, c)^u)^l$. Let $z \in (a, b)^l$. Then $z \leq a, z \leq b$. Hence $z \leq a, z \leq t$ where $t \in (b, c)^u$. Therefore $z \in (a, (b, c)^u)^l$. Thus $(a, b)^l \supseteq (a, (b, c)^u)^l$. Now, we prove that $((a, b)^l, c) \supseteq (a, (b, c)^u)^l$. Let $z \in ((a, b)^l, c)$. Then $z \in (a, b)^l$ or $z = c$. If $z = c$, then $c \in (a, (b, c)^u)^l$. Since we have $a \leq c$, so that $c \geq a, c \geq (b, c)^u$. If $t \in (b, c)^u$, then $t \geq b, t \geq c$ and hence $c \in (a, (b, c)^u)^l$. Therefore $((a, b)^l, c) \supseteq (a, (b, c)^u)^l$. By the poset property, we have $A \subseteq B \Rightarrow A^u \supseteq B^u$. Therefore $((a, b)^l, c)^u \subseteq (a, (b, c)^u)^{lu}$ and hence $(a, (b, c)^u)^{lu} = ((a, b)^l, c)^u$. Thus $P*$ is dual modular poset. ∎

The following results are true in posets. To proving these results modular posets is not necessary.

Theorem 3.7. *Let P is a poset, then $a \leq c$ implies $((a, b)^u, (b, c)^u, (c, a)^u)^l \supseteq ((a, b)^l, (b, c)^l, (c, a)^l)^{ul}$.*

Proof. Let P be a poset. It is enough to prove that $((a,b)^u,(b,c)^u,\ (c,a)^u) \subseteq ((a,b)^l,(b,c)^l,(c,a)^l)^u$. We have $((a,b)^u,c)^l \subseteq (a,(b,c)^l)^{ul}$ by Lemma 3.3 and let $z \in (a,b)^u \cup (b,c)^u \cup (c,a)^u$. Then $z \in (a,b)^u$ or $z \in (b,c)^u$ or $z \in (c,a)^u$. Now, we have to prove that $z \in ((a,b)^l,(b,c)^l,(c,a)^l)^u$ i.e., $z \geq t$ for any $t \in (a,b)^l \cup (b,c)^l \cup (c,a)^l$. Suppose $t \in (a,b)^l \cup (b,c)^l \cup (c,a)^l$. Then $t \in (a,b)^l$ or $t \in (b,c)^l$ or $t \in (c,a)^l$. Suppose $z \geq a,b$ and $t \in (a,b)^l$. Then $t \leq a,b$ and hence $a \geq t$ and $z \geq a$. Therefore $z \geq t$. If $t \in (b,c)^l$. Then $t \leq b, t \leq c$ and hence $b \geq t$ and $z \geq b$. Therefore $z \geq t$. Suppose $t \in (c,a)^l$. Then $t \leq c, t \leq a$ and hence $z \geq a$ and $a \geq t$. Therefore $z \geq t$. If $t \in (a,b)^u \cup (b,c)^u \cup (c,a)^u$, then $z \geq t$. Thus $z \in ((a,b)^l \cup (b,c)^l \cup (c,a)^l)^u$. If $z \geq b,c$. Suppose $t \in (a,b)^l$, then $t \leq a, t \leq b$ and hence $b \geq t$ and $z \geq b$. Therefore $z \geq t$. Suppose $t \in (b,c)^l$, then $t \leq b, t \leq c$ and hence $b \geq t$ and $z \geq b$. Therefore $z \geq t$. Suppose $t \in (c,a)^l$, then $t \leq c, t \leq a$ so that $z \geq c$ and $t \leq c$. Therefore $z \geq t$. If $t \in (a,b)^l \cup (b,c)^l \cup (c,a)^l$, then $z \geq t$ and hence $z \in ((a,b)^l \cup (b,c)^l \cup (c,a)^l)^u$. If $z \geq c,a$. Suppose $t \in (a,b)^l$, then $t \leq a, t \leq b$. Therefore $z \geq a$ and $t \leq a$. Thus $z \geq t$. Suppose $t \in (b,c)^l$, then $t \leq b, t \leq c$ and hence $z \geq c$ and $t \leq c$. Therefore $z \geq t$. Suppose $t \in (c,a)^l$, then $t \leq c$ and $t \leq a$. Therefore $z \geq c$ and $t \leq c$. Thus $z \geq t$. If $t \in (a,b)^l \cup (b,c)^l \cup (c,a)^l$, then $z \geq t$. Therefore $z \in ((a,b)^l \cup (b,c)^l \cup (c,a)^l)^u$. Thus we get, if $a \leq c$, then $((a,b)^u,(b,c)^u,\ (c,a)^u)^l \supseteq ((a,b)^l,(b,c)^l,(c,a)^l)^{ul}$. ■

