Discussiones Mathematicae
General Algebra and Applications 41 (2021) 15–22
doi:10.7151/dmgaa.1346

ON ISOCLINIC EXTENSIONS OF LIE ALGEBRAS AND NILPOTENT LIE ALGEBRAS

Homayoon Arabyani

and

Mohammad Javad Sadeghifard

Department of Mathematics
Neyshabur Branch, Islamic Azad University, Neyshabur, Iran

e-mail: h.arabyani@iau-neyshabur.ac.ir
math.sadeghifard85@gmail.com

Abstract

In this paper, we present the concept of isoclinism of Lie algebras and its relationship to the Schur multiplier of Lie algebras. Moreover, we prove some properties of a pair of nilpotent Lie algebras.

Keywords: Schur multiplier, isoclinism, nilpotent Lie algebras.

2010 Mathematics Subject Classification: Primary 17B40; Secondary 17B30.

1. Introduction

The notion of isoclinism for groups was introduced by Hall [11] in 1940, it was generalized by several authors. Moneyhun [13] gave a Lie algebra analogue of isoclinism and proved that the concepts isoclinism and isomorphism between Lie algebras with the same finite dimension are the same. Let L and K be Lie algebras. Then an isoclinism between L and K is a pair of isomorphisms $\alpha : \frac{L}{Z(L)} \to \frac{K}{Z(K)}$ and $\beta : L^2 \to K^2$ such that the following diagram is commutative:

$$\begin{array}{ccc} \frac{L}{Z(L)} \oplus \frac{L}{Z(L)} & \longrightarrow & L^2 \\ \downarrow \alpha & & \downarrow \beta \\ \frac{K}{Z(K)} \oplus \frac{K}{Z(K)} & \longrightarrow & K^2 \end{array}$$

where horizontal maps are defined by $(\overline{l_1}, \overline{l_2}) \mapsto [l_1, l_2]$. In other words, $\beta([l_1, l_2])$ $= [k_1, k_2]$, whenever $l_i \in L$ and $k_i \in \alpha(l_i + Z(L))$ for $i = 1, 2$. In such a situation, we say that L is isoclinic to K and write $L \sim K$. The class of all abelian Lie algebras, whose classification is completely known, constitutes the isoclinism family of the zero Lie algebra. In [13], it has been proven that each isoclinism families of Lie algebras contains a special Lie algebra, called a stem algebra, such that its center is contained in its derived subalgebra. This proves that the concepts of isoclinism and isomorphism between dimension are identical. See [20] for more information. Also, Salemkar *et al.* [21] generalized the concept of isoclinism to the notion of n-isoclinism that is the isoclinism with respect to the variety of nilpotent Lie algebras of class at most n. In 2012, Salemkar *et al.* [15], introduced the concept of isoclinism on the central extensions of Lie algebras, which is a generalization of the above mentioned work of Moneyhun and gived some equivalent conditions under which two central extensions are isoclinic. Also, they showed that under some conditions, the concepts of isoclinism and isomorphism between the central extensions of finite dimensional Lie algebras are identical. See [14] for more information. In this paper, section 2 is devoted to the study of the connection between isoclinism and the subalgebras of the Schur multipliers. In section 3, we prove some properties of a pair of nilpotent Lie algebras.

2. CONNECTION BETWEEN ISOCLINIC EXTENSIONS AND THE SCHUR MULTIPLIER OF LIE ALGEBRAS

This section is devoted to present some properties of isoclinism of Lie algebras. Recall that the Schur multiplier of a Lie algebra L is the abelian Lie algebra

$$\mathcal{M}(L) = \frac{R \cap F^2}{[R, F]},$$

where $0 \to R \to F \to L \to 0$ is a free presentation of L. Note that the Schur multiplier of L is independent of the choice of the free presentation of L. (See [9, 13, 17] for more information.) An exact sequence $e : 0 \to M \xrightarrow{\subseteq} K \to L \to o$ of Lie algebras is a central extension of L if $M \subseteq Z(K)$. Obviously, $e_K : 0 \to Z(K) \to K \to K/Z(K) \to 0$ is always a central extension. It is easily checked that any Lie homomorphism $\gamma : L_1 \to L_2$ induces a homomorphism $\mathcal{M}(\gamma) : \mathcal{M}(L_1) \to \mathcal{M}(L_2)$. Furthermore, for any central extension $e : 0 \to M \xrightarrow{\subseteq} K \xrightarrow{\pi} L \to 0$, it follows that there exists a homomorphism $\theta(e) : \mathcal{M}(L) \to K^2$ such that the following sequence is exact

$$0 \to ker\theta(e) \to \mathcal{M} \xrightarrow{\theta(e)} K^2 \xrightarrow{\pi|} L^2 \to 0.$$

