Discussiones Mathematicae
General Algebra and Applications 40 (2020) 187–199
doi:10.7151/dmgaa.1343

A NOTE ON GENERALIZED HYBRID TRIBONACCI NUMBERS

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Abstract

In this paper, we introduce the generalized hybrid Tribonacci numbers. These numbers can be considered as a generalization of the generalized complex Tribonacci, generalized hyperbolic Tribonacci and generalized dual Tribonacci numbers. We also obtain some identities for these numbers.

Keywords: complex numbers, hyperbolic numbers, dual numbers, hybrid numbers, generalized Tribonacci numbers.

2010 Mathematics Subject Classification: Primary: 11B37; Secondary: 11B39, 97F50.

1. Introduction

Complex numbers were first introduced in 1545 by Girolamo Cardano in his *Ars Magna*. In addition, the imaginary unit **i** was introduced by Euler in 1777. Moreover, the term complex number was introduced by Gauss. The set of complex numbers is defined as

$$\mathbb{C} = \left\{a + b\mathbf{i} : a, b \in \mathbb{R},\, \mathbf{i}^2 = -1\right\}.$$

Hyperbolic numbers, which are also called split-complex numbers or perplex numbers or double numbers, were introduced in 1848 by James Cockle. Also, William K. Clifford used hyperbolic numbers to represent sums of spins. The set of hyperbolic numbers is defined as

$$\mathbb{H} = \left\{a + b\mathbf{h} : a, b \in \mathbb{R}, \mathbf{h}^2 = 1\right\}.$$

Dual numbers were first introduced in 1873 by William K. Clifford. Furthermore, at the beginning of the twentieth century, Eduard Study applied the principle of transference between spherical and spatial motion by using dual numbers. The set of dual numbers is defined as

$$\mathbb{D} = \left\{ a + b\varepsilon : a, b \in \mathbb{R},\, \varepsilon^2 = 0 \right\}.$$

Complex, hyperbolic and dual numbers are the examples of two-dimensional number systems. For a survey on these numbers, we refer to [1, 4, 8, 13, 15, 22, 24, 25]. Moreover, complex, hyperbolic and dual numbers have been extensively used in science and engineering. Some applications of these numbers can be found in [3, 7, 9–12, 18, 23].

In [16], Özdemir introduced a new non-commutative number system called hybrid numbers. The hybrid number system can be considered as a generalization of the complex, hyperbolic and dual number systems. The set of hybrid numbers is defined as

$$\mathbb{K} = \left\{ z = a + b\mathbf{i} + c\varepsilon + d\mathbf{h} : a, b, c, d \in \mathbb{R}, \mathbf{i}^2 = -1, \varepsilon^2 = 0, \mathbf{h}^2 = 1, \right. \\ \left. \mathbf{ih} = -\mathbf{hi} = \varepsilon + \mathbf{i} \right\}.$$

For the hybrid number $z = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$, a is called the scalar part and $b\mathbf{i} + c\varepsilon + d\mathbf{h}$ is called the vector part.

Using the relation $\mathbf{ih} = -\mathbf{hi} = \varepsilon + \mathbf{i}$, the multiplication table for the hybrid units $\mathbf{i}$, ε, $\mathbf{h}$ can be obtained as follows:

.	1	$\mathbf{i}$	ε	$\mathbf{h}$
1	1	$\mathbf{i}$	ε	$\mathbf{h}$
$\mathbf{i}$	$\mathbf{i}$	-1	$1 - \mathbf{h}$	$\varepsilon + \mathbf{i}$
ε	ε	$\mathbf{h} + \mathbf{1}$	0	$-\varepsilon$
$\mathbf{h}$	$\mathbf{h}$	$-\varepsilon - \mathbf{i}$	ε	1

The addition and multiplication of two hybrid numbers are defined in a natural way: given $z_1 = a_1 + b_1\mathbf{i} + c_1\varepsilon + d_1\mathbf{h}$ and $z_2 = a_2 + b_2\mathbf{i} + c_2\varepsilon + d_2\mathbf{h}$ in $\mathbb{K}$, then

$$z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)\mathbf{i} + (c_1 + c_2)\varepsilon + (d_1 + d_2)\mathbf{h},$$

$$\begin{aligned} z_1 \times z_2 &= (a_1a_2 - b_1b_2 + b_1c_2 + c_1b_2 + d_1d_2) + (a_1b_2 + b_1a_2 + b_1d_2 - d_1b_2)\mathbf{i} \\ &\quad + (a_1c_2 + b_1d_2 - d_1b_2 + c_1a_2 - c_1d_2 + d_1c_2)\varepsilon \\ &\quad + (a_1d_2 + d_1a_2 - b_1c_2 + c_1b_2)\mathbf{h}. \end{aligned}$$

It must be noted that the addition operation in the hybrid numbers is commutative and also associative. On the other hand, the multiplication operation in the hybrid numbers is associative but not commutative.

