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ANALYTIC PROPERTIES OF THE APOSTOL-VU MULTIPLE FIBONACCI ZETA FUNCTIONS

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Abstract

In this note we study the analytic continuation of the Apostol-Vu multiple Fibonacci zeta functions

$$\zeta_{AVF,k}(s_1,\dots,s_k;s_{k+1}) \quad = \quad \sum_{1\le m_1<\cdots<m_k} \frac{1}{F_{m_1}^{s_1}F_{m_2}^{s_2}\cdots F_{m_k}^{s_k}F_{m_1+m_2+\cdots+m_k}^{s_{k+1}}},$$

where $s_1,\dots,s_{k+1}$ are complex variables and F_n is the n-th Fibonacci number. We find a complete list of poles and their corresponding residues.

Keywords: analytic continuation, Fibonacci numbers, multiple Fibonacci zeta function, Apostol-Vu multiple Fibonacci zeta function.

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1. Introduction

The Euler-Zagier's multiple zeta function $\zeta_{EZ,k}$ of depth k is defined by

$$\zeta_{EZ,k}(s_1,\dots,s_k) = \sum_{1\le m_1<m_2\cdots<m_k<\infty} \frac{1}{{m_1}^{s_1}{m_2}^{s_2}\cdots {m_k}^{s_k}}, \tag{1.1}$$

where $s_1,\dots,s_k$ are complex variables [10]. Zhao [11] gave an analytic continuation of (1.1) as a function of $s_i (i=1,\dots,k)$ to $\mathbb{C}^k$ using Gelfand and Shilov's generalized functions. Mehta *et al.* [6] showed the meromorphic continuation of multiple zeta functions (1.1) with their poles and residues by means of an elementary and simple translation formula for specified multiple zeta functions. The series

$$\sum_{m=1}^{\infty}\sum_{n<m}\frac{1}{m^{s_1}n^{s_2}(m+n)}, \tag{1.2}$$

was first introduced by Apostol and Vu [1] and they obtained the partial results on its analytic continuation. The meromorphic continuation of (1.2) and more general series

$$\sum_{m=1}^{\infty}\sum_{n<m}\frac{1}{m^{s_1}n^{s_2}(m+n)^{s_3}}, \tag{1.3}$$

to the whole space was first proved in [3]. In [4], Matsumoto generalized (1.3) and introduced Apostol-Vu r-ple zeta function

$$\begin{aligned}&\zeta_{AV,r}(s_1,\ldots,s_r;s_{r+1})\\ &=\sum_{1\le m_1<m_2<\cdots<m_r<\infty}\frac{1}{m_1^{s_1}m_2^{s_2}\cdots m_r^{s_r}(m_1+m_2+\cdots+m_r)^{s_{r+1}}}.\end{aligned} \tag{1.4}$$

He proved that the series (1.4) is convergent absolutely when $\mathrm{Re}(s_i)>1$ $(1\le i\le r)$, $\mathrm{Re}(s_{r+1})>0$ and can be continued meromorphically to the whole space $\mathbb{C}^{r+1}$.

Let $\{F_n\}_{n\ge 0}$ be the sequence of Fibonacci numbers and is recursively defined by

$$F_0=0,\ F_1=1,\ F_{n+1}=F_n+F_{n-1},\ \text{for } n\ge 1.$$

The closed form expression for $\{F_n\}_{n\ge0}$ is $F_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}$, where $\alpha=\frac{1+\sqrt{5}}{2}$ and $\beta=\frac{1-\sqrt{5}}{2}$ are the zeros of Fibonacci characteristic equation $x^2-x-1=0$. In [7], Navas presented Fibonacci Dirichlet series $\zeta_F(s)=\sum_{m=1}^{\infty}F_m^{-s}$, $Re(s)>0$ for $s\in\mathbb{C}$, where F_m denotes the m-th Fibonacci number and obtained that $\zeta_F(s)$ is analytically continued to a meromorphic function on the complex plane $\mathbb{C}$. Rout and Panda [9] considered balancing zeta function $\zeta_B(s)=\sum_{m=1}^{\infty}B_m^{-s}$, $Re(s)>0$ for $s\in\mathbb{C}$, where B_m denotes the m-th balancing number and gave that $\zeta_B(s)$ is meromorphically continued all over $\mathbb{C}$. They also obtained that $\zeta_B(-m)$ is an irrational number when m is an odd natural number. In a consequent paper, Behera *et al.* [2] proved the analytical continuation of $\zeta_C(s)=\sum_{m=1}^{\infty}C_m^{-s}$, $Re(s)>0$ for $s\in\mathbb{C}$, where C_m denotes the m-th Lucas-balancing number and $\zeta_C(-m)$ is a rational number for any odd natural number m. In [8], Rout and Meher defined the multiple Fibonacci zeta function of depth k as:

$$\zeta_F(s_1,\ldots,s_k)=\sum_{1\le m_1<m_2<\cdots<m_k<\infty}\frac{1}{F_{m_1}^{s_1}F_{m_2}^{s_2}\cdots F_{m_k}^{s_k}}. \tag{1.5}$$

They derived the analytic continuation of $\zeta_F(s_1, \ldots, s_k)$ of depth 2 and found a complete list of poles and their corresponding residues. Recently, Meher and Rout [5] proved the meromorphic continuation of multiple Lucas zeta functions of depth d :

$$\zeta_U(s_1, \ldots, s_d) = \sum_{0<n_1<n_2\cdots<n_d} \frac{1}{U_{n_1}^{s_1} \cdots U_{n_d}^{s_d}},$$

where (U_n) is the Lucas sequence of first kind.