Lemma 3.8. *Let P be a poset. If $a \leq c$, then $(a,(b,c)^l)^{ul} \subseteq ((a,b)^u,(b,c)^u,$ $(c,a)^u)^l$.*

Proof. We prove that $(a,(b,c)^l)^u \supseteq ((a,b)^u,(b,c)^u,(c,a)^u)$. Let $x \in ((a,b)^u,$ $(b,c)^u,(c,a)^u)$. Then $x \in (a,b)^u$ or $x \in (b,c)^u$ or $x \in (c,a)^u$, so that $x \geq a,b$ or $x \geq b,c$ or $x \geq c,a$.

Case (i). If $x \geq c,a$. Now, we have to prove that $x \in (a,(b,c)^l)^u$, i.e., $x \geq \{a\} \cup (b,c)^l$(ie $x \leq a$ or $x \geq t$ for any $t \in (b,c)^l$). Let $t \in (b,c)^l$. Then $t \leq b, t \leq c$ and hence $x \geq c \geq t$. Therefore $x \geq t$ and hence $x \geq a$ and $x \geq t$. Thus $x \in (a,(b,c)^l)^u$.

Case (ii). If $x \geq b,c$. Now, we have to prove that $x \in (a,(b,c)^l)^u$, i.e., $x \geq \{a\} \cup (b,c)^l$ (i.e., $x \geq a$ and $x \geq t$ for any $t \in (b,c)^l$). Let $t \in (b,c)^l$. Then $t \leq b, t \leq c$ and hence $x \geq c \geq t$. Therefore $x \geq t$. Hence $x \geq a$ and $x \geq t$. Thus $x \in (a,(b,c)^l)^u$.

Case (iii). If $x \geq a,b$. Now, we have to prove that $x \in (a,(b,c)^l)^u$, i.e., $x \geq \{a\} \cup (b,c)^l$ (i.e., $x \geq a$ and $x \geq t$ for any $t \in (b,c)^l$). Let $t \in (b,c)^l$. Then $t \leq b, t \leq c$ and hence $x \geq b \geq t$. Therefore $x \geq t$. Hence $x \geq a$ and $x \geq t$. Thus $x \in (a,(b,c)^l)^u$. Therefore $(a,(b,c)^l)^{ul} \subseteq ((a,b)^u \cup (b,c)^u \cup (c,a)^u)^l$. ■

Lemma 3.9. *If P is a poset, and $a \leq c$, then $((a,b)^u,c)^l \supseteq ((a,b)^l,(b,c)^l,(c,a)^l)^{ul}$.*

Proof. Now, we have to prove that $((a,b)^u, c) \subseteq ((a,b)^l, (b,c)^l, (c,a)^l)^u$. Let $x \in ((a,b)^u, c)$. Then $x \in (a,b)^u \cup \{c\}$. Thus $x \in (a,b)^u$ or $x = c$. Now, we have to prove that $x \geq t$ for any $t \in (a,b)^l \cup (b,c)^l \cup (c,a)^l$. If $x = c$. Suppose $t \in (a,b)^l$. Then $t \leq a, t \leq b$.Therefore $t \leq a \leq c$.Hence $t \leq c = x$. Thus $x \geq t$. If $t \in (b,c)^l$, then $t \leq b, t \leq c$. Therefore $x \geq t$. If $t \in (c,a)^l$, then $t \leq c, t \leq a$. Hence $t \leq c = x$. Therefore $x \geq t$ for all $t \in (a,b)^l \cup (b,c)^l \cup (c,a)^l$. Thus $x \in ((a,b)^l \cup (b,c)^l \cup (c,a)^l)^u$. Suppose $x \in (a,b)^u$. Then $x \geq a, b$. Now, we have to prove that $x \in ((a,b)^l, (b,c)^l, (c,a)^l)^u$, i.e., $x \geq t$ for any $t \in (a,b)^l \cup (b,c)^l \cup (c,a)^l$. If $t \in (a,b)^l$, then $t \leq a, t \leq b$ and hence $x \geq a$ and $a \geq t$. Therefore $x \geq t$. If $t \in (b,c)^l$, then $t \leq b, t \leq c$ and hence $x \geq b \geq t$. Therefore $x \geq t$. If $t \in (c,a)^l$, then $t \leq c, t \leq a$ and hence $x \geq a \geq t$. Therefore $x \geq t$. Thus $x \geq t$ for any $t \in (a,b)^l \cup (b,c)^l \cup (c,a)^l$. Hence $x \in ((a,b)^l, (b,c)^l, (c,a)^l)^u$. Thus, we have $((a,b)^u, c) \subseteq ((a,b)^l, (b,c)^l, (c,a)^l)^u$. Hence $((a,b)^l, (b,c)^l, (c,a)^l)^{ul} \subseteq ((a,b)^u, c)^l$. Thus, we get that if P is a modular poset and $a \leq c$, then $((a,b)^u, (b,c)^u, (c,a)^u)^l \subseteq ((a,b)^l, (b,c)^l, (c,a)^l)^{ul}$. ∎

