

## COMMUTATIVITY WITH DERIVATIONS OF SEMIPRIME RINGS

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### Abstract

Let  $R$  be a 2-torsion free semiprime ring with the centre  $Z(R)$ ,  $U$  be a non-zero ideal and  $d: R \rightarrow R$  be a derivation mapping. Suppose that  $R$  admits

- (1) a derivation  $d$  satisfying one of the following conditions:
  - (i)  $[d(x), d(y)] - [x, y] \in Z(R)$  for all  $x, y \in U$ ,
  - (ii)  $[d^2(x), d^2(y)] - [x, y] \in Z(R)$  for all  $x, y \in U$ ,
  - (iii)  $[d(x)^2, d(y)^2] - [x, y] \in Z(R)$  for all  $x, y \in U$ ,
  - (iv)  $[d(x^2), d(y^2)] - [x, y] \in Z(R)$  for all  $x, y \in U$ ,
  - (v)  $[d(x), d(y)] - [x^2, y^2] \in Z(R)$  for all  $x, y \in U$ .
- (2) a non-zero derivation  $d$  satisfying one of the following conditions:
  - (i)  $d([d(x), d(y)]) - [x, y] \in Z(R)$  for all  $x, y \in U$ ,
  - (ii)  $d([d(x), d(y)]) + [x, y] \in Z(R)$  for all  $x, y \in U$ .

Then  $R$  contains a non-zero central ideal.

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### 1. INTRODUCTION

Many authors have investigated the properties of prime or semiprime rings with derivations and commutativity. The first result in this direction is due to Posner [20] who proved that  $R$  is a prime ring and  $d$  is a non-zero derivation of  $R$ . If

$[d(x), x] \in Z(R)$ , for all  $x \in R$ , then  $R$  is commutative. This result was subsequently refined and extended by many algebraists. Bell and Martindal III [2] showed that the center of semiprime ring contains no non-zero nilpotent elements. Also, in [4] it is shown that a semiprime ring  $R$  admits a derivation  $d$  for which either  $xy + d(xy) = yx + d(yx)$  for all  $x, y \in R$  or  $xy - d(xy) = yx - d(yx)$  for all  $x, y \in R$ , then  $R$  is commutative. Majeed and Atteya [9] have studied a commutativity and central ideal for a 2-torsion free semiprime ring  $R$  and  $U$  is a non-zero ideal of  $R$ . If a non-zero derivation  $d$  satisfies the identity  $d([d(x), d(y)]) = [x, y]$  for all  $x, y \in U$  and acts as a homomorphism, then  $R$  contains a non-zero central ideal.

In [6] De Filippis studied the following identities. Let  $R$  be a prime ring,  $d$  be a non-zero derivation of  $R$  and  $U$  be a non-zero two-sided ideal of  $R$ . If  $d([x, y]) = [x, y]$  for all  $x, y \in U$ , then  $R$  is commutative. Another contribution was made in [5] where  $R$  is a semiprime ring and  $d$  is a derivation of  $R$  such that  $d^3 \neq 0$ . If  $[d(x), d(y)] = 0$  for all  $x, y \in R$ , then  $R$  contains a non-zero central ideal.

The following result is due to Atteya and Majeed [10]. Suppose that  $R$  is a 2-torsion free semiprime ring and  $U$  is a non-zero ideal of  $R$ . If  $R$  admits a non-zero derivation  $d$  satisfying  $[d^2(x^2), x] = 0$  for all  $x \in U$ , then  $R$  contains a non-zero central ideal. In addition to that, Oukhtite and Salhi [13] proved that  $R$  is a 2-torsion free  $\sigma$ -prime ring and  $d$  is a non-zero derivation. If  $[d(x), x] \in Z(R)$  for all  $x \in R$ , then  $R$  is commutative. A prime ring  $R$  has an involution  $\sigma$  is said to be  $\sigma$ -prime if  $aRb = aR\sigma(b) = 0$  implies  $a = 0$  or  $b = 0$ .

