Discussiones Mathematicae
General Algebra and Applications 41 (2021) 139–152
doi:10.7151/dmgaa.1349

INTRODUCTION TO THIRD-ORDER JACOBSTHAL AND MODIFIED THIRD-ORDER JACOBSTHAL HYBRINOMIALS

Gamaliel Cerda-Morales

Instituto de Matemáticas
Pontificia Universidad Católica de Valparaíso
Blanco Viel 596, Cerro Barón, Valparaíso, Chile

e-mail: gamaliel.cerda.m@mail.pucv.cl

Abstract

The hybrid numbers are generalization of complex, hyperbolic and dual numbers. In this paper, we introduce and study the third-order Jacobsthal and modified third-order Jacobsthal hybrinomials, i.e., polynomials, which are a generalization of the Jacobsthal hybrid numbers and the Jacobsthal-Lucas hybrid numbers, respectively.

Keywords: third-order Jacobsthal numbers, recurrence relations, complex numbers, hyperbolic numbers, dual numbers, polynomials.

2010 Mathematics Subject Classification: 11B37, 11B39.

1. Introduction

Let $J_n^{(3)}$ be the n-th third-order Jacobsthal number defined recursively by

$$J_n^{(3)} = J_{n-1}^{(3)} + J_{n-2}^{(3)} + 2J_{n-3}^{(3)}, \quad n \geq 3,$$

for $n \geq 3$ with the initial terms $J_0^{(3)} = 0$, $J_1^{(3)} = J_2^{(3)} = 1$.

The n-th modified third-order Jacobsthal number $K_n^{(3)}$ is defined recursively by $K_n^{(3)} = K_{n-1}^{(3)} + K_{n-2}^{(3)} + 2K_{n-3}^{(3)}$ for $n \geq 3$ with the initial terms $K_0^{(3)} = 3$, $K_1^{(3)} = 1$, $K_2^{(3)} = 3$.

The direct formulas for the n-th third-order Jacobsthal number and the n-th modified third-order Jacobsthal number are named as Binet formulas and have the form

$$J_n^{(3)} = \frac{2^{n+1}}{7} - \frac{\omega_1^{n+1}}{(2-\omega_1)(\omega_1-\omega_2)} + \frac{\omega_2^{n+1}}{(2-\omega_2)(\omega_1-\omega_2)},$$

$$K_n^{(3)} = 2^n + \omega_1^n + \omega_2^n,$$

where $\omega_1 + \omega_2 = -1$ and $\omega_1\omega_2 = 1$ (see [3–5]).

For any variable quantity x such that $x^2+x+1 \neq 0$, the third-order Jacobsthal polynomial $J_n^{(3)}(x)$ is defined as $J_n^{(3)}(x) = (x-1)J_{n-1}^{(3)}(x) + (x-1)J_{n-2}^{(3)}(x) + xJ_{n-3}^{(3)}(x)$ for $n \geq 3$ with $J_0^{(3)}(x) = 0$, $J_1^{(3)}(x) = 1$, $J_2^{(3)}(x) = x-1$.

The modified third-order Jacobsthal polynomial $K_n^{(3)}(x)$ is defined as $K_n^{(3)}(x) = (x-1)K_{n-1}^{(3)}(x) + (x-1)K_{n-2}^{(3)}(x) + xK_{n-3}^{(3)}(x)$ for $n \geq 3$ with the initial terms $K_0^{(3)}(x) = 3$, $K_1^{(3)}(x) = x-1$, $K_2^{(3)}(x) = x^2-1$.

For $x = 2$, the third-order Jacobsthal and modified third-order Jacobsthal polynomials give the third-order Jacobsthal and modified third-order Jacobsthal numbers, respectively.

Based on the properties of sequences defined by the third-order linear recurrence relations, we can give direct formulas for $J_n^{(3)}(x)$ and $K_n^{(3)}(x)$. Then,

$$J_n^{(3)}(x) = \frac{x^{n+1}}{x^2+x+1} - \frac{\omega_1^{n+1}}{(x-\omega_1)(\omega_1-\omega_2)} + \frac{\omega_2^{n+1}}{(x-\omega_2)(\omega_1-\omega_2)} \tag{1}$$

and

$$K_n^{(3)}(x) = x^n + \omega_1^n + \omega_2^n, \tag{2}$$

where $\omega_1 = \frac{-1+i\sqrt{3}}{2}$ and $\omega_2 = \frac{-1-i\sqrt{3}}{2}$. Equations (1) and (2) are named as Binet formulas for the third-order Jacobsthal and modified third-order Jacobsthal polynomials, respectively.

Cook was a pioneer in studying the third-order Jacobsthal numbers, see for details [6]. Many papers have studied properties of the third-order Jacobsthal type numbers, for example [1–5]. Note that the third-order Jacobsthal numbers are related to the Jacobsthal numbers. These numbers have many applications in algebra, geometry, numbers theory and other branches of mathematics, see [7–10].