In this note, we present Apostol-Vu multiple Fibonacci zeta function and explore its analytic continuation. We also estimate the residues corresponding to their respective poles.

2. Analytic continuation of Apostol-Vu multiple Fibonacci zeta functions

In this section, the Apostol-Vu multiple Fibonacci zeta function is introduced and its analytic continuation is examined.

Definition 1. We define Apostol-Vu multiple Fibonacci zeta functions $\zeta_{AVF,k}$ as

$$\zeta_{AVF,k}(s_1, \ldots, s_k; s_{k+1}) = \sum_{1\le m_1<\cdots<m_k<\infty} \frac{1}{F_{m_1}^{s_1} F_{m_2}^{s_2} \cdots F_{m_k}^{s_k} F_{m_1+m_2+\cdots+m_k}^{s_{k+1}}}, \tag{2.1}$$

where $s_1, \ldots, s_{k+1}$ are complex variables.

The sum $s_1 + \cdots + s_{k+1}$ is called the weight of $\zeta_{AVF,k}(s_1, \ldots, s_k; s_{k+1})$ and k is called its depth. One can observe that $\zeta_{AVF,k}(s_1, \ldots, s_k; 0)$ is the multiple Fibonacci zeta function (1.5).

Proposition 2. The series (2.1) is absolutely convergent in

$$D_{k+1} = \left\{(s_1, \ldots, s_k; s_{k+1}) \in \mathbb{C}^{k+1} \middle| \sum_{j=d}^{k} Re(s_j) + (k+1-d) Re(s_{k+1}) > 0, 1 \le d \le k\right\}.$$

Proof. Let $s_j = \sigma_j + it_j \in \mathbb{C}$ and $\sigma_j = Re(s_j) > 0$, $j = 1, 2, \ldots, k+1$. From (2.1), we can write

$$\begin{aligned}
&\sum_{1\le m_1<\cdots<m_k} \frac{1}{F_{m_1}^{s_1} F_{m_2}^{s_2} \cdots F_{m_k}^{s_k} F_{m_1+m_2+\cdots+m_k}^{s_{k+1}}} \\
&= \sum_{m_1=1}^{\infty} \frac{1}{F_{m_1}^{s_1}} \sum_{m_2=1}^{\infty} \frac{1}{F_{m_1+m_2}^{s_1}} \cdots \\
&\sum_{m_{k-1}=1}^{\infty} \frac{1}{F_{m_1+m_2+\cdots+m_{k-1}}^{s_{k-1}}} \sum_{m_k=1}^{\infty} \frac{1}{F_{m_1+m_2+\cdots+m_k}^{s_k} F_{km_1+(k-1)m_2+\cdots+m_k}^{s_{k+1}}}.
\end{aligned} \tag{2.2}$$

Now,

$$\begin{aligned}\sum_{m_1=1}^{\infty}\left|\frac{1}{F_{m_1}^{s_1}}\right| &= (\alpha-\beta)^{\sigma_1}\sum_{m_1=1}^{\infty}\left|\frac{1}{\big(\alpha^{m_1}-\beta^{m_1}\big)^{s_1}}\right|\\ &\leq (\alpha-\beta)^{\sigma_1}\sum_{m_1=1}^{\infty}\frac{1}{\alpha^{m_1\sigma_1}\big(1-(|\frac{\beta}{\alpha}|)^{m_1}\big)^{s_1}}\\ &\leq \left(\frac{\alpha-\beta}{1-|\beta/\alpha|}\right)^{\sigma_1}\sum_{m_1=1}^{\infty}\frac{1}{\alpha^{\sigma_1 m_1}} = \Lambda_{\sigma_1}(\alpha-\beta)^{\sigma_1}\frac{1}{\alpha^{\sigma_1}-1}\end{aligned} \tag{2.3}$$

and similarly,

$$\left|\frac{1}{F_{m_1+\cdots+m_d}^{s_d}}\right| \leq \Lambda_{\sigma_d}(\alpha-\beta)^{\sigma_d}\frac{1}{\alpha^{\sigma_d\big(m_1+\cdots+m_d\big)}},\ 2\leq d\leq k, \tag{2.4}$$

and

$$\left|\frac{1}{F_{km_1+(k-1)m_2+\cdots+m_k}^{s_{k+1}}}\right| \leq \Lambda_{\sigma_{k+1}}(\alpha-\beta)^{\sigma_{k+1}}\frac{1}{\alpha^{\sigma_{k+1}\big(km_1+(k-1)m_2+\cdots+m_k\big)}}, \tag{2.5}$$

where $\Lambda_{\sigma_j} = \frac{1}{(1-|\beta/\alpha|)^{\sigma_j}}$. By virtue of (2.2), (2.3), (2.4) and (2.5),