Let $z = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$ be a hybrid number. Then the inverse element of z is

$$-z = -a - b\mathbf{i} - c\varepsilon - d\mathbf{h}.$$

The conjugate of z, denoted by $\overline{z}$, is defined as

$$\overline{z} = a - b\mathbf{i} - c\varepsilon - d\mathbf{h},$$

and the character of z is

$$C(z) = z\overline{z} = \overline{z}z = a^2 + (b-c)^2 - c^2 - d^2.$$

In [6], Cerda-Morales defined the nth hybrid (p, q)-Fibonacci and hybrid (p, q)-Lucas numbers, respectively, by the relations

$$\mathbb{H}\mathcal{F}_n = \mathcal{F}_n + \mathcal{F}_{n+1}\mathbf{i} + \mathcal{F}_{n+2}\varepsilon + \mathcal{F}_{n+3}\mathbf{h}$$

and

$$\mathbb{H}\mathcal{L}_n = \mathcal{L}_n + \mathcal{L}_{n+1}\mathbf{i} + \mathcal{L}_{n+2}\varepsilon + \mathcal{L}_{n+3}\mathbf{h},$$

where $\mathcal{F}_n$ is the nth (p, q)-Fibonacci number and $\mathcal{L}_n$ is the nth (p, q)-Lucas number. Moreover, in [20], Szynal-Liana introduced Horadam hybrid numbers. Then, Szynal-Liana and Wloch studied Jacobsthal and Jacobsthal-Lucas hybrid numbers in [21]. Furthermore, in [2], Catarino studied k-Pell hybrid numbers.

The generalized Tribonacci sequence $\{V_n(V_0, V_1, V_2; p, q, r)\}_{n\geq 0}$ or briefly $\{V_n\}_{n\geq 0}$ is defined through the recurrence relation

$$V_n = pV_{n-1} + qV_{n-2} + rV_{n-3}, \quad n \geq 3, \tag{1}$$

where $V_0 = a$, $V_1 = b$, $V_2 = c$ are any integers, and p, q, r are real numbers.

The nth term of the generalized Tribonacci sequence $\{V_n\}_{n\geq 0}$ is

$$V_n = \frac{\mathsf{P}\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\mathsf{Q}\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{\mathsf{R}\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}, \tag{2}$$

where $\mathsf{P} = c - (\beta+\gamma)b + \beta\gamma a$, $\mathsf{Q} = c - (\alpha+\gamma)b + \alpha\gamma a$, $\mathsf{R} = c - (\alpha+\beta)b + \alpha\beta a$, and α, β, γ are the roots of the equation $x^3 - px^2 - qx - r = 0$.

This sequence has been studied by many authors [5, 14, 17, 19, 26].

The generalized Tribonacci sequence is the generalization of the familiar third-order recurrent sequences, that is, for special values of p, q, r, a, b and c, the following results hold:

- If $p = q = r = 1$ and $V_0 = 0, V_1 = V_2 = 1$, then $\{V_n\}_{n\geq 0}$ is the (usual) Tribonacci sequence $\{T_n\}_{n\geq 0}$.
- If $p = q = 1, r = 2$ and $V_0 = 0, V_1 = V_2 = 1$, then $\{V_n\}_{n\geq 0}$ is the third-order Jacobsthal sequence $\{J_n{}^{(3)}\}_{n\geq 0}$.
- If $p = r = 1, q = 0$ and $V_0 = 0, V_1 = V_2 = 1$, then $\{V_n\}_{n\geq 0}$ is the Narayana sequence $\{S_n\}_{n\geq 0}$.
- If $p = 0, q = r = 1$ and $V_0 = V_1 = V_2 = 1$, then $\{V_n\}_{n\geq 0}$ is the Padovan sequence $\{P_n\}_{n\geq 0}$.