The hybrid numbers were introduced by Özdemir in [11] as a new generalization of complex, hyperbolic and dual numbers. Let $\mathbb{K}$ be the set of hybrid numbers $\mathbf{H}$ of the form

$$\mathbf{H} = a + b\mathbf{i} + c\varepsilon + d\mathbf{h},$$

where $a, b, c, d \in \mathbb{R}$ and $\mathbf{i}$, ε, $\mathbf{h}$ are operators such that

$$\mathbf{i}^2 = -\mathbf{1}, \quad \varepsilon^2 = 0, \quad \mathbf{h}^2 = \mathbf{1}$$

and

$$\mathbf{ih} = -\mathbf{hi} = \varepsilon + \mathbf{i}.$$

The hybrid numbers multiplication is defined using above equations (see Table 1). Note that using above formulas we can find the product of any two hybrid units. Then, the multiplication of hybrid numbers can be made analogously as multiplications of algebraic expressions. Addition operation in the hybrid numbers is both commutative and associative. Zero $\mathbf{0} = 0 + 0\mathbf{i} + 0\varepsilon + 0\mathbf{h}$ is the null element. With respect to the addition operation, the inverse element of $\mathbf{H}$ is $-\mathbf{H} = -a - b\mathbf{i} - c\varepsilon - d\mathbf{h}$. The multiplication is not commutative, but associative. Moreover, $(\mathbb{K}, +, \cdot)$ is non-commutative ring (with identity element $\mathbf{1} = 1 + 0\mathbf{i} + 0\varepsilon + 0\mathbf{h}$).

Table 1. The multiplication table for the basis of $\mathbb{K}$.

$\times$	$\mathbf{1}$	$\mathbf{i}$	ε	$\mathbf{h}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{i}$	ε	$\mathbf{h}$
$\mathbf{i}$	$\mathbf{i}$	$-\mathbf{1}$	$\mathbf{1}-\mathbf{h}$	$\varepsilon+\mathbf{i}$
ε	ε	$\mathbf{1}+\mathbf{h}$	0	$-\varepsilon$
$\mathbf{h}$	$\mathbf{h}$	$-(\varepsilon+\mathbf{i})$	ε	$\mathbf{1}$

A special kind of hybrid numbers, namely Horadam numbers, were introduced in [12]. Interesting results of the Fibonacci and Lucas hybrid numbers obtained recently can be found in [13]. Furthermore, some identities of Jacobshal and Jacobsthal-Lucas hybrid numbers can be found in [14] and Fibonacci and Lucas hybrinomials in [15].

In this paper, we introduce the third-order Jacobsthal and modified third-order Jacobsthal hybrinomials, i.e., polynomials, which can be considered as a generalization of the third-order Jacobsthal hybrid numbers and the modified third-order Jacobsthal hybrid numbers.

For $n \geq 0$, the third-order Jacobsthal and modified third-order Jacobsthal hybrinomials are defined by

$$JH_n^{(3)}(x) = J_n^{(3)}(x) + \mathbf{i}J_{n+1}^{(3)}(x) + \varepsilon J_{n+2}^{(3)}(x) + \mathbf{h}J_{n+3}^{(3)}(x) \tag{3}$$

and

$$KH_n^{(3)}(x) = K_n^{(3)}(x) + \mathbf{i}K_{n+1}^{(3)}(x) + \varepsilon K_{n+2}^{(3)}(x) + \mathbf{h}K_{n+3}^{(3)}(x), \tag{4}$$

where $J_n^{(3)}(x)$ is the n-th third-order Jacobsthal polynomial, $K_n^{(3)}(x)$ is the n-th modified third-order Jacobsthal polynomial and $\mathbf{i}$, ε, $\mathbf{h}$ are hybrid units.

For $x = 2$, we obtain the third-order Jacobsthal hybrid numbers and the modified third-order Jacobsthal hybrid numbers, respectively.

2. Main results

Theorem 1. *For any variable quantity x such that $x^2+x+1\neq 0$ and $n\geq 3$, we have*

$$JH_n^{(3)}(x)=(x-1)JH_{n-1}^{(3)}(x)+(x-1)JH_{n-2}^{(3)}(x)+xJH_{n-3}^{(3)}(x), \tag{5}$$

with

$$\begin{aligned}
JH_0^{(3)}(x)&=\mathbf{i}+\varepsilon\cdot(x-1)+\mathbf{h}\cdot(x^2-x),\\
JH_1^{(3)}(x)&=1+\mathbf{i}\cdot(x-1)+\varepsilon\cdot(x^2-x)+\mathbf{h}\cdot(x^3-x^2+1),\\
JH_2^{(3)}(x)&=x-1+\mathbf{i}\cdot(x^2-x)+\varepsilon\cdot(x^3-x^2+1)+\mathbf{h}\cdot(x^4-x^3+x-1),
\end{aligned}$$

where $\mathbf{i}$*,* ε*,* $\mathbf{h}$ *are hybrid units.*

Proof. If $n=3$, we have

$$\begin{aligned}
JH_3^{(3)}(x)&=(x-1)JH_2^{(3)}(x)+(x-1)JH_1^{(3)}(x)+xJH_0^{(3)}(x)\\
&=(x-1)^2+\mathbf{i}\cdot x(x-1)^2+\varepsilon\cdot(x^3-x^2+1)(x-1)\\
&+\mathbf{h}\cdot(x^3+1)(x-1)^2+(x-1)+\mathbf{i}\cdot(x-1)^2+\varepsilon\cdot x(x-1)^2\\
&+\mathbf{h}\cdot(x^3-x^2+1)(x-1)+\mathbf{i}\cdot x+\varepsilon\cdot x(x-1)+\mathbf{h}\cdot x^2(x-1)\\
&=x^2-x+\mathbf{i}\cdot(x^3-x^2+1)+\varepsilon\cdot(x^4-x^3+x-1)\\
&+\mathbf{h}\cdot(x^5-x^4+x^2-x)\\
&=J_3^{(3)}(x)+\mathbf{i}J_4^{(3)}(x)+\varepsilon J_5^{(3)}(x)+\mathbf{h}J_6^{(3)}(x).
\end{aligned}$$