$$\begin{aligned}&\sum_{1\leq m_1<\cdots<m_k}\left|\frac{1}{F_{m_1}^{s_1}\cdots F_{m_k}^{s_k}F_{m_1+\cdots+m_k}^{s_{k+1}}}\right|\\ &\leq \sum_{m_1=1}^{\infty}\left|\frac{1}{F_{m_1}^{s_1}}\right|\sum_{m_2=1}^{\infty}\left|\frac{1}{F_{m_1+m_2}^{s_1}}\right|\cdots\\ &\quad\sum_{m_{k-1}=1}^{\infty}\left|\frac{1}{F_{m_1+\cdots+m_{k-1}}^{s_{k-1}}}\right|\sum_{m_k=1}^{\infty}\left|\frac{1}{F_{m_1+\cdots+m_k}^{s_k}F_{km_1+(k-1)m_2+\cdots+m_k}^{s_{k+1}}}\right|\\ &\leq \Lambda_{\sigma_1}\Lambda_{\sigma_2}\cdots\Lambda_{\sigma_k}\Lambda_{\sigma_{k+1}}(\alpha-\beta)^{\sigma_1}\cdots(\alpha-\beta)^{\sigma_{k+1}}\sum_{m_1=1}^{\infty}\frac{1}{\alpha^{\sigma_1 m_1}}\sum_{m_2=1}^{\infty}\frac{1}{\alpha^{\sigma_2(m_1+m_2)}}\\ &\quad\times\cdots\times\sum_{m_{k-1}=1}^{\infty}\frac{1}{\alpha^{\sigma_{k-1}(m_1+m_2+\cdots+m_{k-1})}}\sum_{m_k=1}^{\infty}\frac{1}{\alpha^{\sigma_k(m_1+\cdots+m_k)}}\frac{1}{\alpha^{\sigma_{k+1}(km_1+(k-1)m_2+\cdots+m_k)}}\\ &= \Lambda_{\sigma}(\alpha-\beta)^{\sum_{j=1}^{k+1}\sigma_j}\sum_{m_1=1}^{\infty}\frac{1}{\alpha^{(\sigma_1+\cdots+\sigma_k+k\sigma_{k+1})m_1}}\sum_{m_2=1}^{\infty}\frac{1}{\alpha^{(\sigma_2+\cdots+\sigma_k+(k-1)\sigma_{k+1})m_2}}\\ &\quad\times\cdots\times\sum_{m_k=1}^{\infty}\frac{1}{\alpha^{(\sigma_k+\sigma_{k+1})m_k}}\end{aligned}$$

$$= \Lambda_\sigma(\alpha-\beta)^{\frac{\sum_{j=1}^{k+1}\sigma_j}{2}} \frac{1}{\big(\alpha^{\sigma_1+\cdots+\sigma_k+k\sigma_{k+1}}-1\big)} \frac{1}{\big(\alpha^{\sigma_2+\cdots+\sigma_k+(k-1)\sigma_{k+1}}-1\big)}$$
$$\times\cdots\times \frac{1}{\big(\alpha^{\sigma_k+\sigma_{k+1}-1}\big)} < \infty,$$

as $\alpha > 1$, where $\Lambda_\sigma = \Lambda_{\sigma_1}\ldots\Lambda_{\sigma_k}\Lambda_{\sigma_{k+1}}$ and $\Lambda_{\sigma_1},\ldots,\Lambda_{\sigma_k}$ and $\Lambda_{\sigma_{k+1}}$ are the positive constants depending on $\sigma_1,\ldots,\sigma_k$ and σ_{k+1} respectively. Therefore, the series (2.1) converges absolutely in the domain D_{k+1}. This completes the proof. ■

Theorem 3. *The Apostol-Vu multiple Fibonacci zeta function $\zeta_{AVF,k}(s_1,\ldots,s_k;s_{k+1})$ can be analytically continued to a meromorphic function on $\mathbb{C}^{k+1}$. It has possible simple poles on the hyperplanes*

$$\begin{aligned} s_d+\cdots+s_k+(k+1-d)s_{k+1} &= -2(r_d+\cdots+r_k+(k+1-d)r_{k+1}) \\ &+ \frac{\pi i\big(r_d+\cdots+r_k+(k+1-d)r_{k+1}+2a\big)}{\log\alpha}, \\ &1 \le d \le k, \end{aligned}$$

with $r_1,\ldots,r_{k+1}\in\mathbb{Z}_{\ge o}$ and $a\in\mathbb{Z}$.