Note. If P is a modular Poset, then for any $a \leq c$,we have from the Definition 3.1, Lemmas 3.5, 3.8, 3.9, $((a,b)^u, (b,c)^u, (c,a)^u)^l \supseteq (a, (b,c)^l)^{ul} = ((a,b)^u, c)^l \supseteq ((a,b)^l, (b,c)^l, (c,a)^l)^{ul}$.

Theorem 3.10. *Let P be a poset. Then P is modular if and only if $a \geq c$ implies $(a, (b,c)^u)^{lu} = ((a,b)^l, c)^u$ for all $a, b, c \in P$.*

Proof. Suppose P is a modular poset. Let $a \geq c$. We have $(c, (b,a)^l)^{ul} = ((c,b)^u, a)^l$. Therefore $(c, (a,b)^l)^{ulu} = (a, (b,c)^u)^{lu}$. Hence $(c, (a,b)^l)^u = (a, (b,c)^u)^{lu}$. Thus $(a, (b,c)^u)^{lu} = ((a,b)^l, c)^u$ for $a, b, c \in P$. Conversely, assume that $a \geq c$ implies $(a, (b,c)^u)^{lu} = ((a,b)^l, c)^u$ for $a, b, c \in P$. Now, we have to prove that $x \leq z$ implies $(x, (y,z)^l)^{ul} = ((x,y)^u, z)^l$ for all $x, y, z \in P$. Let $x \leq z$. Then $(z, (y,x)^u)^{lu} = ((z,y)^l, x)^{ul}$. Therefore $(z, (x,y)^u)^{lul} = ((y,z)^l, x)^{ul}$ and hence $(z, (x,y)^u)^l = ((y,z)^l, x)^{ul}$. Thus,we have $(x, (y,z)^l)^{ul} = ((x,y)^u, z)^l$ for all $x, y, z \in P$. ∎

Theorem 3.11. *Let P be a poset. Then P is a modular poset if and only if $a, b, c \in P$, $a \in c^l, c^l \subseteq (a,b)^u$, then $(a, (b,c)^l)^{ul} = c^l$.*

Proof. Suppose P is a poset. Assume that P is a modular poset and $a \leq c$. Let $a \in c^l, c^l \subseteq (a,b)^u$. Then $(a, (b,c)^l)^{ul} = ((a,b)^u, c)^l$. Now, we have to prove that $((a,b)^u, c)^l = c^l$. Let $x \in ((a,b)^u, c)^l$. Then $x \leq c, x \leq t$ for any $t \in (a,b)^u$. Therefore $x \in c^l$. Hence $((a,b)^u, c)^l \subseteq c^l$. suppose $x \in c^l$. Then $x \leq c, c^l \subseteq (a,b)^u$ and hence $x \in ((a,b)^u, c)^l$. Therefore $c^l \subseteq ((a,b)^u, c)^l$. Thus $((a,b)^u, c)^l = c^l$. Conversely, assume that $a \leq c$. Now, we have to prove that $(a, (b,c)^l)^{ul} = ((a,b)^u, c)^l$. We know that $a \in ((a,b)^u, c)^l$, so that $((a,b)^u, c)^l \subseteq (a,b)^u$. By the above condition, we get that $(a, (b, ((a,b)^u, c)^l))^{ul} = ((a,b)^u, c)^l$.