In this paper, we define the generalized hybrid Tribonacci numbers by combining hybrid numbers and generalized Tribonacci numbers. These numbers can be accepted as a generalization of the generalized complex, generalized hyperbolic and generalized dual Tribonacci numbers. We give generating function, Binet formula and summation formula for the generalized hybrid Tribonacci numbers. We also obtain some properties of these numbers.

2. The generalized hybrid Tribonacci numbers

The nth generalized hybrid Tribonacci number $\mathbb{H}V_n$ is defined by the relation

$$\mathbb{H}V_n = V_n + V_{n+1}\mathbf{i} + V_{n+2}\varepsilon + V_{n+3}\mathbf{h}, \tag{3}$$

where V_n is the nth generalized Tribonacci number, and i, ε, h are hybrid units.

Let $\mathbb{H}V_n$ be the generalized hybrid Tribonacci number. Then after some necessary calculations, one can obtain the following recurrence relation:

$$\mathbb{H}V_n = p\mathbb{H}V_{n-1} + q\mathbb{H}V_{n-2} + r\mathbb{H}V_{n-3}, \quad n \geq 3 \tag{4}$$

with initial conditions

$$\mathbb{H}V_0 = a + b\mathbf{i} + c\varepsilon + (pc + qb + ra)\mathbf{h}, \tag{5}$$

$$\mathbb{H}V_1 = b + c\mathbf{i} + (pc + qb + ra)\varepsilon + ((p^2 + q)c + (pq + r)b + pra)\mathbf{h}, \tag{6}$$

$$\begin{aligned} \mathbb{H}V_2 &= c + (pc + qb + ra)\mathbf{i} + ((p^2 + q)c + (pq + r)b + pra)\varepsilon \\ &+ ((p^3 + 2pq + r)c + (p^2q + q^2 + pr)b + (p^2r + qr)a)\mathbf{h}. \end{aligned} \tag{7}$$

Particular cases of the above definition are

- Hybrid (usual) Tribonacci numbers are

$$\mathbb{H}T_n = T_n + T_{n+1}\mathbf{i} + T_{n+2}\varepsilon + T_{n+3}\mathbf{h},$$

where T_n is the nth Tribonacci number, with initial conditions

$$\begin{aligned} \mathbb{H}T_0 &= \mathbf{i} + \varepsilon + 2\mathbf{h}, \\ \mathbb{H}T_1 &= 1 + \mathbf{i} + 2\varepsilon + 4\mathbf{h}, \\ \mathbb{H}T_2 &= 1 + 2\mathbf{i} + 4\varepsilon + 7\mathbf{h}. \end{aligned}$$

- Hybrid third-order Jacobsthal numbers are

$$\mathbb{H}J_n{}^{(3)} = J_n{}^{(3)} + J_{n+1}{}^{(3)}\mathbf{i} + J_{n+2}{}^{(3)}\varepsilon + J_{n+3}{}^{(3)}\mathbf{h},$$

where $J_n{}^{(3)}$ is the nth third-order Jacobsthal number, with initial conditions

$$\begin{aligned} \mathbb{H}J_0{}^{(3)} &= \mathbf{i} + \varepsilon + 2\mathbf{h}, \\ \mathbb{H}J_1{}^{(3)} &= 1 + \mathbf{i} + 2\varepsilon + 5\mathbf{h}, \\ \mathbb{H}J_2{}^{(3)} &= 1 + 2\mathbf{i} + 5\varepsilon + 9\mathbf{h}. \end{aligned}$$

- Hybrid Narayana numbers are

$$\mathbb{H}S_n = S_n + S_{n+1}\mathbf{i} + S_{n+2}\varepsilon + S_{n+3}\mathbf{h},$$

where S_n is the nth Narayana number, with initial conditions

$$\begin{aligned} \mathbb{H}S_0 &= \mathbf{i} + \varepsilon + \mathbf{h}, \\ \mathbb{H}S_1 &= 1 + \mathbf{i} + \varepsilon + 2\mathbf{h}, \\ \mathbb{H}S_2 &= 1 + \mathbf{i} + 2\varepsilon + 3\mathbf{h}. \end{aligned}$$