Proof. For any $s=\sigma+it\in\mathbb{C}$, we have

$$\begin{aligned} F_m^s = \left(\frac{\alpha^m-\beta^m}{\sqrt{5}}\right)^s &= (\sqrt{5})^{-s}\alpha^{ms}\left(1-\left(\frac{\beta}{\alpha}\right)^m\right)^s \\ &= 5^{-\frac{s}{2}}\alpha^{ms}\sum_{k=0}^{\infty}\binom{s}{k}(-1)^k\left(\frac{\beta}{\alpha}\right)^{mk} \\ &= 5^{-\frac{s}{2}}\sum_{k=0}^{\infty}\binom{s}{k}(-1)^{\big(m+1\big)k}\alpha^{m\big(s-2k\big)}. \end{aligned} \tag{2.6}$$

As $\alpha>1$, the series (2.6) converges. Using the above identity in (2.2), we get

$$\begin{aligned} &\zeta_{AVF,k}(s_1,\ldots,s_k;s_{k+1}) \\ &= \sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\cdots\sum_{m_k=1}^{\infty}\frac{1}{F_{m_1}^{s_1}}\frac{1}{F_{m_1+m_2}^{s_2}}\cdots\frac{1}{F_{m_1+m_2+\cdots+m_k}^{s_k}}\frac{1}{F_{km_1+(k-1)m_2+\cdots+m_k}^{s_{k+1}}} \\ &= \sum_{m_1,m_2,\ldots,m_k=1}^{\infty} 5^{\frac{s_1}{2}}\sum_{r_1=0}^{\infty}\binom{-s_1}{r_1}(-1)^{(m_1+1)r_1}\alpha^{-m_1(s_1+2r_1)} \\ &\times 5^{\frac{s_2}{2}}\sum_{r_2=0}^{\infty}\binom{-s_2}{r_2}(-1)^{(m_1+m_2+1)r_2}\alpha^{-(m_1+m_2)(s_2+2r_2)}\times\cdots \end{aligned}$$

$$\times\, 5^{\frac{s_k}{2}} \sum_{r_k=0}^{\infty} \binom{-s_k}{r_k} (-1)^{\left(1+\sum_{d=1}^{k} m_d\right) r_k} \alpha^{-\left(\sum_{d=1}^{k} m_d\right)(s_k+2r_k)}$$
$$\times\, 5^{\frac{s_{k+1}}{2}} \sum_{r_{k+1}=0}^{\infty} \binom{-s_{k+1}}{r_{k+1}} (-1)^{\left(1+\sum_{d=1}^{k}(k+1-d)m_d\right) r_{k+1}} \tag{2.7}$$
$$\times\, \alpha^{-\left(\sum_{d=1}^{k}(k+1-d)m_d\right)(s_{k+1}+2r_{k+1})}.$$

Using the fact $\left|\binom{-s_i}{r_i}\right| \leq (-1)^{r_i}\binom{-|s_i|}{r_i}$ for $1 \leq i \leq k+1$, we have

$$\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\cdots\sum_{m_k=1}^{\infty}\left|\frac{1}{F_{m_1}^{s_1}}\frac{1}{F_{m_1+m_2}^{s_2}}\cdots\frac{1}{F_{m_1+m_2+\cdots+m_k}^{s_k}}\frac{1}{F_{km_1+(k-1)m_2+\cdots+m_k}^{s_{k+1}}}\right|$$
$$= \sum_{m_1,m_2,\ldots,m_k=1}^{\infty}\left| 5^{\frac{s_1}{2}} \sum_{r_1=0}^{\infty}\binom{-s_1}{r_1}(-1)^{(m_1+1)r_1}\alpha^{-m_1(s_1+2r_1)}\right.$$
$$\times\, 5^{\frac{s_2}{2}} \sum_{r_2=0}^{\infty}\binom{-s_2}{r_2}(-1)^{(m_1+m_2+1)r_2}\alpha^{-(m_1+m_2)(s_2+2r_2)} \times\cdots$$
$$\times\, 5^{\frac{s_k}{2}} \sum_{r_k=0}^{\infty} \binom{-s_k}{r_k} (-1)^{\left(1+\sum_{d=1}^{k} m_d\right) r_k} \alpha^{-\left(\sum_{d=1}^{k} m_d\right)(s_k+2r_k)}$$
$$\left.\times\, 5^{\frac{s_{k+1}}{2}} \sum_{r_{k+1}=0}^{\infty} \binom{-s_{k+1}}{r_{k+1}} (-1)^{\left(1+\sum_{d=1}^{k}(k+1-d)m_d\right) r_{k+1}} \alpha^{-\left(\sum_{d=1}^{k}(k+1-d)m_d\right)(s_{k+1}+2r_{k+1})}\right|$$
$$\leq \sum_{m_1,m_2,\ldots,m_k=1}^{\infty} 5^{\frac{\sum_{n=1}^{k+1}\sigma_n}{2}}\alpha^{-m_1\sigma_1}\sum_{r_1=0}^{\infty}\binom{-|s_1|}{r_1}(-1)^{r_1}\alpha^{-2m_1r_1}$$
$$\times\, \alpha^{-(m_1+m_2)\sigma_2}\sum_{r_2=0}^{\infty}\binom{-|s_2|}{r_2}(-1)^{r_2}\alpha^{-2(m_1+m_2)r_2}\times\cdots$$
$$\times \alpha^{-\left(\sum_{d=1}^{k} m_d\right)\sigma_k}\sum_{r_k=0}^{\infty}\binom{-|s_k|}{r_k}(-1)^{r_k}\alpha^{-2\left(\sum_{d=1}^{k} m_d\right)r_k}$$
$$\times\, \alpha^{\left(\sum_{d=1}^{k}(k+1-d)m_d\right)\sigma_{k+1}}\sum_{r_{k+1}=0}^{\infty}\binom{-|s_{k+1}|}{r_{k+1}}(-1)^{r_{k+1}}\alpha^{-2\left(\sum_{d=1}^{k}(k+1-d)m_d\right)r_{k+1}}$$
$$= \sum_{m_1,m_2,\ldots,m_k=1}^{\infty} 5^{\frac{\sum_{n=1}^{k+1}\sigma_n}{2}}\alpha^{-m_1\sigma_1}\left(1-\alpha^{-2m_1}\right)^{-|s_1|}\alpha^{-(m_1+m_2)\sigma_2}$$
$$\times\left(1-\alpha^{-2(m_1+m_2)}\right)^{-|s_2|}\times\cdots\times\ \alpha^{\left(\sum_{d=1}^{k} m_d\right)\sigma_k}\left(1-\alpha^{-2\left(\sum_{d=1}^{k} m_d\right)}\right)^{-|s_k|}$$