Obviously, every a prime ring equipped with an involution  $\sigma$  is  $\sigma$ -prime the converse need not be true in general. Furthermore, Chaudhry and Allah-Bakhsh Thaheem [1] proved that  $(\alpha, \beta)$  is an epimorphisms of a semiprime ring  $R$  such that  $\beta$  is a centralizing. If  $d$  is a commuting  $(\alpha, \beta)$ -derivation of  $R$ , then  $[x, y]d(u) = 0 = d(u)[x, y]$  for all  $x, y, u \in R$ . An additive mapping  $d: R \rightarrow R$  is called an  $(\alpha, \beta)$ -derivation if  $d(xy) = \beta(x)d(y) + d(x)\alpha(y)$  for all  $x, y \in R$ .

By using Atteya's result in [11], it follows that a semiprime ring  $R$  admits a generalized derivation  $D$  associated with a non-zero derivation  $d$  satisfying  $D(xy) - xy \in Z(R)$  for all  $x, y \in U$ , where  $U$  is a non-zero ideal. Then  $R$  contains a non-zero central ideal. Other authors moved to develop the derivations e.g. the concept of a generalized derivation was first time introduced by Brešar [3]. He stated that, an additive mapping  $D: R \rightarrow R$  is said to be a generalized derivation if there exists a derivation  $d: R \rightarrow R$  such that  $D(xy) = D(x)y + xd(y)$  for all  $x, y \in R$ . Hence, the concept of a generalized derivation covers both the concepts of a derivation and of a left multiplier, i.e., an additive map  $d$  satisfies  $d(xy) = d(x)y$  for all  $x, y \in R$  [12]. For more information see that [7, 14 and 15].

Throughout  $R$  will represent an associative ring with center  $Z(R)$ ,  $R$  is said to be an  $n$ -torsion free, where  $n \neq 0$  is an integer, if whenever  $nx = 0$ , with

$x \in R$ , then  $x = 0$ . We recall that  $R$  is a semiprime if  $xRx = (0)$  implies  $x = 0$  and it is a prime if  $xRy = (0)$  implies  $x = 0$  or  $y = 0$ . A prime ring is a semiprime but the converse need not be true in general.

An additive mapping  $d: R \rightarrow R$  is called a derivation if  $d(xy) = d(x)y + xd(y)$  holds for all  $x, y \in R$ , and is said to be  $n$ -centralizing on  $U$  (resp.  $n$ -commuting on  $U$ ), if  $[x^n, d(x)] \in Z(R)$  holds for all  $x \in U$  (resp.  $[x^n, d(x)] = 0$  holds for all  $x \in U$ , where  $n$  is a positive integer and  $U$  is a subset of  $R$ ). We write  $[x, y]$  for  $xy - yx$  and make extensive use of basic commutator identities  $[xy, z] = x[y, z] + [x, z]y$  and  $[x, yz] = y[x, z] + [x, y]z$ . In this paper, we study and investigate results concerning a derivation  $d$  of a semiprime ring  $R$ .

To achieve our purpose we mention the following results.

**Lemma 1** ([8], Main Theorem). *Let  $R$  be a semiprime ring,  $U$  be a non-zero left ideal of  $R$  and  $d$  be a non-zero derivation of  $R$ . If for some positive integers  $t_0, t_1, \dots, t_n$  and all  $x \in U$ , the identity  $[[\dots [d(x^{t_0}), x^{t_1}], x^{t_2}], \dots], x^{t_n}] = 0$  holds, then either  $d(U) = 0$  or else  $d(U)$  and  $d(R)U$  are contained in non-zero central ideal of  $R$ . In particular when  $R$  is a prime ring,  $R$  is commutative.*

**Lemma 2** ([17], Lemma 2.4). *Let  $R$  be a semiprime ring and  $a \in R$ . Then  $[a, [a, x]] = 0$  holds for all  $x \in R$  if and only if  $a^2, 2a \in Z(R)$ .*

**Lemma 3** ([19], Theorem). *Let  $R$  be a ring in which, given  $a, b \in R$ , there exist integers  $m = m(a, b), n = n(a, b)$  greater than or equal to 1 such that  $a^m b^n = b^n a^m$ . Then, the commutator ideal of  $R$  is nil. In particular, if  $R$  has no non-zero nil ideals, then  $R$  must be commutative.*