If $n\geq 4$, then using the definition of the third-order Jacobsthal polynomials, we have

$$\begin{aligned}
JH_n^{(3)}(x)&=J_n^{(3)}(x)+\mathbf{i}J_{n+1}^{(3)}(x)+\varepsilon J_{n+2}^{(3)}(x)+\mathbf{h}J_{n+3}^{(3)}(x)\\
&=(x-1)J_{n-1}^{(3)}(x)+(x-1)J_{n-2}^{(3)}(x)+xJ_{n-3}^{(3)}(x)\\
&+\mathbf{i}\cdot\Big((x-1)J_n^{(3)}(x)+(x-1)J_{n-1}^{(3)}(x)+xJ_{n-2}^{(3)}(x)\Big)\\
&+\varepsilon\cdot\Big((x-1)J_{n+1}^{(3)}(x)+(x-1)J_n^{(3)}(x)+xJ_{n-1}^{(3)}(x)\Big)\\
&+\mathbf{h}\cdot\Big((x-1)J_{n+2}^{(3)}(x)+(x-1)J_{n+1}^{(3)}(x)+xJ_n^{(3)}(x)\Big)\\
&=(x-1)\cdot JH_{n-1}^{(3)}(x)+(x-1)\cdot JH_{n-2}^{(3)}(x)+x\cdot JH_{n-3}^{(3)}(x),
\end{aligned}$$

which ends the proof. ∎

In the same way, we obtain the next result for modified third-order Jacobsthal hybrinomials.

Theorem 2. *For any variable quantity x such that $x^2+x+1\neq 0$, we have*

$$KH_n^{(3)}(x)=(x-1)KH_{n-1}^{(3)}(x)+(x-1)KH_{n-2}^{(3)}(x)+xKH_{n-3}^{(3)}(x),\ n\geq 3, \tag{6}$$

with $KH_0^{(3)}(x)=3+\mathbf{i}\cdot(x-1)+\varepsilon\cdot(x^2-1)+\mathbf{h}\cdot(x^3+2)$, $KH_1^{(3)}(x)=x-1+\mathbf{i}\cdot(x^2-1)+\varepsilon\cdot(x^3+2)+\mathbf{h}\cdot(x^4-1)$ *and* $KH_2^{(3)}(x)=x^2-1+\mathbf{i}\cdot(x^3+2)+\varepsilon\cdot(x^4-1)+\mathbf{h}\cdot(x^5-1)$.

Now we give the Binet formulas for the third-order Jacobsthal and modified third-order Jacobsthal hybrinomials.

Theorem 3 (Binet formulas)**.** *For any variable quantity x such that $x^2+x+1\neq 0$ and $n\geq 0$ be an integer. Then,*

$$\begin{aligned} JH_n^{(3)}(x)=&\frac{x^{n+1}}{x^2+x+1}(1+x\mathbf{i}+x^2\varepsilon+x^3\mathbf{h})\\ &-\frac{\omega_1^{n+1}}{(x-\omega_1)(\omega_1-\omega_2)}(1+\omega_1\mathbf{i}+\omega_2\varepsilon+\mathbf{h})\\ &+\frac{\omega_2^{n+1}}{(x-\omega_2)(\omega_1-\omega_2)}(1+\omega_2\mathbf{i}+\omega_1\varepsilon+\mathbf{h}), \end{aligned} \tag{7}$$

$$\begin{aligned} KH_n^{(3)}(x)=&x^n(1+x\mathbf{i}+x^2\varepsilon+x^3\mathbf{h})+\omega_1^n(1+\omega_1\mathbf{i}+\omega_2\varepsilon+\mathbf{h})\\ &+\omega_2^n(1+\omega_2\mathbf{i}+\omega_1\varepsilon+\mathbf{h}), \end{aligned} \tag{8}$$

where $\omega_1=\frac{-1+i\sqrt{3}}{2}$ *and* $\omega_2=\frac{-1-i\sqrt{3}}{2}$.

Proof. Using Equations (2) and (4), we have

$$\begin{aligned} KH_n^{(3)}(x)&=K_n^{(3)}(x)+\mathbf{i}K_{n+1}^{(3)}(x)+\varepsilon K_{n+2}^{(3)}(x)+\mathbf{h}K_{n+3}^{(3)}(x)\\ &=x^n+\omega_1^n+\omega_2^n+\mathbf{i}\left(x^{n+1}+\omega_1^{n+1}+\omega_2^{n+1}\right)\\ &+\varepsilon\left(x^{n+2}+\omega_1^{n+2}+\omega_2^{n+2}\right)+\mathbf{h}\left(x^{n+3}+\omega_1^{n+3}+\omega_2^{n+3}\right) \end{aligned}$$

and after calculations result (8) follows. In the same way, using Equations (1) and (3), we obtain Binet formula (7) for the third-order Jacobsthal hybrinomials. ■

Theorem 4. *For $n\geq 2$, we have*

$$KH_n^{(3)}(x)=(x-1)JH_n^{(3)}(x)+2(x-1)JH_{n-1}^{(3)}(x)+3xJH_{n-2}^{(3)}(x).$$