$$
\begin{aligned}
&\times\ \alpha^{\left(\sum_{d=1}^{k}(k+1-d)m_d\right)\sigma_{k+1}}\left(1-\alpha^{-2\left(\sum_{d=1}^{k}(k+1-d)m_d\right)}\right)^{-|s_{k+1}|}\\
\leq\ & \sum_{m_1,m_2,\ldots,m_k=1}^{\infty} 5^{\frac{\sum_{n=1}^{k+1}\sigma_n}{2}}\alpha^{-m_1\sigma_1}\left(1-\alpha^{-2}\right)^{-|s_1|}\alpha^{-(m_1+m_2)\sigma_2}\left(1-\alpha^{-2}\right)^{-|s_2|}\\
&\times\cdots\times\ \alpha^{\left(\sum_{d=1}^{k}m_d\right)\sigma_k}\left(1-\alpha^{-2}\right)^{-|s_k|}\times\alpha^{\left(\sum_{d=1}^{k}(k+1-d)m_d\right)\sigma_{k+1}}\left(1-\alpha^{-2}\right)^{-|s_{k+1}|}\\
=\ & 5^{\frac{\sum_{n=1}^{k+1}\sigma_n}{2}}\left(1-\alpha^{-2}\right)^{-\left(\sum_{n=1}^{k+1}|s_n|\right)}\sum_{m_1=1}^{\infty}\alpha^{-(\sigma_1+\cdots+\sigma_k+k\sigma_{k+1})m_1}\\
&\sum_{m_2=1}^{\infty}\alpha^{-(\sigma_2+\cdots+\sigma_k+(k-1)\sigma_{k+1})m_2}\cdots\sum_{m_k=1}^{\infty}\alpha^{-(\sigma_k+\sigma_{k+1})m_k}<\infty.
\end{aligned}
$$

Thus, the above series absolutely converges for a fixed point $(s_1,\ldots,s_k;s_{k+1})$ in D_{k+1}. Then, by interchanging the order of summation in (2.7), we get

$$
\begin{aligned}
&\zeta_{AVF,k}(s_1,\ldots,s_k;s_{k+1})\\
=\ & 5^{\frac{\sum_{n=1}^{k+1}s_n}{2}}\sum_{r_1=0}^{\infty}\binom{-s_1}{r_1}(-1)^{r_1}\sum_{r_2=0}^{\infty}\binom{-s_2}{r_2}(-1)^{r_2}\cdots\sum_{r_{k+1}=0}^{\infty}\binom{-s_{k+1}}{r_{k+1}}(-1)^{r_{k+1}}\\
&\times\sum_{m_1,m_2,\ldots,m_k=1}^{\infty}(-1)^{m_1r_1}\alpha^{-m_1(s_1+2r_1)}(-1)^{(m_1+m_2)r_2}\alpha^{-(m_1+m_2)(s_2+2r_2)}\\
&\cdots\ (-1)^{\left(\sum_{d=1}^{k}m_d\right)r_k}\\
&\times\ \alpha^{-\left(\sum_{d=1}^{k}m_d\right)(s_k+2r_k)}(-1)^{\left(\sum_{d=1}^{k}(k+1-d)m_d\right)r_{k+1}}\alpha^{-\left(\sum_{d=1}^{k}(k+1-d)m_d\right)(s_{k+1}+2r_{k+1})}\\
=\ & 5^{\frac{\sum_{n=1}^{k+1}s_n}{2}}\sum_{r_1=0}^{\infty}\binom{-s_1}{r_1}(-1)^{r_1}\sum_{r_2=0}^{\infty}\binom{-s_2}{r_2}(-1)^{r_2}\cdots\sum_{r_{k+1}=0}^{\infty}\binom{-s_{k+1}}{r_{k+1}}(-1)^{r_{k+1}}\\
&\times\sum_{m_1,m_2,\ldots,m_k=1}^{\infty}\left((-1)^{r_1+\cdots+r_k+kr_{k+1}}\alpha^{-\left(s_1+\cdots+s_k+ks_{k+1}+2(r_1+\cdots+r_k+kr_{k+1})\right)}\right)^{m_1}\\
&\times\left((-1)^{r_2+\cdots+r_k+(k-1)r_{k+1}}\alpha^{-\left(s_2+\cdots+s_k+(k-1)s_{k+1}+2(r_2+\cdots+r_k+(k-1)r_{k+1})\right)}\right)^{m_2}\\
&\times\cdots\times\left((-1)^{r_k+r_{k+1}}\alpha^{-\left(s_k+s_{k+1}+2(r_k+r_{k+1})\right)}\right)^{m_k}. \qquad (2.8)
\end{aligned}
$$