**Lemma 4** ([18], Proposition 8.5.3). *Let  $R$  be a ring. Then every intersection of prime ideals is a semiprime. Conversely, every semiprime ideal is an intersection of prime ideals.*

**Lemma 5** ([16], Remarks). *Let  $R$  be a prime ring. If  $R$  contains a non-zero commutative left ideal, then  $R$  is a commutative ring.*

## 2. THE MAIN RESULTS

**Theorem 6.** *Let  $R$  be a 2-torsion free semiprime ring and  $U$  be a non-zero ideal of  $R$ . If  $d$  satisfies the identity  $[d(x), d(y)] - [x, y] \in Z(R)$  for all  $x, y \in U$ , then  $R$  contains a non-zero central ideal.*

**Proof.** When  $d \neq 0$ , by the hypothesis, we have  $[d(x), d(y)] - [x, y] \in Z(R)$  for all  $x, y \in U$ . Replacing  $x$  by  $xt$ , we expand the main relation into  $[d(x)t, d(y)] + [xd(t), d(y)] - [xt, y] \in Z(R)$  for all  $x, y, t \in U$ .

Simplifying this relation, we find that  $d(x)[t, d(y)] + [d(x), d(y)]t + x[d(t), d(y)] + [x, d(y)]d(t) - x[t, y] + [x, y]t \in Z(R)$  for all  $x, y, t \in U$ .

Writing  $t$  to  $y$ , we see that  $d(x)[y, d(y)] + [d(x), d(y)]y + [x, d(y)]d(y) - [x, y]y \in Z(R)$  for all  $x, y \in U$ . Thus, we conclude that  $[d(x)[y, d(y)] + [x, d(y)]d(y), r] + [[d(x), d(y)] - [x, y]]y, r] = 0$  for all  $x, y \in U, r \in R$ . Furthermore, we obtain  $[d(x)[y, d(y)] + [x, d(y)]d(y), r] + ([d(x), d(y)] - [x, y])[y, r] = 0$  for all  $x, y \in U, r \in R$ . Replacing  $r$  by  $y$ , we find that  $[d(x)[y, d(y)] + [x, d(y)]d(y), y] = 0$  for all  $x, y \in U$ .

Taking  $y$  instead of  $x$  in this relation, we see that  $[[y, d(y)^2], y] = 0$  for all  $y \in U$ . Linearizing  $y$  by  $x + y$ , we obtain  $16[[d(y^2), y], y] = 0$  for all  $y \in U$ . By reason of  $R$  is 2-torsion free semiprime and using Lemma 1. The proof of this case is completed.  $\blacksquare$

Now, for  $d = 0$ , note that  $[x, y] \in Z(R)$  for all  $x, y \in U$ . It is clear that  $R$  contains a non-zero central ideal. In the following theorem, the ideal contains the intersection of prime ideals.

**Theorem 7.** *Let  $R$  be a 2-torsion free semiprime ring and  $U$  be a non-zero ideal of  $R$ . If  $d$  satisfies the identity  $[d^2(x), d^2(y)] - [x, y] \in Z(R)$  for all  $x, y \in U$ , then  $R$  contains a non-zero central ideal.*

**Proof.** Suppose that  $d \neq 0$ . Then, the main relation becomes  $[d^2(x), d^2(y)] - [x, y] \in Z(R)$  for all  $x, y \in U$ . Replacing  $x$  by  $xy$ , we conclude that  $[d^2(xy), d^2(y)] - [xy, y] \in Z(R)$  for all  $x, y \in U$ . Moreover, we obtain  $[d(d(xy)), d^2(y)] - [x, y]y \in Z(R)$  for all  $x, y \in U$ . By a simple calculation, we deduce that  $d^2(x)[y, d^2(y)] + [d^2(x), d^2(y)]y + 2[d(x)d(y), d^2(y)] + [x, d^2(y)]d^2(y) - [x, y]y \in Z(R)$  for all  $x, y \in U$ . Thus,

$$(1) \quad \begin{aligned} & [d^2(x)[y, d^2(y)] + 2[d(x)d(y), d^2(y)] + [x, d^2(y)]d^2(y), r] \\ & + ([d^2(x), d^2(y)] - [x, y])[y, r] = 0 \end{aligned}$$

for all  $x, y \in U, r \in R$ .