Proof. Using Equation (4), we obtain

$$\begin{aligned}
KH_n^{(3)}(x) &= K_n^{(3)}(x) + \mathbf{i}K_{n+1}^{(3)}(x) + \varepsilon K_{n+2}^{(3)}(x) + \mathbf{h}K_{n+3}^{(3)}(x) \\
&= (x-1)J_n^{(3)}(x) + 2(x-1)J_{n-1}^{(3)}(x) + 3xJ_{n-2}^{(3)}(x) \\
&+ \mathbf{i}\left((x-1)J_{n+1}^{(3)}(x) + 2(x-1)J_n^{(3)}(x) + 3xJ_{n-1}^{(3)}(x)\right) \\
&+ \varepsilon\left((x-1)J_{n+2}^{(3)}(x) + 2(x-1)J_{n+1}^{(3)}(x) + 3xJ_n^{(3)}(x)\right) \\
&+ \mathbf{h}\left((x-1)J_{n+3}^{(3)}(x) + 2(x-1)J_{n+2}^{(3)}(x) + 3xJ_{n+1}^{(3)}(x)\right) \\
&= (x-1)JH_n^{(3)}(x) + 2(x-1)JH_{n-1}^{(3)}(x) + 3xJH_{n-2}^{(3)}(x).
\end{aligned}$$

Thus, the result follows. ■

Now, we will give some identities related to the well-known identities for the third-order Jacobsthal and modified third-order Jacobsthal numbers

$$\begin{aligned}
J_{n+1}^{(3)} \cdot J_{n-1}^{(3)} - \left(J_n^{(3)}\right)^2 &= \frac{1}{49}\left(2^n(8Z_n + 3Z_{n+1}) - 7\right) \\
&= \begin{cases} 2^n - 1 & \text{if } n \equiv 0 \pmod 3 \\ -3 \cdot 2^n - 1 & \text{if } n \equiv 1 \pmod 3 \\ 2^{n+1} - 1 & \text{if } n \equiv 2 \pmod 3, \end{cases}
\end{aligned}$$

$$\begin{aligned}
K_{n+1}^{(3)} \cdot K_{n-1}^{(3)} - \left(K_n^{(3)}\right)^2 &= -2^{n-1}(3Y_{n+1} + 8Y_n) - 3 \\
&= \begin{cases} -13 \cdot 2^{n-1} - 3 & \text{if } n \equiv 0 \pmod 3 \\ 11 \cdot 2^{n-1} - 3 & \text{if } n \equiv 1 \pmod 3 \\ 2^n - 3 & \text{if } n \equiv 2 \pmod 3, \end{cases}
\end{aligned}$$

where $Y_n = \omega_1^n + \omega_2^n$ and $Z_n = \frac{1}{\omega_1 - \omega_2}\left((-3 - 2\omega_2)\omega_1^n - (-3 - 2\omega_1)\omega_2^n\right)$.

We give their versions for the third-order Jacobsthal and modified third-order Jacobsthal hybrinomials.

For simplicity of notation, let

$$\begin{aligned}
\underline{x} &= 1 + x\mathbf{i} + x^2\varepsilon + x^3\mathbf{h}, \\
\underline{\omega_1} &= 1 + \omega_1\mathbf{i} + \omega_2\varepsilon + \mathbf{h}, \\
\underline{\omega_2} &= 1 + \omega_2\mathbf{i} + \omega_1\varepsilon + \mathbf{h}, \\
Z_{H,n} &= \frac{1}{\omega_1 - \omega_2}\left((x - \omega_2)\underline{\omega_1}\omega_1^{n+1} - (x - \omega_1)\underline{\omega_2}\omega_2^{n+1}\right), \\
Y_{H,n} &= \underline{\omega_1}\omega_1^n + \underline{\omega_2}\omega_2^n.
\end{aligned}$$

Then, we can write Equations (7) and (8) as

$$JH_n^{(3)}(x) = \frac{1}{x^2+x+1}\left(\underline{x}x^{n+1} - Z_{H,n}\right)$$

and

$$KH_n^{(3)}(x) = \underline{x}x^n + Y_{H,n}.$$

Lemma 5 (Cassini like-identity for the sequence $Z_{H,n}$)**.** *For $n \geq 2$, we have*

$$\begin{aligned} &Z_{H,n+1}^{(3)}(x) \cdot Z_{H,n-1}^{(3)}(x) - \left(Z_{H,n}^{(3)}(x)\right)^2 \\ &\qquad = -\frac{x^2+x+1}{3}\left(\underline{\omega_1\omega_2}(1-\omega_2) + \underline{\omega_2\omega_1}(1-\omega_1)\right). \end{aligned}$$

Proof.

$$\begin{aligned} &Z_{H,n+1}^{(3)}(x) \cdot Z_{H,n-1}^{(3)}(x) - \left(Z_{H,n}^{(3)}(x)\right)^2 \\ &= \frac{1}{(\omega_1-\omega_2)^2}\left(A\underline{\omega_1}\omega_1^{n+2} - B\underline{\omega_2}\omega_2^{n+2}\right)\left(A\underline{\omega_1}\omega_1^{n} - B\underline{\omega_2}\omega_2^{n}\right) \\ &- \frac{1}{(\omega_1-\omega_2)^2}\left(A\underline{\omega_1}\omega_1^{n+1} - B\underline{\omega_2}\omega_2^{n+1}\right)\left(A\underline{\omega_1}\omega_1^{n+1} - B\underline{\omega_2}\omega_2^{n+1}\right) \\ &= \frac{AB}{3}\left(\underline{\omega_1\omega_2}\omega_1^2 + \underline{\omega_2\omega_1}\omega_2^2 - \underline{\omega_1\omega_2} - \underline{\omega_2\omega_1}\right). \end{aligned}$$