Now for any $(r_1,\ldots,r_k;r_{k+1})\in\mathbb{Z}_{\geq 0}^{k+1}$, we have

$$
\sum_{m_1,m_2,\ldots,m_k=1}^{\infty}\left((-1)^{r_1+\cdots+r_k+kr_{k+1}}\alpha^{-\left(s_1+\cdots+s_k+ks_{k+1}+2(r_1+\cdots+r_k+kr_{k+1})\right)}\right)^{m_1}
$$

$$\times\ \Big((-1)^{r_2+\cdots+r_k+(k-1)r_{k+1}}\alpha^{-\left(s_2+\cdots+s_k+(k-1)s_{k+1}+2(r_2+\cdots+r_k+(k-1)r_{k+1})\right)}\Big)^{m_2}$$
$$\times\cdots\times\Big((-1)^{r_k+r_{k+1}}\alpha^{-\left(s_k+s_{k+1}+2(r_k+r_{k+1})\right)}\Big)^{m_k}$$
$$= \frac{(-1)^{r_1+\cdots+r_k+kr_{k+1}}\alpha^{-\left(s_1+\cdots+s_k+ks_{k+1}+2(r_1+\cdots+r_k+kr_{k+1})\right)}}{\left(1-(-1)^{r_1+\cdots+r_k+kr_{k+1}}\alpha^{-\left(s_1+\cdots+s_k+ks_{k+1}+2(r_1+\cdots+r_k+kr_{k+1})\right)}\right)}$$
$$\times\ \frac{(-1)^{r_2+\cdots+r_k+(k-1)r_{k+1}}\alpha^{-\left(s_2+\cdots+s_k+(k-1)s_{k+1}+2(r_2+\cdots+r_k+(k-1)r_{k+1})\right)}}{\left(1-(-1)^{r_2+\cdots+r_k+(k-1)r_{k+1}}\alpha^{-\left(s_2+\cdots+s_k+(k-1)s_{k+1}+2(r_2+\cdots+r_k+(k-1)r_{k+1})\right)}\right)}$$
$$\times\cdots\times\frac{(-1)^{r_k+r_{k+1}}\alpha^{-\left(s_k+s_{k+1}+2(r_k+r_{k+1})\right)}}{\left((1-(-1)^{r_k+r_{k+1}}\alpha^{-\left(s_k+s_{k+1}+2(r_k+r_{k+1})\right)}\right)}$$
$$= \frac{(-1)^{r_1+\cdots+r_k+kr_{k+1}}}{\left(\alpha^{s_1+\cdots+s_k+ks_{k+1}+2(r_1+\cdots+r_k+kr_{k+1})}-(-1)^{r_1+\cdots+r_k+kr_{k+1}}\right)}$$
$$\times\ \frac{(-1)^{r_2+\cdots+r_k+(k-1)r_{k+1}}}{\left(\alpha^{s_2+\cdots+s_k+(k-1)s_{k+1}+2(r_2+\cdots+r_k+(k-1)r_{k+1})}-(-1)^{r_2+\cdots+r_k+(k-1)r_{k+1}}\right)}$$
$$\times\cdots\times\frac{(-1)^{r_k+r_{k+1}}}{\left(\alpha^{s_k+s_{k+1}+2(r_k+r_{k+1})}-(-1)^{r_k+r_{k+1}}\right)}. \tag{2.9}$$

By virtue of (2.8) and (2.9), we have

$$\zeta_{AVF,k}(s_1,\ldots,s_k;s_{k+1})$$
$$= 5^{\frac{\sum_{n=1}^{k+1}s_n}{2}}\sum_{r_1=0}^{\infty}\binom{-s_1}{r_1}\sum_{r_2=0}^{\infty}\binom{-s_2}{r_2}\cdots\sum_{r_{k+1}=0}^{\infty}\binom{-s_{k+1}}{r_{k+1}}(-1)^{\sum_{n=1}^{k+1}r_n}$$
$$\times\ \frac{(-1)^{r_1+\cdots+r_k+kr_{k+1}}}{\left(\alpha^{s_1+\cdots+s_k+ks_{k+1}+2(r_1+\cdots+r_k+kr_{k+1})}-(-1)^{r_1+\cdots+r_k+kr_{k+1}}\right)}$$
$$\times\ \frac{(-1)^{r_2+\cdots+r_k+(k-1)r_{k+1}}}{\left(\alpha^{s_2+\cdots+s_k+(k-1)s_{k+1}+2(r_2+\cdots+r_k+(k-1)r_{k+1})}-(-1)^{r_2+\cdots+r_k+(k-1)r_{k+1}}\right)}$$
$$\times\cdots\times\frac{(-1)^{r_k+r_{k+1}}}{\left(\alpha^{s_k+s_{k+1}+2(r_k+r_{k+1})}-(-1)^{r_k+r_{k+1}}\right)}. \tag{2.10}$$

Thus, this bound is uniform when $(s_1,\ldots,s_k;s_{k+1})$ varies over compact subsets of $\mathbb{C}^{k+1}$. The infinite series (2.10) determines the holomorphic function on $\mathbb{C}^{k+1}$ except for the poles derived from the functions

$$\begin{aligned}&\alpha^{s_d+\cdots+s_k+(k+1-d)s_{k+1}+2(r_d+\cdots+r_k+(k+1-d)r_{k+1})}\\&-(-1)^{r_d+\cdots+r_k+(k+1-d)r_{k+1}}=0\end{aligned} \tag{2.11}$$

for $d = 1, \ldots, k$.