Since  $[d^2(x), d^2(y)] - [x, y] \in Z(R)$  for all  $x, y \in U$ , we arrive to

$$(2) \quad \begin{aligned} & [d^2(x)[y, d^2(y)] + 2[d(x)d(y), d^2(y)] + [x, d^2(y)]d^2(y), r] \\ & + [y, r]([d^2(x), d^2(y)] - [x, y]) = 0. \end{aligned}$$

Subtracting relations (2) and (1), we conclude that

$$(3) \quad [[d^2(x), d^2(y)] - [x, y], [y, r]] = 0.$$

Replacing  $r$  with  $x$ , we obtain  $[[d^2(x), d^2(y)], [x, y]] = 0$ .

Right-multiplying side by  $t[y, x], t \in R$ , we find that  $[d^2(x), d^2(y)], [x, y] t[y, r] = 0$ .

In other words, we have  $[d^2(x), d^2(y)], [x, y]R[y, r] = (0)$ . According to the hypothesis that  $R$  is a semiprime, we consider the set  $\{P_\alpha\}$  of a prime ideals of  $R$  such that  $\cap P_\alpha = \{0\}$ . Applying Lemma 4, we obtain the set  $\{P_\alpha\}$  of a prime ideals of  $R$  is a semiprime ideal.

Let  $\cap P_\alpha \subseteq U$ . In what follows, we find that either  $[x, y] \in \cap P_\alpha \subseteq U$  or  $[[d^2(x), d^2(y)], [x, y]] \in \cap P_\alpha \subseteq U$ , i.e., we have two options.

Firstly, when  $[x, y] \in \cap P_\alpha \subseteq U$  yields  $[x, y] = 0$  for all  $x, y \in U$ . Basically, every commutative ideal is a central ideal. Hence,  $U$  is a central ideal.

While the second option  $[[d^2(x), d^2(y)], [x, y]] \in \cap P_\alpha \subseteq U$ . This implies that  $[[d^2(x), d^2(y)], [x, y]] \in U \subseteq Z(R)$  for all  $x, y \in U$ . Obviously,  $R$  contains a non-zero central ideal.

If we take  $d = 0$ , then we obtain  $[x, y] \in Z(R)$  for all  $x, y \in U$ . It is sufficient to obtain the desired result. ■

Here, we state the consequence of Theorem 7.

**Theorem 8.** *Let  $R$  be a 2-torsion free semiprime ring and  $U$  be a non-zero ideal of  $R$ . If  $d$  satisfies the relation  $[d^2(x), d^2(y)] - [x, y] \in Z(R)$  for all  $x, y \in R$ , then  $R$  is commutative.*

The proof straightforward, hence we omitted.

**Corollary 9.** *Let  $R$  be a 2-torsion free prime ring and  $U$  be a non-zero left ideal of  $R$ . If  $d$  satisfies  $[d^2(x), d^2(y)] - [x, y] \in Z(R)$  for all  $x, y \in U$ , then either  $R$  contains a non-zero central ideal or  $R$  is commutative.*

**Proof.** The first case of the proof of Theorem 2 gives us  $U$  is a commutative ideal and applying Lemma 5, we conclude that  $R$  is commutative. While the second case satisfies that  $R$  contains a non-zero central left ideal. ■

**Theorem 10.** *Let  $R$  be a 2-torsion free semiprime ring and  $U$  be a non-zero ideal of  $R$ . If  $R$  admits a derivation  $d$  satisfying  $[d(x)^2, d(y)^2] - [x, y] \in Z(R)$  for all  $x, y \in U$ , then  $R$  contains a non-zero central ideal.*

**Proof.** When  $d \neq 0$ , according to the hypothesis, we have  $[d(x)^2, d(y)^2] - [x, y] \in Z(R)$  for all  $x, y \in U$ . Linearizing  $x$  by  $x + y$ , we obtain  $[d(x+y)^2, d(y)^2] - [x, y] \in Z(R)$  for all  $x, y \in U$ .