Then, we have

$$\begin{aligned} &Z_{H,n+1}^{(3)}(x) \cdot Z_{H,n-1}^{(3)}(x) - \left(Z_{H,n}^{(3)}(x)\right)^2 \\ &\qquad = -\frac{AB}{3}\left(\underline{\omega_1\omega_2}(1-\omega_2) + \underline{\omega_2\omega_1}(1-\omega_1)\right) \\ &\qquad = -\frac{x^2+x+1}{3}\left(\underline{\omega_1\omega_2}(1-\omega_2) + \underline{\omega_2\omega_1}(1-\omega_1)\right), \end{aligned}$$

where $A = x - \omega_2$ and $B = x - \omega_1$. ∎

Theorem 6 (Cassini like-identity for the third-order Jacobsthal hybrinomials)**.** *Let $n \geq 0$, $r \geq 0$ be integers such that $n \geq r$. Then,*

$$\begin{aligned} &JH_{n+1}^{(3)}(x) \cdot JH_{n-1}^{(3)}(x) - \left(JH_n^{(3)}(x)\right)^2 \\ &= \frac{1}{(x^2+x+1)^2}\left(\underline{x}x^{n+1}(Z_{H,n} - xZ_{H,n-1}) + (xZ_{H,n} - Z_{H,n+1})\underline{x}x^n\right) \\ &- \frac{1}{3(x^2+x+1)}\left(\underline{\omega_1\omega_2}(1-\omega_2) + \underline{\omega_2\omega_1}(1-\omega_1)\right), \end{aligned}$$

where $Z_{H,n}$ is as in Lemma 5.

Proof. Applying Theorem 3 and Lemma 5, we have that

$$
\begin{aligned}
&JH_{n+1}^{(3)}(x)\cdot JH_{n-1}^{(3)}(x)-\left(JH_n^{(3)}(x)\right)^2\\
&\qquad=\frac{1}{(x^2+x+1)^2}\left(\underline{x}x^{n+2}-Z_{H,n+1}\right)\left(\underline{x}x^{n}-Z_{H,n-1}\right)\\
&\qquad-\frac{1}{(x^2+x+1)^2}\left(\underline{x}x^{n+1}-Z_{H,n}\right)\left(\underline{x}x^{n+1}-Z_{H,n}\right).
\end{aligned}
$$

Finally, we obtain

$$
\begin{aligned}
&JH_{n+1}^{(3)}(x)\cdot JH_{n-1}^{(3)}(x)-\left(JH_n^{(3)}(x)\right)^2\\
&=\frac{1}{(x^2+x+1)^2}\left(\underline{x}x^{n+1}(Z_{H,n}-xZ_{H,n-1})+(xZ_{H,n}-Z_{H,n+1})\underline{x}x^{n}\right)\\
&+\frac{1}{(x^2+x+1)^2}\left(Z_{H,n+1}^{(3)}(x)\cdot Z_{H,n-1}^{(3)}(x)-\left(Z_{H,n}^{(3)}(x)\right)^2\right)\\
&=\frac{1}{(x^2+x+1)^2}\left(\underline{x}x^{n+1}(Z_{H,n}-xZ_{H,n-1})+(xZ_{H,n}-Z_{H,n+1})\underline{x}x^{n}\right)\\
&-\frac{1}{3(x^2+x+1)}\left(\underline{\omega_1\omega_2}(1-\omega_2)+\underline{\omega_2\omega_1}(1-\omega_1)\right).
\end{aligned}
$$

■

Note that

$$
\begin{aligned}
Z_{H,n}&=\frac{1}{\omega_1-\omega_2}\left((x-\omega_2)\underline{\omega_1}\omega_1^{n+1}-(x-\omega_1)\underline{\omega_2}\omega_2^{n+1}\right)\\
&=x\left(\frac{\underline{\omega_1}\omega_1^{n+1}-\underline{\omega_2}\omega_1^{n+1}}{\omega_1-\omega_2}\right)-\left(\frac{\underline{\omega_1}\omega_1^{n}-\underline{\omega_2}\omega_1^{n}}{\omega_1-\omega_2}\right)\\
&=\begin{cases} x-(x+1)\mathbf{i}+\varepsilon+x\mathbf{h} & \text{if } n\equiv 0 \pmod 3\\ -(x+1)+\mathbf{i}+x\varepsilon-(x+1)\mathbf{h} & \text{if } n\equiv 1 \pmod 3\\ 1+x\mathbf{i}-(x+1)\varepsilon+\mathbf{h} & \text{if } n\equiv 2 \pmod 3). \end{cases}
\end{aligned}
$$

Next we shall give the generating function for the third-order Jacobsthal hybrinomials.

Theorem 7. *The generating function for the third-order Jacobsthal hybrinomial sequence* $\left(JH_n^{(3)}(x)\right)_{n\geq 0}$ *is*

$$
j(t)=\frac{\left\{\begin{array}{c}\mathbf{i}+\varepsilon\cdot(x-1)+\mathbf{h}\cdot(x^2-x)+\left(1+\varepsilon\cdot(x-1)+\mathbf{h}\cdot(x^2-x+1)\right)t\\ +\left(\varepsilon\cdot x+\mathbf{h}\cdot(x^2-x)\right)t^2\end{array}\right\}}{1-(x-1)t-(x-1)t^2-xt^3}.
$$