Hence, the series (2.10) converges uniformly and absolutely on compact subsets of $\mathbb{C}^{k+1}$ not containing any poles of the functions (2.11). Therefore, $\zeta_{AVF,k}(s_1, \ldots, s_k; s_{k+1})$ can be analytically continued to a meromorphic function on $\mathbb{C}^{k+1}$ and its simple poles are on the hyperplanes

$$\begin{aligned} s_d + \cdots + s_k + (k+1-d)s_{k+1} &= -2(r_d + \cdots + r_k + (k+1-d)r_{k+1}) \\ &+ \frac{\pi i\big(r_d + \cdots + r_k + (k+1-d)r_{k+1} + 2a\big)}{\log \alpha}, \end{aligned} \tag{2.12}$$

for $1 \leq d \leq k$, with $r_1, \ldots, r_{k+1} \in \mathbb{Z}_{\geq o}$ and $a \in \mathbb{Z}$. This finishes the proof. ■

3. Residues of Apostol-Vu multiple Fibonacci zeta functions at poles

We now calculate the residues of $\zeta_{AVF,k}(s_1, \ldots, s_k; s_{k+1})$ at the poles that lie on the hyperplanes derived from Theorem 3. For $1 \leq d \leq k$, let us denote

$$s_{k+1}(d) = s_d + \cdots + s_k + (k+1-d)s_{k+1}, r_{k+1}(d) = r_d + \cdots + r_k + (k+1-d)r_{k+1},$$

and $r'_{k+1}(d) = r'_d + \cdots + r'_k + (k+1-d)r'_{k+1}$, with the assumption that

$$s_{k+1}(d) = 0, r_{k+1}(d) = 0,\ r'_{k+1}(d) = 0 \text{ for } d \geq k+1.$$

We define the residue of the Apostol-Vu multiple Fibonacci zeta functions $\zeta_{AVF,k}(s_1, \ldots, s_k; s_{k+1})$ along the hyperplanes (2.12) to be the restriction of the meromorphic function

$$\left(s_{k+1}(d) + 2r_{k+1}(d) - \big(r_{k+1}(d) + 2a\big)\frac{\pi i}{\log \alpha}\right)\zeta_{AVF,k}(s_1, \ldots, s_k; s_{k+1})$$

to the hyperplanes (2.12).

Theorem 4. *Let $r'_d, \ldots, r'_k, r'_{k+1}$ be non-negative integers for $1 \leq d \leq k$ and $e_{k+1}(d) = -2r'_{k+1}(d) + \big(r'_{k+1}(d) + 2a\big)\frac{\pi i}{\log \alpha}$. Then the residue of $\zeta_{AVF,k}(s_1, \ldots, s_k;$ $s_{k+1})$ at $s_{k+1}(d) = e_{k+1}(d)$ is*

$$\begin{aligned} 5^{\frac{e_{k+1}(d)}{2}} \frac{(-1)^{r'_{k+1}(d)}}{\log \alpha} &\sum_{\substack{r_1, \ldots, r_k, r_{k+1} \geq 0 \\ r_{k+1}(d) = r'_{k+1}(d)}} \binom{-s_1}{r_1} \cdots \binom{-s_{k+1}}{r_{k+1}} (-1)^{(\sum_{i=1}^{d-1} r_i) + (d-k)r_{k+1}} \\ &\times 5^{\frac{(\sum_{i=1}^{d-1} s_i) + (d-k)s_{k+1}}{2}} \prod_{\substack{l=1 \\ l \neq d}}^{k} \frac{(-1)^{r_{k+1}(l)}}{\big(\alpha^{s_{k+1}(l) + 2r_{k+1}(l)} - (-1)^{r_{k+1}(l)}\big)}. \end{aligned}$$

Proof. The functions $\alpha^{s_{k+1}(d)+2r'_{k+1}(d)} - (-1)^{r'_{k+1}(d)}$ are analytic functions which have simple poles at $e_{k+1}(d)$ for $1 \leq d \leq k$.
Now,

$$\begin{aligned}
&\lim_{s_{k+1}(d)\to e_{k+1}(k)} \frac{(-1)^{r_{k+1}(d)}(s_{k+1}(d)-e_{k+1}(d))}{\alpha^{s_{k+1}(d)+2r_{k+1}(d)} - (-1)^{r_{k+1}(d)}} \\
&= (-1)^{r_{k+1}(d)} \text{ Res }_{s_{k+1}(d)=e_{k+1}(d)} \frac{1}{\alpha^{s_{k+1}(d)+2r_{k+1}(d)} - (-1)^{r_{k+1}(d)}} \\
&= \frac{(-1)^{r_{k+1}(d)}}{(-1)^{r_{k+1}(d)}\log\alpha} = \frac{1}{\log\alpha}.
\end{aligned}$$