Furthermore, we find that  $[d(x^2), d(y)^2] + [d(x)x, d(y)^2] + [xd(y), d(y)^2] + [d(y)x, d(y)^2] + [yd(x), d(y)^2] - [x, y] \in Z(R)$ . Replacing  $y$  by  $x$ , we conclude that  $d(x)[x, d(x)^2] + [x, d(x)^2]d(x) + d(x)[x, d(x)^2] + [x, d(x)^2]d(x) \in Z(R)$ .

Consequently, we show that  $2[x, d(x)^3] \in Z(R)$  for all  $x, y \in U$ . After that, we obtain  $2[[x, d(x)^3], r] = 0$  for all  $x, y \in U, r \in R$ . Based on the fact that  $R$  is 2-torsion free, we find that  $[[x, d(x)^3], r] = 0$  for all  $x, y \in U, r \in R$ . Linearizing  $x$  by  $x + y$  with replacing  $r$  by  $x$ , we see that  $16[[x, d(x)^3], x] = 0$  for all  $x \in U$ . In

view of  $R$  is 2-torsion free with using Lemma 1, we obtain  $R$  contains a non-zero central ideal. If we take  $d = 0$ , then we have  $[x, y] \in Z(R)$  for all  $x, y \in U$ . Clearly,  $U$  is a non-zero central ideal. We completed the proof. ■

**Theorem 11.** *Let  $R$  be a 2-torsion free semiprime ring and  $U$  be a non-zero ideal of  $R$ . If  $d$  satisfies  $[d(x^2), d(y^2)] - [x, y] \in Z(R)$  for all  $x, y \in U$ , then  $R$  contains a non-zero central ideal.*

**Proof.** When  $d \neq 0$ , we have the main relation  $[d(x^2), d(y^2)] - [x, y] \in Z(R)$  for all  $x, y \in U$ . Linearizing  $x$  by  $x + y$ , we obtain  $[d((x + y)^2), d(y^2)] - [x, y] \in Z(R)$ . After some calculation, we arrive to

$$\begin{aligned} & d(x)d(y)[x, y] + d(x)[x, d(y)]y + d(y)[d(x), y]x + [d(x), d(y)]yx + d(x)y[x, d(y)] \\ & + d(x)[x, y]d(y) + y[d(x), d(y)]x + [d(x), y]d(y)x + xd(y)[d(x), y] + x[d(x), d(y)]y \\ & + d(y)[x, y]d(x) + [x, d(y)]yd(x) + xy[d(x), d(y)] + x[d(x), y]d(y) + y[x, d(y)]d(x) \\ & + [x, y]d(y)d(x) - [x, y] \in Z(R). \end{aligned}$$

Replacing  $y$  by  $x$ , we obtain

$$d(x)x[x, d(x)] + [d(x), x]d(x)x + xd(x)[d(x), x] + [x, d(x)]xd(x) \in Z(R)$$

yields  $2[d(x), x] \in Z(R)$  for all  $x \in U$ . Of course, we conclude that  $2[[d(x), x], r] = 0$  for all  $x \in U, r \in R$ . Applying the fact that  $R$  has 2-torsion free property, we find that  $[[d(x), x], r] = 0$  for all  $x \in U, r \in R$ . Replacing  $r$  by  $x$ , we observe that  $[[d(x), x], x] = 0$  for all  $x \in U$ . Applying Lemma 1, we complete the proof. ■

When  $d = 0$ . Obviously,  $R$  contains a non-zero central ideal.