- Hybrid Padovan numbers are

$$\mathbb{H}P_n = P_n + P_{n+1}\mathbf{i} + P_{n+2}\varepsilon + P_{n+3}\mathbf{h},$$

where P_n is the nth Padovan number, with initial conditions

$$\begin{aligned} \mathbb{H}P_0 &= 1 + \mathbf{i} + \varepsilon + 2\mathbf{h}, \\ \mathbb{H}P_1 &= 1 + \mathbf{i} + 2\varepsilon + 2\mathbf{h}, \\ \mathbb{H}P_2 &= 1 + 2\mathbf{i} + 2\varepsilon + 3\mathbf{h}. \end{aligned}$$

Let $\mathbb{H}V_n$ and $\mathbb{H}V_m$ be two generalized hybrid Tribonacci numbers. Then the addition and subtraction of two generalized hybrid Tribonacci numbers are defined by

$$\begin{aligned} \mathbb{H}V_n \pm \mathbb{H}V_m &= (V_n \pm V_m) + (V_{n+1} \pm V_{m+1})\mathbf{i} + (V_{n+2} \pm V_{m+2})\varepsilon \\ &\quad + (V_{n+3} \pm V_{m+3})\mathbf{h}. \end{aligned} \tag{8}$$

The multiplication of a generalized hybrid Tribonacci number by a real scalar λ is defined by

$$\lambda \mathbb{H}V_n = \lambda V_n + \lambda V_{n+1}\mathbf{i} + \lambda V_{n+2}\varepsilon + \lambda V_{n+3}\mathbf{h}. \tag{9}$$

The multiplication of two generalized hybrid Tribonacci numbers is defined by

$$\begin{aligned} \mathbb{H}V_n \times \mathbb{H}V_m &= (V_nV_m - V_{n+1}V_{m+1} + V_{n+2}V_{m+3} + V_{n+2}V_{m+1} + V_{n+3}V_{m+3}) \\ &+ (V_nV_{m+1} + V_{n+1}V_m + V_{n+1}V_{m+3} - V_{n+3}V_{m+1})\mathbf{i} \\ &+ (V_nV_{m+2} + V_{n+1}V_{m+3} - V_{n+3}V_{m+1} - V_{n+2}V_m \\ &- V_{n+2}V_{m+3} + V_{n+3}V_{m+2})\varepsilon \\ &+ (V_nV_{m+3} + V_{n+3}V_m - V_{n+1}V_{m+2} + V_{n+2}V_{m+1})\mathbf{h}. \end{aligned} \tag{10}$$

Moreover, the conjugate of the generalized hybrid number $\mathbb{H}V_n$ is

$$\overline{\mathbb{H}V_n} = V_n - V_{n+1}\mathbf{i} - V_{n+2}\varepsilon - V_{n+3}\mathbf{h}. \tag{11}$$

The following theorem gives the character of a generalized hybrid Tribonacci number.

Theorem 1. *Let $\mathbb{H}V_n$ be the nth generalized hybrid Tribonacci number. Then the character of $\mathbb{H}V_n$ is*

$$\begin{aligned} C(\mathbb{H}V_n) = (1-r^2){V_n}^2 + (1-q^2){V_{n+1}}^2 - p^2{V_{n+2}}^2 - 2(qrV_nV_{n+1} + prV_nV_{n+2} \\ + (1+pq)V_{n+1}V_{n+2}). \end{aligned}$$

Proof. From the definition of the character of a hybrid number, we can write

$$C(\mathbb{H}V_n) = {V_n}^2 + (V_{n+1} - V_{n+2})^2 - {V_{n+2}}^2 - {V_{n+3}}^2. \tag{12}$$

From the recurrence relation (1), we have

$$V_{n+3} = pV_{n+2} + qV_{n+1} + rV_n. \tag{13}$$

Considering this relation (13) in Eq.(12), after some necessary calculation we obtain the desired result. ∎

Particular cases of Theorem 1 are

- The character of the nth hybrid (usual) Tribonacci number $\mathbb{H}T_n$ is

$$C(\mathbb{H}T_n) = -{T_{n+2}}^2 - 2(T_nT_{n+1} + T_nT_{n+2} + 2T_{n+1}T_{n+2}).$$