Now, we prove that $(a,(b,((a,b)^u,c)^l)) = (a,(b,c)^l)$. It is enough to prove that $(b,((a,b)^u,c)^l) = (b,c)^l$. Let $x \in (b,((a,b)^u,c)^l)$. Then $x \leq b, x \leq t$ for any $t \in ((a,b)^u,c)$ and hence $x \leq b, x \leq t, t \leq c$. Therefore $x \leq b, x \leq c$. Hence $x \in (b,c)^l$. Thus $(b,((a,b)^u,c)^l) \subseteq (b,c)^l$. On the other hand $x \in (b,c)^l$. Then $x \leq b, x \leq c$. Now, we have to prove that $x \in ((a,b)^u,c)^l$. Suppose $t \in (a,b)^u$. Then $t \geq a, t \geq b$ and hence $t \geq b$. Therefore $x \leq b \leq t$ (since $x \leq b$). Hence $x \leq t$. Thus $x \leq c$ and $x \leq t$ for any $t \in (a,b)^u$. Therefore $x \in (c,(a,b)^u)^l$. Hence $(b,c)^l \subseteq (b,((a,b)^u,c)^l)$. Thus we get that $(b,((a,b)^u,c)^l) = (b,c)^l$. Therefore $(a,((a,b)^u,c)^l) = (a,(b,c)^l)$. ∎

Theorem 3.12. *Let P be a poset. Then P is a modular poset if and only if $a,b,c \in P, (a,b)^l \subseteq c^u$, $c^u \subseteq a^u$, then $(a,(b,c)^u)^{lu} = c^u$.*

Proof. Suppose P is a poset. Assume that P is a modular poset and $a \leq c$. Let $(a,b)^l \subseteq c^u$, $c^u \subseteq a^u$. Then $(a,(b,c)^u)^{lu} = ((a,b)^l,c)^u$. Now, we have to prove that $((a,b)^l,c)^u = c^u$. Let $x \in ((a,b)^l,c)^u$. Then $x \geq c, x \geq t$ for any $t \in (a,b)^l$ and hence $x \in c^u$. Therefore $((a,b)^l,c)^u \subseteq c^u$. Suppose $x \in c^u$. Then $x \geq c, c^u \supseteq (a,b)^l$ and hence $x \in ((a,b)^l,c)^u$. Therefore $c^u \supseteq ((a,b)^l,c)^u$. Thus $((a,b)^l,c)^u = c^u$. Conversely, assume that $a \leq c$. Now, we have to prove that $(a,(b,c)^u)^{lu} = ((a,b)^l,c)^u$. We have $(a,b)^l \subseteq ((a,b)^l,c)^u$. By the above condition we get that $(a,(b,((a,b)^l,c)^u)^{lu} = ((a,b)^l,c)^u$. Now we prove $(a,(b,((a,b)^l,c)))^u = (a,(b,c)^u)$. It is enough to prove that $(b,((a,b)^l,c))^u = (b,c)^u$. Let $x \in (b,((a,b)^l,c))^u$. Then $x \geq b, x \geq t$ for any $t \in ((a,b)^l,c)$ and hence $x \geq b, x \geq t, t \geq c$. Therefore $x \geq b, x \geq c$. Hence $x \in (b,c)^u$. Thus $(b,((a,b)^l,c))^u \subseteq (b,c)^u$. On the other hand,let $x \in (b,c)^u$. Then $x \geq b, x \geq c$. Now, we have to prove that $x \in ((a,b)^l,c)^u$ i.e., $x \geq c$ and $x \geq t$ for any $t \in (a,b)^l$. Suppose $t \in (a,b)^l$. Then $t \leq a, t \leq b$ and hence $t \geq b$. We have $x \geq b$, so that $x \geq b \geq t$. Therefore $x \geq t$. Hence $x \leq c$ and $x \geq t$ for any $t \in (a,b)^l$. Thus $(b,c)^u \subseteq (b,((a,b)^l,c)^u)$. Therefore $(b,((a,b)^l,c))^u = (b,c)^u$. Thus $(a,(b,((a,b)^l,c))^u) = (a,(b,c)^u)$. ∎