**Theorem 12.** *Let  $R$  be a 2-torsion free semiprime ring. If  $d$  satisfies  $[d(x), d(y)] - [x^2, y^2] \in Z(R)$  for all  $x, y \in R$ , then  $R$  is commutative.*

**Proof.** Firstly, when  $d \neq 0$ , we have  $[d(x), d(y)] - [x^2, y^2] \in Z(R)$  for all  $x, y \in R$ . Replacing  $x$  by  $x + t, t \in R$ , we see that

$$[d(x), d(y)] + [d(t), d(y)] - [x^2, y^2] - [t^2, y^2] - [xt, y^2] - [tx, y^2] \in Z(R).$$

Moreover, we conclude that

$$(4) \quad [[xt + tx, y^2], r] = 0.$$

Now we add the term  $\mp tx$  to the internal bracket of relation (4), we conclude that

$$(5) \quad [[[x, t], y^2] + 2[tx, y^2], r] = 0.$$

Furthermore, we find that

$$(6) \quad [[[x, t], y^2], r] + 2(t[[x, y^2], r] + [t, r][x, y^2] + [t, y^2][x, r] + [[t, y^2], r]x) = 0.$$

In relation (5), replacing  $t$  by  $x$  and using the fact that  $R$  is a 2-torsion free semiprime, we see that  $[[x^2, y^2], r] = 0$  for all  $x, y, r \in R$ . In relation (6), we apply this result and replacing  $t$  and  $x$  with  $x^2$  and using the fact that  $R$  is 2-torsion free, we arrive to

$$[x^2, r][x^2, y^2] + [x^2, y^2][x^2, r] = 0.$$

Replacing  $r$  by  $y^2$  and using that  $R$  is 2-torsion free semiprime, we notice that  $[x^2, y^2]^2 = 0$ . Applying Lemma 2 and  $R$  is a 2-torsion free semiprime, we observe that  $[x^2, y^2] \in Z(R)$ . Clearly, we find that  $[[x^2, y^2], r] = 0$ . Linearizing  $x$  by  $x + y$ , we see that  $[[x^2 + xy + yx + y^2, y^2], r] = 0$ . By reason of  $[x^2, y^2] \in Z(R)$  for all  $x, y \in R$ , we obtain  $[[xy + yx, y^2], r] = 0$ . This can be written as  $[[x, y^2]y + y[x, y^2], r] = 0$  for all  $x, y \in R$ . Consequently, we obtain  $[[x, y^2]y, r] + [y[x, y^2], r] = 0$  for all  $x, y, r \in R$ . This implies

$$[x, y^2][y, r] + [[x, y^2], r] + y[[x, y^2], r] + [y, r][x, y^2] = 0$$

for all  $x, y, r \in R$ . Substituting  $x$  by  $x^2$  and using  $[x^2, y^2] \in Z(R)$ , this relation reduces to  $[x^2, y^2][y, r] = 0$ . Replacing  $y$  by  $y^2$  and  $r$  by  $rx^2$ , we obtain that  $[x^2, y^4]r[y^2, x^2] = 0$ . Simplifying this relation, we see that

$$y^2[x^2, y^2]r[y^2, x^2] + [x^2, y^2]y^2r[y^2, x^2] = 0$$

for all  $x, y, r \in R$ . In consideration of  $[x^2, y^2] \in Z(R)$ , we find that  $2y^2[x^2, y^2]r[y^2, x^2] = 0$  for all  $x, y, r \in R$ . Since  $R$  is 2-torsion free, we find that

$$(7) \quad y^2[x^2, y^2]r[y^2, x^2] = 0.$$

Left-multiplying relation (7) by  $x^2$  and replacing  $r$  by  $rx^2y^2$  with using the semiprimeness of  $R$ , we see that

$$(8) \quad x^2y^2[x^2, y^2] = 0.$$

In view of the relation  $[x^2, y^2] \in Z(R)$ , from relation (7), we obtain

$$(9) \quad [x^2, y^2]y^2r[y^2, x^2] = 0$$

Right-multiplying relation (9) by  $y^2x^2$  and replacing  $r$  by  $x^2r$  with using the semiprimeness of  $R$ , we arrive at

$$(10) \quad [x^2, y^2]y^2x^2 = 0.$$

Based on  $[x^2, y^2] \in Z(R)$  and relation (10), we conclude that

$$(11) \quad y^2x^2[x^2, y^2] = 0.$$

Subtracting (8) and (11), we obtain  $[x^2, y^2]^2 = 0$  for all  $x, y \in R$ . Applying Lemma 2 and using that  $R$  is 2-torsion free semiprime ring, we see that  $[x^2, y^2] = 0$ . Employing Lemma 3, we obtain the result as the desired. By using similar manner with using relation (4) the conclusion can be obtained for the case  $d = 0$ . ■

By using similar technique with some necessary variations, we can prove the following.