See [19, 20]. If the following diagram of Lie algebras and Lie homomorphisms is commutative

$$\begin{array}{llllllllll} e_1 & : 0 & \longrightarrow & M_1 & \longrightarrow & K_1 & \xrightarrow{\pi_1} & L_1 & \longrightarrow & 0 \\ & & & \downarrow \beta|_{M_1} & & \downarrow \beta & & \downarrow \gamma & & \\ e_2 & : 0 & \longrightarrow & M_2 & \longrightarrow & K_2 & \xrightarrow{\pi_2} & L_2 & \longrightarrow & 0, \end{array}$$

where the rows are central extensions and $\beta\ |_{M_1}$ is the restriction of β to M_1, then the triple $(\beta\ |_{M_1}, \beta, \gamma) : e_1 \to e_2$ is called a morphism from e_1 to e_2. In particular, if β, γ are isomorphisms then e_1 and e_2 are said to be isomorphic and denoted by $e_1 \cong e_2$.

Definition 2.1. Let $e_i : 0 \to M_i \xrightarrow{\subseteq} K_i \xrightarrow{\pi_i} L_i \to 0, i = 1, 2$, be two central extensions.

(i) The extensions e_1 and e_2 are said to be isoclinic if there exist Lie isomorphisms $\gamma : L_1 \to L_2$ and $\beta' : K_1^2 \to K_2^2$ such that for all $k_1, k_2 \in K_1$ we have $\beta'([k_1, k_2]) = [k_1', k_2']$, where $k_i' \in K_2$ and $\gamma\pi_1(k_i) = \pi_2(k_i'), i = 1, 2$. In this case the pair (γ, β') is called an isoclinism from e_1 to e_2 and we write $(\gamma, \beta') : e_1 \sim e_2$. In particular, K_1 and K_2 are isoclinic as [13] if their corresponding relative central extensions e_{K_1} and e_{K_2} are isoclinic.

(ii) A morphism $(\beta\ |_{M_1}, \beta, \gamma) : e_1 \to e_2$ is called isoclinic if the pair $(\gamma, \beta\ |_{K_1^2})$ is an isoclinism from e_1 to e_2. Moreover, $(\beta\ |_{M_1}, \beta, \gamma)$ is said to be an isoclinic epimorphism or monomorphism if β is onto or one-to-one, respectively.

Recall from [19] that a Lie algebra L is said to be capable if there exists a Lie algebra H such that $L \cong H/Z(H)$. (See [6, 7, 16, 18] for more information). Also, let (γ, β') be an isoclinism between two extensions e_1 and e_2 and $\alpha : M_1 \to M_2$ an isomorphism. Then the extensions e_1 and e_2 are said to be strongly isoclinic of the first kind if α and β' coincide on $M_1 \cap K_1^2$. Now, let $\alpha' : K_1/Z(K_1) \to K_2/Z(K_2)$ be an isomorphism. Then the extensions e_1 and e_2 are called strongly isoclinism of the second kind if α' induces an isomorphism from $\frac{K_1}{M_1+K_1^2}$ to $\frac{K_2}{M_2+K_2^2}$, which coincides with the one induced by γ.

The following Lemma is useful for the proof of our results.

Lemma 2.2 ([15], Theorem 4.3)**.** *Let $e_i : 0 \to M_i \xrightarrow{\subseteq} K_i \xrightarrow{\pi_i} L_i \to 0, i = 1, 2$, be two central extensions, and $\gamma : L_1 \to L_2$ be an isomorphism. Then the following statements are equivalent:*

(i) *e_1 and e_2 are isoclinic.*

(ii) *There is an isomorphism $\beta' : K_1^2 \to K_2^2$ with $\beta'\theta(e_1) = \theta(e_2)\mathcal{M}(\gamma)$.*

(iii) *$\mathcal{M}(\gamma)(ker\theta(e_1)) = ker\theta(e_2)$.*

Theorem 2.3. *With the notation of this section, let $\gamma : L_1 \to L_2$ be an isomorphism. Then the following statements are equivalent:*

(i) *There exists a subalgebra N of M_1 such that*

$$e_1 |_N : 0 \to \frac{M_1}{N} \to \frac{K_1}{N} \to L_1 \to 0,$$

and the extension e_2 are isoclinic.