Proof. Assume that the generating function of the third-order Jacobsthal hybrinomial sequence $\left(JH_n^{(3)}(x)\right)_{n\geq 0}$ has the form $j(t)=\sum_{n=0}^{\infty} JH_n^{(3)} t^n$. Then,

$$j(t) = JH_0^{(3)} + JH_1^{(3)}t + JH_2^{(3)}t^2 + \cdots .$$

Multiply the above equality on both sides by $-(x-1)t$, $-(x-1)t^2$ and then by $-xt^3$, we obtain

$$\begin{aligned}
-(x-1)tj(t) &= -(x-1)JH_0^{(3)}t - (x-1)JH_1^{(3)}t^2 - (x-1)JH_2^{(3)}t^3 - \cdots \\
-(x-1)t^2 j(t) &= -(x-1)JH_0^{(3)}t^2 - (x-1)JH_1^{(3)}t^3 - (x-1)JH_2^{(3)}t^4 - \cdots \\
-xt^3 j(t) &= -xJH_0^{(3)}t^3 - xJH_1^{(3)}t^4 - xJH_2^{(3)}t^5 - \cdots .
\end{aligned}$$

By adding the four equalities above, we will get

$$\begin{aligned}
&j(t)(1-(x-1)t-(x-1)t^2-xt^3) \\
&= JH_0^{(3)} + \left(JH_1^{(3)} - (x-1)JH_0^{(3)}\right)t \\
&+ \left(JH_2^{(3)} - (x-1)JH_1^{(3)} - (x-1)JH_0^{(3)}\right)t^2,
\end{aligned}$$

since $JH_n^{(3)}(x) = (x-1)JH_{n-1}^{(3)}(x) + (x-1)JH_{n-2}^{(3)}(x) + xJH_{n-3}^{(3)}(x)$, (see Theorem 1) and the coefficients of t^n for $n \geq 3$ are equal to zero. Moreover, $JH_0^{(3)}(x) = \mathbf{i} + \varepsilon \cdot (x-1) + \mathbf{h} \cdot (x^2 - x)$, $JH_1^{(3)}(x) = 1 + \mathbf{i} \cdot (x-1) + \varepsilon \cdot (x^2 - x) + \mathbf{h} \cdot (x^3 - x^2 + 1)$ and $JH_2^{(3)}(x) = x - 1 + \mathbf{i} \cdot (x^2 - x) + \varepsilon \cdot (x^3 - x^2 + 1) + \mathbf{h} \cdot (x^4 - x^3 + x - 1)$, and the result follows. ∎

In the same way, we obtain the next theorem.

Theorem 8. *The generating function for the modified third-order Jacobsthal hybrinomial sequence* $\left(KH_n^{(3)}(x)\right)_{n\geq 0}$ *is*

$$k(t) = \frac{\left\{ \begin{array}{c} KH_0^{(3)} + \left(KH_1^{(3)} - (x-1)KH_0^{(3)}\right)t \\ + \left(KH_2^{(3)} - (x-1)KH_1^{(3)} - (x-1)KH_0^{(3)}\right)t^2 \end{array} \right\}}{1-(x-1)t-(x-1)t^2-xt^3},$$

where $KH_0^{(3)}(x) = 3 + \mathbf{i} \cdot (x-1) + \varepsilon \cdot (x^2 - 1) + \mathbf{h} \cdot (x^3 + 2)$, $KH_1^{(3)}(x) = x - 1 + \mathbf{i} \cdot (x^2 - 1) + \varepsilon \cdot (x^3 + 2) + \mathbf{h} \cdot (x^4 - 1)$ *and* $KH_2^{(3)}(x) = x^2 - 1 + \mathbf{i} \cdot (x^3 + 2) + \varepsilon \cdot (x^4 - 1) + \mathbf{h} \cdot (x^5 - 1)$.

There are some analogies between properties of third-order Jacobsthal and modified third-order Jacobsthal polynomials and third-order Jacobsthal and modified third-order Jacobsthal hybrinomials.

Lemma 9. *Let $n \geq 1$ be an integer and $x \neq 1$. Then,*

$$\sum_{l=0}^{n} J_l^{(3)}(x) = \frac{1}{3(x-1)}\left(J_{n+2}^{(3)}(x) - (x-2)J_{n+1}^{(3)}(x) + xJ_n^{(3)}(x) - 1\right). \tag{9}$$

Proof. By induction on n. Then, if $n = 1$, we have

$$\sum_{l=0}^{1} J_l^{(3)}(x) = 1 = \frac{1}{3(x-1)}\left(J_3^{(3)}(x) - (x-2)J_2^{(3)}(x) + xJ_1^{(3)}(x) - 1\right).$$

If $n \geq 2$, then using the definition of the third-order Jacobsthal polynomials, we have

$$\begin{aligned}
\sum_{l=0}^{n+1} J_l^{(3)}(x) &= \sum_{l=0}^{n} J_l^{(3)}(x) + J_{n+1}^{(3)}(x) \\
&= \frac{1}{3(x-1)}\left(J_{n+2}^{(3)}(x) - (x-2)J_{n+1}^{(3)}(x) + xJ_n^{(3)}(x) - 1\right) + J_{n+1}^{(3)}(x) \\
&= \frac{1}{3(x-1)}\left(J_{n+2}^{(3)}(x) + (2x-1)J_{n+1}^{(3)}(x) + xJ_n^{(3)}(x) - 1\right) \\
&= \frac{1}{3(x-1)}\left(J_{n+3}^{(3)}(x) - (x-2)J_{n+2}^{(3)}(x) + xJ_{n+1}^{(3)}(x) - 1\right),
\end{aligned}$$

which ends the proof. ■

Theorem 10. *Let $n \geq 0$ be an integer. Then, we have*

$$\sum_{l=0}^{n} JH_l^{(3)}(x) = \frac{1}{3(x-1)}\left\{\begin{array}{l} JH_{n+2}^{(3)}(x) - (x-2)JH_{n+1}^{(3)}(x) \\ + xJH_n^{(3)}(x) - 1 - \mathbf{i} - (3x-2)\varepsilon - (3x^2-3x+1)\mathbf{h} \end{array}\right\}. \tag{10}$$