**Corollary 13.** *Let  $R$  be a 2-torsion free semiprime ring and  $U$  be a non-zero ideal of  $R$ . Suppose that a derivation  $d$  satisfies  $[d(x), d(y)] - [x^2, y^2] \in Z(R)$  for all  $x, y \in U$ . Then  $R$  contains a non-zero central ideal.*

**Theorem 14.** *Let  $R$  be a 2-torsion free semiprime ring and  $U$  be a non-zero ideal of  $R$ . Suppose that a derivation  $d$  satisfies  $[d^2(x), d^2(y)] - [x^2, y^2] \in Z(R)$  for all  $x, y \in U$ . Then  $R$  contains a non-zero central ideal.*

**Proof.** We begin with the situation  $d \neq 0$ , then  $[d^2(x), d^2(y)] - [x^2, y^2] \in Z(R)$  for all  $x, y \in U$ . Linearizing  $x$  by  $x + y$  yields  $[d(d(x + y)), d^2(y)] - [x, y] - [xy, y^2] - [yx, y^2] \in Z(R)$ , for all  $x, y \in U$ . According to the hypothesis, we arrive to  $[[xy, y^2] + [yx, y^2], r] = 0$ , for all  $x, y \in U, r \in R$ . This can be written as  $[[xy + yx, y^2], r] = 0$ . By using similar argument as we have used in Theorem 12, we get the result. ■

**Theorem 15.** *Let  $R$  be a 2-torsion free semiprime ring and  $U$  be a non-zero ideal of  $R$ . Suppose that a derivation  $d$  satisfies  $d([d(x), d(y)]) - [x, y] \in Z(R)$  for all  $x, y \in U$  and acts as a homomorphism. Then  $R$  contains a non-zero central ideal.*

**Proof.** Let us start with  $d \neq 0$ , that is mean, we have  $d([d(x), d(y)]) - [x, y] \in Z(R)$  for all  $x, y \in U$ . Replacing  $x$  by  $x^2$ , give us the relation  $d([d(x)x, d(y)]) + d([xd(x), d(y)]) - [x^2, y] \in Z(R)$ .

We extend this expression to  $d(d(x)[x, d(y)]) + d([d(x), d(y)]x) + d(x[d(x), d(y)]) + d([x, d(y)]d(x)) - [x^2, y] \in Z(R)$ .

Replacing  $y$  by  $x$ , we conclude that  $d(d(x)[x, d(x)]) + d([x, d(x)]d(x)) \in Z(R)$ , for all  $x \in U$ .

This implies that  $d([d(x)^2, x]) \in Z(R)$ . Linearizing  $x$  by  $x + y$ , for all  $x, y \in U$ , we obtain  $8d(d(x^2)x - xd(x^2)) \in Z(R)$  which implies to  $8[d^2(x^2), x] + [d(x^2), d(x)] \in Z(R)$ . Since  $d$  is a homomorphism this relation reduces to  $8[d^2(x^2), x] \in Z(R)$ . Simplify and using  $R$  is 2-torsion free, we conclude that  $[d^2(x)x + 2d(x)^2 + xd^2(x), x], r] = 0$ . Furthermore, we see that  $[[d^2(x), x^2] + 2[d(x)^2, x], r] = 0$ , for all  $x \in U, r \in R$ . Writing  $x$  by  $x^2$ , we obtain  $[[d^2(x^2), x^4] + 2[d(x^2)^2, x^2], r] = 0$ , for all  $x \in U, r \in R$ .