(ii) $\mathcal{M}(\gamma)(ker\theta(e_1)) \subseteq ker\theta(e_2)$.

Moreover, if condition (ii) *holds, then the following identity is also true for some subalgebra N of M_1:*

$$Im\theta(e_1) \cap N = \theta(e_1)\mathcal{M}\left(\gamma^{-1}\right)(ker\theta(e_2)).$$

Proof. By Lemma 2.2, condition (i) is equivalent to

$$(I) \quad ker\theta\left(\frac{e_1}{N}\right) = \mathcal{M}\left(\gamma^{-1}\right)(ker\theta(e_2)).$$

If φ is projective from M_1 onto $\frac{M_1}{N}$, then $\theta\left(\frac{e_1}{N}\right) = \theta(e_1)$. So, the equality (I) is equivalent to

$$\mathcal{M}\left(\gamma^{-1}\right)(ker\theta(e_2)) = \{x \mid x \in \mathcal{M}(L_1), \ \ \theta(e_1)x \in N\}.$$

This holds when,

$$Im\theta(e_1) \cap N = \theta(e_1)\mathcal{M}\left(\gamma^{-1}\right)(ker\theta(e_2))$$

and $ker\theta(e_1) \subseteq \mathcal{M}\left(\gamma^{-1}\right)(ker\theta(e_2))$. Which completes the proof. ■

The following result gives a criterion for a Lie algebra to be capable.

Corollary 2.4. *Let $e : 0 \to M \to K \to L \to 0$ be a stem cover of a Lie algebra L. Then the following conditions are equivalent:*

(i) *L is capable;*

(ii) $M = Z(K)$.

Proof. (ii)⇒(i) is clear.

(i)⇒(ii) We can see that, there exists an extension $e_1 : 0 \to M_1 \to K_1 \to L \to 0$ of a Lie algebra L such that $M_1 = Z(K_1)$. Using Theorem 2.3(i), there exists a subalgebra N of M such that the extensions e/N and e_1 are isoclinic. Thus by Definition 2.1(ii), M/N is mapped onto $Z(K_1/N)$. Hence, $M = Z(K)$. ■

In the following we give equivalent conditions for two extensions to be isoclinism of the first kind.

Proposition 2.5. *Let $\alpha : M_1 \to M_2$ and $\gamma : L_1 \to L_2$ be two isomorphisms. Then the following conditions are equivalent:*

(i) *α and γ induce the same strong isoclinism of the first kind from e_1 to e_2;*

(ii) *$\alpha\theta(e_1) = \theta(e_2)\mathcal{M}(\gamma)$.*

Proof. By [15, Lemma 4.2], γ induces a isoclinism from e_1 to e_2 if and only if there exists an isomorphism $\beta : K_1^2 \to K_2^2$ such that

$$\beta\theta(e_1) = \theta(e_2)\mathcal{M}(\gamma).$$

If (α, β, γ) is a strong isoclinism, then by the definition, α and β coincide on $M_1 \cap K_1^2$ and so the result holds. ■

By the above proposition, we obtain the following criterion for two Lie algebras to be strongly isoclinic of the second kind.

Corollary 2.6. *Two Lie algebras K_1 and K_2 are strongly isoclinic of the second kind if and only if the corresponding stem extensions*

$$0 \to Z(K_i) \cap K_i^2 \to K_i \to \frac{K_i}{Z(K_i) \cap K_i^2} \to 0, \quad i = 1, 2,$$

are isoclinic.

3. Some properties of a pair of nilpotent Lie algebras

In this section we prove some properties of a pair of nilpotent Lie algebras. Kayvanfar *et al.* [12] introduced a concept of nilpotency for pair of groups. Also, Gholamian and Eghdami [10], proved some results on the nilpotency of a pair of Lie algebras. See [22] for more information. Let (N, L) be a pair of Lie algebras, in which N is an ideal of L. Then (N, L) is called nilpotent if it has a central series

$$0 = N_0 \leq N_1 \leq \cdots \leq N_t = N,$$

such that N_i is an ideal of L and $\frac{N_{i+1}}{N_i} \leq Z\big(\frac{N}{N_i}, \frac{L}{N_i}\big)$ for all i.