- The character of the nth hybrid third-order Jacobsthal number $\mathbb{H}J_n{}^{(3)}$ is

$$C\left(\mathbb{H}J_n{}^{(3)}\right) = -3J_n{}^{(3)2} - J_{n+2}{}^{(3)2} - 4\Big(J_n{}^{(3)}J_{n+1}{}^{(3)} + J_n{}^{(3)}J_{n+2}{}^{(3)} + J_{n+1}{}^{(3)}J_{n+2}{}^{(3)}\Big).$$

- The character of the nth hybrid Narayana number $\mathbb{H}S_n$ is

$$C(\mathbb{H}S_n) = S_{n+1}{}^2 - S_{n+2}{}^2 - 2(S_nS_{n+2} + S_{n+1}S_{n+2}).$$

- The character of the nth hybrid Padovan number $\mathbb{H}P_n$ is

$$C(\mathbb{H}P_n) = -2P_nP_{n+1} - 2P_{n+1}P_{n+2}.$$

The generating function for the generalized hybrid Tribonacci numbers is given in the following theorem.

Theorem 2. *The generating function of the generalized hybrid Tribonacci numbers is given by*

$$G(x) = \frac{\mathbb{H}V_0 + (\mathbb{H}V_1 - p\mathbb{H}V_0)x + (\mathbb{H}V_2 - p\mathbb{H}V_1 - q\mathbb{H}V_0)x^2}{1 - px - qx^2 - rx^3}.$$

Proof. Let $G(x)$ be the generating function for the generalized hybrid Tribonacci numbers. Then we write

$$G(x) = \sum_{n=0}^{\infty} \mathbb{H}V_n x^n = \mathbb{H}V_0 + \mathbb{H}V_1 x + \mathbb{H}V_2 x^2 + \cdots + \mathbb{H}V_n x^n + \cdots . \tag{14}$$

Multiplying the Equation (14) with px, qx^2 and rx^3, respectively, we get

$$pxG(x) = p\mathbb{H}V_0x + p\mathbb{H}V_1x^2 + p\mathbb{H}V_2x^3 + \cdots + p\mathbb{H}V_{n-1}x^n + \cdots ,$$
$$qx^2G(x) = q\mathbb{H}V_0x^2 + q\mathbb{H}V_1x^3 + q\mathbb{H}V_2x^4 + \cdots + q\mathbb{H}V_{n-2}x^n + \cdots$$

and

$$rx^3G(x) = r\mathbb{H}V_0x^3 + r\mathbb{H}V_1x^4 + r\mathbb{H}V_2x^5 + \cdots + r\mathbb{H}V_{n-3}x^n + \cdots .$$

Then we obtain

$$\begin{aligned}(1 - px - qx^2 - rx^3)G(x) &= \mathbb{H}V_0 + (\mathbb{H}V_1 - p\mathbb{H}V_0)x + (\mathbb{H}V_2 - p\mathbb{H}V_1 - q\mathbb{H}V_0)x^2 \\ &\quad + \sum_{n=3}^{\infty}(\mathbb{H}V_n - p\mathbb{H}V_{n-1} - q\mathbb{H}V_{n-2} - r\mathbb{H}V_{n-3})x^n \\ &= \mathbb{H}V_0 + (\mathbb{H}V_1 - p\mathbb{H}V_0)x + (\mathbb{H}V_2 - p\mathbb{H}V_1 - q\mathbb{H}V_0)x^2.\end{aligned}$$

Thus, we get

$$G(x) = \frac{\mathbb{H}V_0 + (\mathbb{H}V_1 - p\mathbb{H}V_0)x + (\mathbb{H}V_2 - p\mathbb{H}V_1 - q\mathbb{H}V_0)x^2}{1 - px - qx^2 - rx^3}.$$

∎

Particular cases of Theorem 2 are

- The generating function of the hybrid (usual) Tribonacci numbers is

$$f(x)=\frac{x+\mathbf{i}+(1+x+x^2)\varepsilon+(2+2x+x^2)\mathbf{h}}{1-x-x^2-x^3}.$$

- The generating function of the hybrid third-order Jacobsthal numbers is

$$g(x)=\frac{x+\mathbf{i}+(1+3x+2x^2)\varepsilon+(2+7x+2x^2)\mathbf{h}}{1-x-x^2-2x^3}.$$

- The generating function of the hybrid Narayana numbers is

$$h(x)=\frac{x+\mathbf{i}+(1+x^2)\varepsilon+(1+x+x^2)\mathbf{h}}{1-x-x^3}.$$

- The generating function of the hybrid Padovan numbers is

$$t(x)=\frac{1+x+(1+x+x^2)\mathbf{i}+(1+2x+x^2)\varepsilon+(2+2x+x^2)\mathbf{h}}{1-x^2-x^3}.$$

The following theorem gives the Binet formula for the generalized hybrid Tribonacci numbers.