Then, we find that  $[[d^2(x^2), x^4], r] + 2[[d(x^2)^2, x^2], r] = 0$  for all  $x \in U$ ,  $r \in R$ . Now we expand this relation to  $[x^2[d^2(x^2), x^2] + [d^2(x^2), x^2]x^2, r] + 2[[d(x^2)^2, x^2], r] = 0$ .

Using simple calculation yields that  $[x^3[d^2(x^2), x] + x^2[d^2(x^2), x]x + x[d^2(x^2), x]x^2 + [d^2(x^2), x]x^3, r] + 2[[d(x^2)^2, x^2], r] = 0$ , for all  $x \in U$ ,  $r \in R$ .

Moreover, we observe that  $[[x^3[d^2(x^2), x], r] + [x^2[d^2(x^2), x]x, r] + [x[d^2(x^2), x]x^2, r] + [[d^2(x^2), x]x^3, r] + 2[[d(x^2)^2, x^2], r] = 0$ .

Based on  $8[d^2(x^2), x] \in Z(R)$  for all  $x \in U$ , we conclude that  $8([x^3, r][d^2(x^2), x] + x^2[d^2(x^2), x][x, r] + [x^2, r][d^2(x^2), x]x + x[d^2(x^2), x][x^2, r] + [x, r][d^2(x^2), x]x^2 + [d^2(x^2), x][x^3, r]) + 16[[d(x^2)^2, x^2], r] = 0$ .

Replacing  $r$  by  $x$ , we obtain  $16[[d(x^2)^2, x^2], x] = 0$  for all  $x \in U$ . Linearizing  $x$  by  $x + y$  and using  $R$  is 2-torsion free, we obtain  $[d(x^4), x^2], x] = 0$  for all  $x \in U$ . We can complete the proof by applying Lemma 1 to this relation. ■

**Remark 16.** In the previous results the condition "torsion free" is essential in the hypothesis. The following example demonstrates that.

**Example 17.** Let  $R = \mathcal{M}_2(\mathbb{F})$  be a ring of  $2 \times 2$  matrices over a field  $\mathbb{F}$  with  $char. = 2$ , choose that:  $R = \mathcal{M}_2(\mathbb{F}) = \left\{ \begin{pmatrix} x & y \\ y & x \end{pmatrix} \mid x, y \in \mathbb{F} \right\}$ . Let  $d$  be the additive mapping of  $R$ , given by  $d(a) = \begin{pmatrix} 0 & y \\ y & 0 \end{pmatrix}$ .

Now, we check whether  $d$  is a derivation of  $R$ . Then, for all  $a, b \in R$ ,  $a = \begin{pmatrix} x & y \\ y & x \end{pmatrix}$ , and  $b = \begin{pmatrix} g & h \\ h & g \end{pmatrix}$ . The left side produces  $d(ab) = d\left(\begin{pmatrix} x & y \\ y & x \end{pmatrix} \begin{pmatrix} g & h \\ h & g \end{pmatrix}\right) = d\left(\begin{pmatrix} xg + yh & xh + yg \\ yg + xh & yh + xg \end{pmatrix}\right)$ .

According to the form of  $d$ , we gain  $d(ad) = \begin{pmatrix} 0 & xh + yg \\ (yg + xh) & 0 \end{pmatrix}$ .

From the right side, we see that

$$\begin{aligned} d(a)b + ad(b) &= \begin{pmatrix} 0 & y \\ y & 0 \end{pmatrix} \begin{pmatrix} g & h \\ h & g \end{pmatrix} + \begin{pmatrix} x & y \\ y & x \end{pmatrix} \begin{pmatrix} 0 & h \\ h & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2yh & yg + xh \\ (yg + xh) & 0 \end{pmatrix}. \end{aligned}$$

By reason of the fact that  $R$  has  $char. = 2$ , the above matrix modifies to

$$= \begin{pmatrix} 0 & yg + xh \\ yg + xh & 0 \end{pmatrix}. \text{ Obviously, we find that } d \text{ is a derivation of } R.$$

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