Theorem 3.1. *Let (N, L) be a pair of Lie algebras, $M \leq Z(N, L)$ and $\big(\frac{N}{M}, \frac{L}{M}\big)$ be nilpotent. Then (N, L) is nilpotent.*

Proof. Since $\big(\frac{N}{M}, \frac{L}{M}\big)$ is a pair of nilpotent Lie algebras, so there exist a series as follows

$$0 = \frac{N_1}{M} \leq \frac{N_2}{M} \leq \cdots \leq \frac{N_n}{M} = \frac{N}{M},$$

such that

$$\frac{\frac{N_{i+1}}{M}}{\frac{N_i}{M}} \leq Z\left(\frac{\frac{N}{M}}{\frac{N_i}{M}}, \frac{\frac{L}{M}}{\frac{N_i}{M}}\right),$$

so, we have $\frac{N_{i+1}}{N_i} \leq Z\big(\frac{N}{N_i}, \frac{L}{N_i}\big)$. On the other hand $M \leq Z(N, L)$. So, the series

$$0 = M_0 \leq M \leq N_1 \leq N_2 \leq \cdots \leq N_n = N,$$

is a central series of (N, L). Therefore, (N, L) is a pair of nilpotent Lie algebras. ■

Theorem 3.2. *If (N, L) is a pair of nilpotent Lie algebras and K is an ideal in L, such that* $\dim K = n$ *then $K \leq Z_n(N, L)$.*

Proof. By induction, if $n = 1$ then $0 \neq K \cap Z(N, L) \leq K$. Thus, $\dim(K \cap Z(N, L)) = \dim K = 1$, hence, $K \cap Z(N, L) = K$ and so, $K \leq Z(N, L)$. Let the result hold for every number less than n, $\dim K = n$ and $M = K \cap Z(N, L) \neq 0$, then $\dim\frac{K}{M} = m$ and $m < n$. Now, we have

$$\frac{K}{M} = \frac{K}{K \cap Z(N, L)} \cong \frac{K + Z(N, L)}{Z(N, L)}.$$

By induction hypothesis,

$$\frac{K + Z(N, L)}{Z(N, L)} \leq Z_m\left(\frac{N}{Z(N, L)}, \frac{L}{Z(N, L)}\right) = \frac{Z_{m+1}(N, L)}{Z(N, L)}.$$

Therefore, $K \leq Z_{m+1}(N, L) \leq Z_n(N, L)$. ■

Let (N, L) be a pair of Lie algebras, in which N is an ideal in L. Also, let $0 \to R \to F \to L \to 0$ be a free presentation of L. We can define the c-nilpotent multiplier of (N, L) as

$$\mathcal{M}^{(c)}(N, L) = \frac{R \cap [S,_c F]}{[R,_c F]} \quad (c \geq 1).$$

Where $N \cong S/R$ for an ideal S in F. (See [1, 2, 3, 4] for more information.) We recall that a Lie algebra L is filiform if L has maximal nilpotency class (See [5]). The following theorem was proved by the first author in [3, Corollary 2.7]. Here, we fix it.

Theorem 3.3. *Let L be an n-dimensional filiform Lie algebra, N be an ideal of L such that $Z(L) \subseteq N$ and M be an ideal of L, which is contained in $N \cap Z(L)$. Then*

$$\dim\mathcal{M}^{(c)}(N, L) + 1 \leq \dim\mathcal{M}^{(c)}(N/M, L/M) + 2^c.$$

Proof. By [1, Proposition 2.3(a)], we can see that

$$\dim\mathcal{M}^{(c)}(N,L) + \dim(Z(L)\cap[N,_c L]) \leq \dim\mathcal{M}^{(c)}(N/Z(L), L/Z(L))$$
$$+ \dim\mathcal{M}^{(c)}(Z(L)) + \dim\mathcal{M}^{(c)}((L/Z(L)^c \otimes Z(L)),$$

where $X^c \otimes Y = \underbrace{X \otimes \cdots \otimes X}_{c-times} \otimes Y$ is the abelian tensor product. On the other hand, by [8, Corollary 1.4], we have $\dim\mathcal{M}^{(c)}(Z(L)) = 0$. Also, we know that

$$\dim((L/Z(L))^c \otimes Z(L)) = 2^c.$$

Hence, $\dim\mathcal{M}^{(c)}(N,L) + 1 \leq \dim\mathcal{M}^{(c)}(N/M, L/M) + 2^c$. ■

References

[1] H. Arabyani, *Bounds for the dimension of the c-nilpotent multiplier of a pair of Lie algebras*, Bull. Iranian Math. Soc. **43** (2017) 2411–2418. doi:10.1142/S1793557119500074

[2] H. Arabyani, *Some results on the c-nilpotent multiplier of a pair of Lie algebras*, Bull. Iranian Math. Soc. **45** (2019) 205–212. doi:10.1007/s41980-018-0126-6

[3] H. Arabyani, *On dimension of c-nilpotent multiplier of a pair of Lie algebras*, Southeast Asian Bull. Math. **43** (2019) 305–312.