Theorem 3.13. *Let$(P,\leq)$ be a poset then P is modular poset if and only if $a \geq c$ implies $(a,(b,c)^u)^{lu} \subseteq ((a,b)^l,c)^u$.*

Proof. Suppose P is a modular poset. Let $a \geq c$. Then $(a,(b,c)^u)^{lu} = ((a,b)^l,c)^u$. Therefore $(a,(b,c)^u)^{lu} \subseteq ((a,b)^l,c)^u$. Conversely, assume that $a \geq c$ implies $(a,(b,c)^u)^{lu} \subseteq ((a,b)^l,c)^u$. Now, we have to prove that $(a,(b,c)^u)^l \supseteq ((a,b)^l,c)$. Let $z \in (a,b)^l$. Then $z \leq a, z \leq b$ and hence $z \leq a, z \leq b \in (b,c)^u$. Therefore $z \in (a,(b,c)^u)^l$. Thus $(a,b)^l \subseteq (a,(b,c)^u)^l$. Now, we prove that $((a,b)^l,c) \subseteq (a,(b,c)^u)^l$. Let $z \in ((a,b)^l,c)$. Then $z \in (a,b)^l$ or $z = c$. If $z = c$, then $c \in (a,(b,c)^u)^l$. If $t \in (b,c)^u$, then $t \geq b, t \geq c$. Therefore $c \in (a,(b,c)^u)^l$. Hence $((a,b)^l,c) \subseteq (a,(b,c)^u)^l$. Therefore, by poset property, $A \subseteq B$ implies $A^u \supseteq B^u$, we get that $((a,b)^l,c)^u \supseteq (a,(b,c)^u)^{lu}$. Thus we get $(a,(b,c)^u)^{lu} = ((a,b)^l,c)^u$. ∎

Lemma 3.14. (A) $(a,(a,c)^l)^{ul} = a^{ul}$.

Proof. Let $x \in (a,(a,c)^l)^u$. Then $x \geq a, x \geq t$ for any $t \in (a,c)^l$ and hence $x \geq a, x \geq t$. Therefore $t \leq a \leq x, a \leq x$. Hence $x \geq a$. Therefore $x \in a^u$, so that $(a,(a,c)^l)^u \subseteq a^u$. Thus $(a,(a,c)^l)^{ul} \supseteq a^{ul}$. Let $x \in a^u$, Then $x \geq a$. Now, we have to prove that $x \geq t$ for any $t \in (a,c)^l$. Suppose $t \in (a,c)^l$. Then $t \leq a, t \leq c$ and hence $x \leq a \leq t$. Therefore $x \geq t$. Thus $x \in (a,(a,c)^l)^u$, so that $a^u \subseteq (a,(a,c)^l)^u$. Therefore $a^u = (a,(a,c)^l)^u$. ■

The following has similar proof as above lemma.

Lemma 3.15. (B) $(a,(a,c)^u)^{lu} = a^{lu}$.

Proof. Let $x \in (a,(a,c)^u)^l$. Then $x \leq a, x \leq t$ for any $t \in (a,c)^u$ and hence $t \leq a, x \leq t$. Therefore $x \geq a \geq x$, so that $x \leq a$. Hence $x \in a^l$. Thus $(a,(a,c)^u)^l \subseteq a^l$, so that $(a,(a,c)^u)^{lu} \supseteq a^{lu}$. Let $x \in a^l$. Then $x \leq a$. Now, we have to prove that $x \leq t$ for any $t \in (a,c)^u$. Suppose that $t \in (a,c)^u$. Then $t \geq a, t \geq c$ and hence $t \geq a \geq x$, so that $t \geq x$. Therefore $x \in (a,(a,c)^u)^l$. Hence $a^l \subseteq (a,(a,c)^u)^l$. Thus we have $(a,(a,c)^u)^l = a^l$. ■

Remark 3.16. Let P be a modular poset such that if $(a,c)^u = (b,c)^u, (a,c)^l = (b,c)^l$ and $a \leq b$ then $a^{ul} = b^l$.

Proof. Let $a, b, c \in P$. Suppose $(a,c)^u = (b,c)^u, (a,c)^l = (b,c)^l$ and $a \leq b$. Now $a \leq b$ *implies* $(a,(c,b)^l)^{ul} = ((a,c)^u, b)^l$. Then $(a,(a,c)^l)^{ul} = ((b,c)^u, b)^l$. By known Lemmas $(A), (B)$, we get $a^{ul} = b^l$. ■

Remark 3.17. If $a^{ul} = b^l$ and $b^{ul} = a^l$, then $a = b$.