Now consider the limit as follows

$$\begin{aligned}
&\lim_{s_{k+1}(d)\to e_{k+1}(d)} (s_{k+1}(d)-e_{k+1}(d)) 5^{\frac{s_{k+1}(d)}{2}} \sum_{r_d,\ldots,r_k,r_{k+1}=0} \binom{-s_d}{r_d}(-1)^{r_d} \\
&\cdots \binom{-s_{k+1}}{r_{k+1}}(-1)^{r_{k+1}} \frac{(-1)^{r_{k+1}(d)}}{(\alpha^{s_{k+1}(d)+2r_{k+1}(d)} - (-1)^{r_{k+1}(d)})} \\
&\cdots \frac{(-1)^{r_{k+1}(k)}}{(\alpha^{s_{k+1}(k)+2r_{k+1}(k)} - (-1)^{r_{k+1}(k)})}.
\end{aligned}$$

In the above calculation, after applying the limit, only those terms containing $r_d, \ldots, r_k, r_{k+1}$ will survive when $r_{k+1}(d) = r'_{k+1}(d)$ and rest of the terms will vanish. Thus the above limit becomes

$$\begin{aligned}
&5^{\frac{e_{k+1}(d)}{2}} \sum_{\substack{r_d,\ldots,r_k,r_{k+1}\geq 0 \\ r_{k+1}(d)=r'_{k+1}(d)}} \binom{-s_d}{r_d}(-1)^{r_d} \cdots \binom{-s_{k+1}}{r_{k+1}}(-1)^{r_{k+1}} \\
&\frac{(-1)^{r_{k+1}(d+1)}}{(\alpha^{s_{k+1}(d+1)+2r_{k+1}(d+1)} - (-1)^{r_{k+1}(d+1)})} \cdots \frac{(-1)^{r_{k+1}(k)}}{(\alpha^{s_{k+1}(k)+2r_{k+1}(k)} - (-1)^{r_{k+1}(k)})} \frac{1}{\log\alpha}.
\end{aligned}$$

Therefore, the residue of $\zeta_{AVF,k}(s_1, \ldots, s_k; s_{k+1})$ along $s_{k+1}(d) = e_{k+1}(d)$ is

$$\begin{aligned}
&\text{Res}_{s_{k+1}(d)=e_{k+1}(d)} \zeta_{AVF,k}(s_1, \ldots, s_k; s_{k+1}) \\
&= \lim_{s_{k+1}(d)\to e_{k+1}(d)} (s_{k+1}(d)-e_{k+1}(d)) 5^{\frac{s_1+\cdots+s_{k+1}}{2}} \sum_{r_1=0}^{\infty} \binom{-s_1}{r_1}(-1)^{r_1} \\
&\quad \cdots \sum_{r_{k+1}=0}^{\infty} \binom{-s_{k+1}}{r_{k+1}}(-1)^{r_{k+1}}
\end{aligned}$$

$$\times \frac{(-1)^{r_{k+1}(1)}}{(\alpha^{s_{k+1}(1)+2r_{k+1}(1)} - (-1)^{r_{k+1}(1)})} \cdots \frac{(-1)^{r_{k+1}(k)}}{(\alpha^{s_{k+1}(k)+2r_{k+1}(k)} - (-1)^{r_{k+1}(k)})}$$

$$= \sum_{r_1,\ldots,r_{k+1}=0}^{\infty} \binom{-s_1}{r_1} \cdots \binom{-s_{k+1}}{r_{k+1}} \lim_{s_{k+1}(d)\to e_{k+1}(d)} (s_{k+1}(d)-e_{k+1}(d)) 5^{\frac{(\sum_{i=1}^{k+1} s_i)}{2}} (-1)^{(\sum_{i=1}^{k+1} r_i)}$$

$$\times \prod_{d=1}^{k} \frac{(-1)^{r_{k+1}(d)}}{(\alpha^{s_{k+1}(d)+2r_{k+1}(d)} - (-1)^{r_{k+1}(d)})}$$

$$= 5^{\frac{e_{k+1}(d)}{2}} \frac{(-1)^{r'_{k+1}(d)}}{\log \alpha} \sum_{\substack{r_1,\ldots,r_k,r_{k+1}\geq 0 \\ r_{k+1}(d)=r'_{k+1}(d)}} \binom{-s_1}{r_1} \cdots \binom{-s_{k+1}}{r_{k+1}} (-1)^{(\sum_{i=1}^{d-1} r_i)+(d-k)r_{k+1}}$$

$$5^{\frac{(\sum_{i=1}^{d-1} s_i)+(d-k)s_{k+1}}{2}} \prod_{\substack{l=1 \\ l\neq d}}^{k} \frac{(-1)^{r_{k+1}(l)}}{(\alpha^{s_{k+1}(l)+2r_{k+1}(l)} - (-1)^{r_{k+1}(l)})}.$$

This ends the proof. ■

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