[4] H. Arabyani, *Bounds for the dimension of Lie algebras*, J. Math. Ext. **13** (2019) 231–239.

[5] H. Arabyani, F. Saeedi, M.R.R. Moghaddam and E. Khamseh, *Characterization of nilpotent Lie algebras pair by their Schur multipliers*, Comm. Algebra **42** (2014) 5474–5483. doi:10.1080/00927872.2012.677081

[6] H. Arabyani, *On the capability and tensor center of Lie algebras*, Southeast Asian Bull. Math. **43** (2019) 631–637.

[7] H. Arabyani and F. Saeedi, *On dimensions of derived algebra and central factor of a Lie algebra*, Bull. Iranian Math. Soc. **41** (2015) 1093–1102.

[8] M. Araskhan, *The dimension of the c-nilpotent multiplier*, J. Algebra **386** (2013) 105–112. doi:10.1016/j.jalgebra.2013.04.010

[9] P. Batten, K. Moneyhun and E. Stitzinger, *On characterizing nilpotent Lie algebras by their multipliers*, Comm. Algebra **24** (1996) 4319–4330. doi:10.1080/00927879608825817

[10] A. Gholamiyan and H. Eghdami, *On the nilpotency of a pair of Lie algebras*, Mathematica **32** (2016) 1–5.

[11] P. Hall, *The classification of prime-power groups*, J. Reine Angew. Math. **182** (1940) 130–141.
doi:10.1515/crll.1940.182.130

[12] M. Hassanzadeh, A. Pourmirzaei and S. Kayvanfar, *On the nilpotency of a pair of groups*, Southeast Asian Bull. Math. **37** (2013) 67–77.

[13] K. Moneyhun, *Isoclinisms in Lie algebras*, Algebras Groups Geom. **11** (1994) 9–22.

[14] M.R.R. Moghaddam, A.R. Salemkar and A. Gholami, *Some properties of Isologism of groups and Baer-invariants*, Southeast Asian Bull. Math. **24** (2000) 255–261.
doi:10.1007/s10012-000-0255-7

[15] H. Mohammadzadeh, A.R. Salemkar and Z. Riyahi, *Isoclinic extensions of Lie algebras*, Turk. J. Math. **37** (2013) 598–606.

[16] P. Niroomand, M. Parvizi and F.G. Russo, *Some criteria for detecting capable Lie algebras*, J. Algebra **384** (2013) 36–44.
doi:10.1016/j.jalgebra.2013.02.033

[17] F. Saeedi, H. Arabyani and P. Niroomand, *On dimension of Schur multiplier of nilpotent Lie algebra* II, Asian-Eur. J. Math **10** (2017), 1750076 (8 pages).
doi:10.1142/S1793557117500760

[18] H. Safa and H. Arabyani, *Capable pairs of Lie algebras*, Math. Proc. R. Ir. Acad. **118 A** (2018) 39–45.
doi:10.3318/pria.2018.118.05

[19] A.R. Salemkar, V. Alamian and H. Mohammadzadeh, *Some properties of the Schur multiplier and covers of Lie algebras*, Comm. Algebra. **36** (2001) 697–707.
doi:10.1080/00927870701724193

[20] A.R. Salemkar, H. Bigdely and V. Alamian, *Some properties on isoclinism of Lie algebras and covers*, J. Algebra Appl. **7** (2008) 507–516.
doi:10.1142/S0219498808002965

[21] A.R. Salemkar and F. Mirzaei, *Characterizing n-isoclinism classes of Lie algebras*, Comm. Algebra. **38** (2010) 3392–3403.
doi:10.1080/00927870903117535

[22] S.M. Taheri and F. Shahini, *Varietal nilpotent groups*, World Appl. Sci. J. **12** (2011) 2314–2316.

Received 12 January 2020
Revised 9 April 2020
Accepted 13 April 2020