Proof. We know that, $a \in a^{ul} = b^l$, then $a \leq b$. Similarly, if $b \in b^{ul} = a^l$, then $b \leq a$. Therefore $a = b$. ■

Remark 3.18. If $a^{lu} = b^u$ and $b^{lu} = a^u$, then $a = b$.

Proof. We know that, $a \in a^{lu} = b^u$, then $a \geq b$. Similarly, if $b \in b^{lu} = a^u$, then $b \geq a$. Therefore $a = b$. ■

Finally we conclude with the following theorem.

Theorem 3.19. *Let* $(P, \leq)$ *be a modular poset, then* $a \geq c$ *implies* $((a,b)^l, (b,c)^l, (c,a)^l)^u \supseteq ((a,b)^u, (b,c)^u, (c,a)^u)^{lu}$.

Proof. Assume P is a modular poset. We prove that, $a \geq c$ implies $(a,b)^l, (b,c)^l, (c,a)^l)^u \supseteq ((a,b)^u, (b,c)^u, (c,a)^u)^{lu}$. It is enough to prove that $((a,b)^l, (b,c)^l, (c,a)^l) \subseteq ((a,b)^u, (b,c)^u, (c,a)^u)^l$ for $a \geq c$. Now, assume $a \geq c$. Let $z \in (a,b)^l \cup (b,c)^l \cup (c,a)^l$. Then $z \in (a,b)^l$ or $z \in (b,c)^l$ or $z \in (c,a)^l$, so that $z \leq a, b$ or

$z \leq b, c$ or $z \leq c, a$. Now, we have to prove that if $z \in ((a,b)^u, (b,c)^u, (c,a)^u)^l$ then, $z \leq t$ for any $t \in ((a,b)^u, (b,c)^u, (c,a)^u)$. Suppose $t \in (a,b)^u \cup (b,c)^u \cup (c,a)^u$. Then $t \in (a,b)^u$ or $t \in (b,c)^u$ or $t \in (c,a)^u$.

Case (i). If $z \leq a, b$. Suppose $t \in (a,b)^u$. Then $t \geq a, b$ and hence $t \geq a \geq z$. Therefore $z \leq t$. Suppose $t \in (b,c)^u$. Then $t \geq b, c$ and hence $z \leq b \leq t$. Therefore $z \leq t$. Suppose $t \in (c,a)^u$. Then $t \geq c, a$ and hence $z \leq a \leq t$. Therefore $z \leq t$. Thus we have, if $t \in (a,b)^u \cup (b,c)^u \cup (c,a)^u$, then $z \leq t$.

Case (ii). If $z \leq b, c$. Suppose $t \in (a,b)^u$. Then $t \geq a, t \geq b$ and hence $z \leq b \leq t$. Therefore $z \leq t$. Suppose $t \in (b,c)^u$. Then $t \geq b, t \geq c$ and hence $z \leq b \leq t$. Therefore $z \leq t$. Suppose $t \in (c,a)^u$. Then $t \geq c, t \geq a$ and hence $z \leq c \leq t$. Therefore $z \leq t$. Thus we have, if $t \in (a,b)^u \cup (b,c)^u \cup (c,a)^u$, then $z \leq t$.

Case (iii). If $z \leq c, a$. Suppose $t \in (a,b)^u$. Then $t \geq a$, $t \geq b$ and hence $z \leq a \leq t$. Therefore $z \leq t$. Suppose $t \in (b,c)^u$. Then $t \geq b, t \geq c$ and hence $z \leq c \leq t$. Therefore $z \leq t$. Suppose $t \in (c,a)^u$. Then $t \geq c, t \geq a$ and hence $z \leq t$. Therefore $z \leq t$. Thus we have, if $t \in ((a,b)^u, (b,c)^u, (c,a)^u)^{lu}$, then $z \leq t$.

Thus we get $((a,b)^l, (b,c)^l, (c,a)^l) \subseteq ((a,b)^u, (b,c)^u, (c,a)^u)^l$ for $a \geq c$. Hence, if $a \geq c \Rightarrow ((a,b)^l, (b,c)^l, (c,a)^l)^u \supseteq ((a,b)^u, (b,c)^u, (c,a)^u)^{lu}$. ■

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