Theorem 3. *The nth term of the generalized hybrid Tribonacci number is given by*

$$\mathbb{H}V_n=\frac{\mathsf{P}\alpha^*\alpha^n}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\mathsf{Q}\beta^*\beta^n}{(\beta-\alpha)(\beta-\gamma)}+\frac{\mathsf{R}\gamma^*\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}, \tag{15}$$

where $\alpha^*=1+\alpha\mathbf{i}+\alpha^2\varepsilon+\alpha^3\mathbf{h}$, $\beta^*=1+\beta\mathbf{i}+\beta^2\varepsilon+\beta^3\mathbf{h}$, $\gamma^*=1+\gamma\mathbf{i}+\gamma^2\varepsilon+\gamma^3\mathbf{h}$, *and* P, Q, R, α, β, γ *as in Equation* (2).

Proof. Using the definition of the generalized hybrid Tribonacci numbers and Binet formula of the generalized Tribonacci sequence, we have

$$\begin{aligned}\mathbb{H}V_n &= \left(\frac{\mathsf{P}\alpha^n}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\mathsf{Q}\beta^n}{(\beta-\alpha)(\beta-\gamma)}+\frac{\mathsf{R}\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}\right)\\ &+\left(\frac{\mathsf{P}\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\mathsf{Q}\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)}+\frac{\mathsf{R}\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}\right)\mathbf{i}\\ &+\left(\frac{\mathsf{P}\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\mathsf{Q}\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)}+\frac{\mathsf{R}\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)}\right)\varepsilon\\ &+\left(\frac{\mathsf{P}\alpha^{n+3}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\mathsf{Q}\beta^{n+3}}{(\beta-\alpha)(\beta-\gamma)}+\frac{\mathsf{R}\gamma^{n+3}}{(\gamma-\alpha)(\gamma-\beta)}\right)\mathbf{h}.\end{aligned}$$

Then we get

$$\begin{aligned}\mathbb{H}V_n &= \frac{\mathsf{P}\alpha^n}{(\alpha-\beta)(\alpha-\gamma)}(1+\alpha\mathbf{i}+\alpha^2\varepsilon+\alpha^3\mathbf{h}) \\ &+ \frac{\mathsf{Q}\beta^n}{(\beta-\alpha)(\beta-\gamma)}(1+\beta\mathbf{i}+\beta^2\varepsilon+\beta^3\mathbf{h}) \\ &+ \frac{\mathsf{R}\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}(1+\gamma\mathbf{i}+\gamma^2\varepsilon+\gamma^3\mathbf{h}).\end{aligned}$$

If we take $\alpha^* = 1+\alpha\mathbf{i}+\alpha^2\varepsilon+\alpha^3\mathbf{h}$, $\beta^* = 1+\beta\mathbf{i}+\beta^2\varepsilon+\beta^3\mathbf{h}$ and $\gamma^* = 1+\gamma\mathbf{i}+\gamma^2\varepsilon+\gamma^3\mathbf{h}$, we obtain the desired result. ■

Particular cases of Therorem 3 are

- The Binet formula of the nth hybrid (usual) Tribonacci number is

$$\mathbb{H}T_n = \frac{\alpha^*\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^*\beta^{n+1}}{(\alpha-\beta)(\beta-\gamma)} + \frac{\gamma^*\gamma^{n+1}}{(\alpha-\gamma)(\beta-\gamma)},$$

  where α, β, γ, which satisfy $\alpha+\beta+\gamma=1$, $\alpha\beta+\alpha\gamma+\beta\gamma=-1$, $\alpha\beta\gamma=1$, are the roots of the equation $x^3-x^2-x-1=0$, and $\alpha^* = 1+\alpha\mathbf{i}+\alpha^2\varepsilon+\alpha^3\mathbf{h}$, $\beta^* = 1+\beta\mathbf{i}+\beta^2\varepsilon+\beta^3\mathbf{h}$, $\gamma^* = 1+\gamma\mathbf{i}+\gamma^2\varepsilon+\gamma^3\mathbf{h}$.