Proof. Let consider the sum $\sum_{l=0}^{n} JH_l^{(3)}(x)$. Then,

$$\begin{aligned}
\sum_{l=0}^{n} JH_l^{(3)}(x) &= JH_0^{(3)}(x) + JH_1^{(3)}(x) + \cdots + JH_n^{(3)}(x) \\
&= J_0^{(3)}(x) + \mathbf{i}J_1^{(3)}(x) + \varepsilon J_2^{(3)}(x) + \mathbf{h}J_3^{(3)}(x) \\
&+ J_1^{(3)}(x) + \mathbf{i}J_2^{(3)}(x) + \varepsilon J_3^{(3)}(x) + \mathbf{h}J_4^{(3)}(x) \\
&\vdots \\
&+ J_n^{(3)}(x) + \mathbf{i}J_{n+1}^{(3)}(x) + \varepsilon J_{n+2}^{(3)}(x) + \mathbf{h}J_{n+3}^{(3)}(x).
\end{aligned}$$

Furthermore, using Lemma 9 we have

$$\begin{aligned}\sum_{l=0}^{n} JH_l^{(3)}(x) &= J_0^{(3)}(x) + J_1^{(3)}(x) + \cdots + J_n^{(3)}(x)\\ &+ \mathbf{i}\left(J_1^{(3)}(x) + J_2^{(3)}(x) + \cdots + J_{n+1}^{(3)}(x)\right)\\ &+ \varepsilon\left(J_2^{(3)}(x) + J_3^{(3)}(x) + \cdots + J_{n+2}^{(3)}(x)\right)\\ &+ \mathbf{h}\left(J_3^{(3)}(x) + J_4^{(3)}(x) + \cdots + J_{n+3}^{(3)}(x)\right)\\ &= \frac{1}{3(x-1)}\left(J_{n+2}^{(3)}(x) - (x-2)J_{n+1}^{(3)}(x) + xJ_n^{(3)}(x) - 1\right)\\ &+ \frac{\mathbf{i}}{3(x-1)}\left(J_{n+3}^{(3)}(x) - (x-2)J_{n+2}^{(3)}(x) + xJ_{n+1}^{(3)}(x) - 1\right)\\ &+ \frac{\varepsilon}{3(x-1)}\left(J_{n+4}^{(3)}(x) - (x-2)J_{n+3}^{(3)}(x) + xJ_{n+2}^{(3)}(x) - 1\right)\\ &+ \frac{\mathbf{h}}{3(x-1)}\left(J_{n+5}^{(3)}(x) - (x-2)J_{n+4}^{(3)}(x) + xJ_{n+3}^{(3)}(x) - 1\right)\\ &- \varepsilon - x\mathbf{h}\end{aligned}$$

and finally

$$\sum_{l=0}^{n} JH_l^{(3)}(x) = \frac{1}{3(x-1)}\left\{\begin{array}{l} JH_{n+2}^{(3)}(x) - (x-2)JH_{n+1}^{(3)}(x)\\ + xJH_n^{(3)}(x) - 1 - \mathbf{i} - (3x-2)\varepsilon\\ - (3x^2 - 3x + 1)\mathbf{h}\end{array}\right\}.$$

■

Theorem 11. *Let $n \geq 0$ be an integer. Then, we have*

$$\sum_{l=0}^{n} KH_l^{(3)}(x) = \frac{1}{3(x-1)}\left\{\begin{array}{l} KH_{n+2}^{(3)}(x) - (x-2)KH_{n+1}^{(3)}(x)\\ + xKH_n^{(3)}(x) + 3(x-2) + 3(x-3)\mathbf{i}\\ + 2(x-4)\varepsilon + (x^2 - 2x + 7)\mathbf{h}\end{array}\right\}. \tag{11}$$

Proof. Using the next identity

$$\sum_{l=0}^{n} K_l^{(3)}(x) = \frac{1}{3(x-1)}\left\{\begin{array}{l} K_{n+2}^{(3)}(x) - (x-2)K_{n+1}^{(3)}(x)\\ + xK_n^{(3)}(x) + 3(x-2)\end{array}\right\}$$

and proceeding in the same way as in Theorem 10, the result follows. ■

Matrix generators play an important role in the theory of the third-order Jacobsthal numbers and the third-order Jacobsthal polynomials (see, for example [4]). We derive the matrix representation of the third-order Jacobsthal hybrinomials.

Definition 12. Third-order Jacobsthal hybrinomial matrix $Jh_n^{(3)}(x)$ is defined by

$$Jh_n^{(3)}(x) = \begin{bmatrix} JH_{n+4}^{(3)}(x) & (x-1)JH_{n+3}^{(3)}(x) + xJH_{n+2}^{(3)}(x) & xJH_{n+3}^{(3)}(x) \\ JH_{n+3}^{(3)}(x) & (x-1)JH_{n+2}^{(3)}(x) + xJH_{n+1}^{(3)}(x) & xJH_{n+2}^{(3)}(x) \\ JH_{n+2}^{(3)}(x) & (x-1)JH_{n+1}^{(3)}(x) + xJH_{n}^{(3)}(x) & xJH_{n+1}^{(3)}(x) \end{bmatrix},$$

for all $n \geq 0$.