- The Binet formula of the nth hybrid third-order Jacobsthal number is

$$\mathbb{H}J_n^{(3)} = \frac{2^*2^{n+1}}{(2-w_1)(2-w_2)} + \frac{{w_1}^*{w_1}^{n+1}}{(2-w_1)(w_1-w_2)} + \frac{{w_2}^*{w_2}^{n+1}}{(2-w_2)(w_1-w_2)},$$

  where 2, w_1, w_2, which satisfy $w_1+w_2=-1$, $w_1w_2=1$, are the roots of the equation $x^3-x^2-x-2=0$, and $2^* = 1+2\mathbf{i}+4\varepsilon+8\mathbf{h}$, ${w_1}^* = 1+w_1\mathbf{i}+{w_1}^2\varepsilon+{w_1}^3\mathbf{h}$, ${w_2}^* = 1+w_2\mathbf{i}+{w_2}^2\varepsilon+{w_2}^3\mathbf{h}$.

- The Binet formula of the nth hybrid Narayana number is

$$\mathbb{H}S_n = \frac{{z_1}^*{z_1}^{n+1}}{(z_1-z_2)(z_1-z_3)} + \frac{{z_2}^*{z_2}^{n+1}}{(z_1-z_2)(z_2-z_3)} + \frac{{z_3}^*{z_3}^{n+1}}{(z_1-z_3)(z_2-z_3)},$$

  where z_1, z_2, z_3, which satisfy $z_1+z_2+z_3=1$, are the roots of the equation $x^3-x-1=0$, and ${z_1}^* = 1+z_1\mathbf{i}+{z_1}^2\varepsilon+{z_1}^3\mathbf{h}$, ${z_2}^* = 1+z_2\mathbf{i}+{z_2}^2\varepsilon+{z_2}^3\mathbf{h}$, ${z_3}^* = 1+z_3\mathbf{i}+{z_3}^2\varepsilon+{z_3}^3\mathbf{h}$.

- The Binet formula of the nth hybrid Padovan number is

$$\begin{aligned}\mathbb{H}P_n &= \frac{(r_2-1)(r_3-1)}{(r_1-r_2)(r_1-r_3)}{r_1}^*{r_1}^n + \frac{(r_1-1)(r_3-1)}{(r_1-r_2)(r_2-r_3)}{r_2}^*{r_2}^n \\ &+ \frac{(r_1-1)(r_2-1)}{(r_1-r_3)(r_2-r_3)}{r_3}^*{r_3}^n,\end{aligned}$$

  where r_1, r_2, r_3 are the roots of the equation $x^3-x^2-1=0$, and ${r_1}^* = 1+r_1\mathbf{i}+{r_1}^2\varepsilon+{r_1}^3\mathbf{h}$, ${r_2}^* = 1+r_2\mathbf{i}+{r_2}^2\varepsilon+{r_2}^3\mathbf{h}$, ${r_3}^* = 1+r_3\mathbf{i}+{r_3}^2\varepsilon+{r_3}^3\mathbf{h}$.

In ([5], Lemma 2.3), Cerda-Morales proved the following lemma dealing with the summation formula for the generalized Tribonacci numbers.

Lemma 4 [5]**.** *For every integer $n \geq 0$, we have*

$$\sum_{l=0}^{n} V_l = \frac{1}{\delta(p,q,r)}\big(V_{n+2} + (1-p)V_{n+1} + rV_n + (p+q-1)a + (p-1)b - c\big), \tag{16}$$

where $\delta = \delta(p,q,r) = p+q+r-1$ and V_n denote the nth term of the generalized Tribonacci numbers.

We now give the summation formula for the generalized hybrid Tribonacci numbers.

Theorem 5. *The summation formula for the generalized hybrid Tribonacci numbers is*

$$\sum_{l=0}^{n} \mathbb{H}V_l = \frac{1}{\delta(p,q,r)}\big(\mathbb{H}V_{n+2} + (1-p)\mathbb{H}V_{n+1} + r\mathbb{H}V_n + R\big),$$

where $R = S + (S-\delta a)\mathbf{i} + (S - \delta(a+b))\varepsilon + (S - \delta(a+b+c))\mathbf{h}$ and $S = (p+q-1)a + (p-1)b - c$.

Proof. From the Equation (3), we get

$$\sum_{l=0}^{n} \mathbb{H}V_l = \sum_{l=0}^{n} V_l + \left(\sum_{l=0}^{n} V_{l+1}\right)\mathbf{i} + \left(\sum_{l=0}^{n} V_{l+2}\right)\varepsilon + \left(\sum_{l=0}^{n} V_{l+3}\right)\mathbf{h}.$$

Then using the Equation (16), we can write

$$\begin{aligned}\sum_{l=0}^{n} \mathbb{H}V_l = {}& \frac{1}{\delta}(V_{n+2} + (1-p)V_{n+1} + rV_n + (p+q-1)a + (p-1)b - c)\\ &+ \frac{1}{\delta}(V_{n+3} + (1-p)V_{n+2} + rV_{n+1} + (p+q-1)a + (p-1)b - c - \delta a)\mathbf{i}\\ &+ \frac{1}{\delta}(V_{n+4} + (1-p)V_{n+3} + rV_{n+2} + (p+q-1)a + (p-1)b - c - \delta(a+b))\varepsilon\\ &+ \frac{1}{\delta}(V_{n+5} + (1-p)V_{n+4} + rV_{n+3} + (p+q-1)a + (p-1)b - c - \delta(a+b+c))\mathbf{h}.\end{aligned}$$

From the Equation (3) and taking S instead of $(p+q-1)a + (p-1)b - c$, we have

$$\begin{aligned}\sum_{l=0}^{n} \mathbb{H}V_l = {}& \frac{1}{\delta}(\mathbb{H}V_{n+2} + (1-p)\mathbb{H}V_{n+1} + r\mathbb{H}V_n + S + (S-\delta a)\mathbf{i} + (S-\delta(a+b))\varepsilon\\ &+ (S - \delta(a+b+c))\mathbf{h}).\end{aligned}$$

Taking R instead of $S+(S-\delta a)\mathbf{i}+(S-\delta(a+b))\varepsilon+(S-\delta(a+b+c))\mathbf{h}$, we obtain

$$\sum_{l=0}^{n}\mathbb{H}V_l=\frac{1}{\delta}\big(\mathbb{H}V_{n+2}+(1-p)\mathbb{H}V_{n+1}+r\mathbb{H}V_n+R\big)$$

which completes the proof. ■

Particular cases of Therorem 5 are

- The summation formula of the hybrid (usual) Tribonacci numbers is

$$\sum_{l=0}^{n}\mathbb{H}T_l=\frac{1}{2}\big(\mathbb{H}T_{n+2}+\mathbb{H}T_n-(1+\mathbf{i}+3\varepsilon+5\mathbf{h})\big).$$

- The summation formula of the hybrid third-order Jacobsthal numbers is

$$\sum_{l=0}^{n}\mathbb{H}J_l{}^{(3)}=\frac{1}{3}\left(\mathbb{H}J_{n+2}{}^{(3)}+2\mathbb{H}J_n{}^{(3)}-(1+\mathbf{i}+4\varepsilon+7\mathbf{h})\right).$$

- The summation formula of the hybrid Narayana numbers is

$$\sum_{l=0}^{n}\mathbb{H}S_l=\mathbb{H}S_{n+3}-(1+\mathbf{i}+2\varepsilon+3\mathbf{h}).$$

- The summation formula of the hybrid Padovan numbers is

$$\sum_{l=0}^{n}\mathbb{H}P_l=\mathbb{H}P_{n+5}-(2+3\mathbf{i}+4\varepsilon+5\mathbf{h}).$$

3. Conclusions

In this study, the generalized hybrid Tribonacci numbers which can be considered as a generalization of the generalized complex Tribonacci, generalized hyperbolic Tribonacci and generalized dual Tribonacci numbers, were introduced. The generalized hybrid Tribonacci numbers include hybrid (usual) Tribonacci numbers, hybrid third-order Jacobsthal numbers, hybrid Narayana numbers, hybrid Padovan numbers. Some properties of these numbers, including generating function, Binet formula and summation formula, were given.

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Received 2 July 2019
Revised 30 January 2020
Accepted 7 June 2020