Theorem 13. *Let $n \geq 0$ be an integer. Then*

$$Jh_n^{(3)}(x) = Jh_0^{(3)}(x) \cdot \begin{bmatrix} x-1 & x-1 & x \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n .$$

Proof. (by induction on n) If $n = 0$ then assuming that the matrix to the power 0 is the identity matrix, the result is obvious. Now assume that for any $n \geq 0$ holds

$$Jh_n^{(3)}(x) = Jh_0^{(3)}(x) \cdot \begin{bmatrix} x-1 & x-1 & x \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n .$$

By simple calculation using induction's hypothesis, we have

$$\begin{aligned} & Jh_0^{(3)}(x) \cdot \begin{bmatrix} x-1 & x-1 & x \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{n+1} \\ &= Jh_0^{(3)}(x) \cdot \begin{bmatrix} x-1 & x-1 & x \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{n} \cdot \begin{bmatrix} x-1 & x-1 & x \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ &= Jh_n^{(3)}(x) \cdot \begin{bmatrix} x-1 & x-1 & x \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ &= Jh_{n+1}^{(3)}(x), \end{aligned}$$

which ends the proof. ■

In the same way, we obtain the matrix representation for the modified third-order Jacobsthal hybrinomials.

Theorem 14. *Let $n \geq 0$ be an integer. Then*

$$Kh_n^{(3)}(x) = Kh_0^{(3)}(x) \cdot \begin{bmatrix} x-1 & x-1 & x \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n ,$$

where

$$Kh_n^{(3)}(x) = \begin{bmatrix} KH_{n+4}^{(3)}(x) & (x-1)KH_{n+3}^{(3)}(x) + xKH_{n+2}^{(3)}(x) & xKH_{n+3}^{(3)}(x) \\ KH_{n+3}^{(3)}(x) & (x-1)KH_{n+2}^{(3)}(x) + xKH_{n+1}^{(3)}(x) & xKH_{n+2}^{(3)}(x) \\ KH_{n+2}^{(3)}(x) & (x-1)KH_{n+1}^{(3)}(x) + xKH_{n}^{(3)}(x) & xKH_{n+1}^{(3)}(x) \end{bmatrix}.$$

3. Conclusion

We defined new numbers by using definitions of hybrinomial sequence, third-order Jacobsthal hybrinomial, modified third-order Jacobsthal hybrinomial. The properties of those numbers were examined. Some theorems about these numbers were presented.

References

[1] G. Cerda-Morales, *Identities for third order Jacobsthal quaternions*, Adv. Appl. Clifford Alg. **27** (2017) 1043–1053.
doi:10.1007/s00006-016-0654-1

[2] G. Cerda-Morales, *On a generalization of Tribonacci quaternions*, Mediterranean J. Math. **14** (2017) 1–12.
doi:10.1007/s00009-017-1042-3

[3] G. Cerda-Morales, *A note on modified third-order Jacobsthal numbers*, Proyecciones J. Math. **39** (2020) 731–747.
doi:10.22199/issn.0717-6279-2020-02-0025

[4] G. Cerda-Morales, *Some results on dual third-order Jacobsthal quaternions*, FILOMAT **33** (2019) 1865–1876.
doi:10.2298/FIL1907865C

[5] G. Cerda-Morales, *Third-order Jacobsthal generalized quaternions*, J. Geom. Symmetry Phys. **50** (2018) 11–27.
doi:10.7546/jgsp-50-2018-11-27

[6] C.K. Cook and M.R. Bacon, *Some identities for Jacobsthal and Jacobsthal-Lucas numbers satisfying higher order recurrence relations*, Ann. Math. Inform. **41** (2013) 27–39.

[7] G.B. Djordjević, *Generalized Jacobsthal polynomials*, Fibonacci Quart. **38** (2000) 239–243.

[8] G.B. Djordjević and H.M. Srivastava, *Incomplete generalized Jacobsthal and Jacobsthal-Lucas numbers*, Math. and Comp. Modelling **42** (2005) 1049–1056.
doi:10.1016/j.mcm.2004.10.026

[9] A.F. Horadam, *Jacobsthal representation polynomials*, Fibonacci Quart. **35** (1997) 137–148.

[10] A.F. Horadam, *Jacobsthal representation numbers*, The Fibonacci Quarterly **43** (1996) 40–54.

[11] M. Özdemir, *Introduction to hybrid numbers*, Adv. Appl. Clifford Alg. **28** (2018). doi:10.1007/s00006-018-0833-3

[12] A. Szynal-Liana, *The Horadam hybrid numbers*, Discuss. Math. Gen. Alg. Appl. **38** (2018) 91–98.
doi:10.7151/dmgaa.1287

[13] A. Szynal-Liana and I. Włoch, *The Fibonacci hybrid numbers*, Util. Math. **110** (2019) 3–10.

[14] A. Szynal-Liana and I. Włoch, *On Jacobsthal and Jacobsthal-Lucas Hybrid Numbers*, Ann. Math. Silesianae **33** (2019) 276–283.
doi:10.2478/amsil-2018-0009

[15] A. Szynal-Liana and I. Włoch, Introduction to Fibonacci and Lucas hybrinomials, Complex Variables and Elliptic Equations (2019).
doi:10.1080/17476933.2019.1681416

Received 20 March 2020
Revised 22 June 2020
Accepted 23 November 2020