

ANALYTIC PROPERTIES OF THE APOSTOL-VU MULTIPLE FIBONACCI ZETA FUNCTIONS

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Abstract

In this note we study the analytic continuation of the Apostol-Vu multiple Fibonacci zeta functions

$$\zeta_{AVF,k}(s_1, \dots, s_k; s_{k+1}) = \sum_{1 \leq m_1 < \dots < m_k} \frac{1}{F_{m_1}^{s_1} F_{m_2}^{s_2} \cdots F_{m_k}^{s_k} F_{m_1+m_2+\dots+m_k}^{s_{k+1}}},$$

where s_1, \dots, s_{k+1} are complex variables and F_n is the n -th Fibonacci number. We find a complete list of poles and their corresponding residues.

Keywords: analytic continuation, Fibonacci numbers, multiple Fibonacci zeta function, Apostol-Vu multiple Fibonacci zeta function.

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1. INTRODUCTION

The Euler-Zagier's multiple zeta function $\zeta_{EZ,k}$ of depth k is defined by

$$(1.1) \quad \zeta_{EZ,k}(s_1, \dots, s_k) = \sum_{1 \leq m_1 < m_2 < \dots < m_k < \infty} \frac{1}{m_1^{s_1} m_2^{s_2} \cdots m_k^{s_k}},$$

where s_1, \dots, s_k are complex variables [10]. Zhao [11] gave an analytic continuation of (1.1) as a function of $s_i (i = 1, \dots, k)$ to \mathbb{C}^k using Gelfand and Shilov's generalized functions. Mehta *et al.* [6] showed the meromorphic continuation of multiple zeta functions (1.1) with their poles and residues by means of an

elementary and simple translation formula for specified multiple zeta functions. The series

$$(1.2) \quad \sum_{m=1}^{\infty} \sum_{n < m} \frac{1}{m^{s_1} n^{s_2} (m+n)},$$

was first introduced by Apostol and Vu [1] and they obtained the partial results on its analytic continuation. The meromorphic continuation of (1.2) and more general series

$$(1.3) \quad \sum_{m=1}^{\infty} \sum_{n < m} \frac{1}{m^{s_1} n^{s_2} (m+n)^{s_3}},$$

to the whole space was first proved in [3]. In [4], Matsumoto generalized (1.3) and introduced Apostol-Vu r -ple zeta function

$$(1.4) \quad \zeta_{AV,r}(s_1, \dots, s_r; s_{r+1}) = \sum_{1 \leq m_1 < m_2 < \dots < m_r < \infty} \frac{1}{m_1^{s_1} m_2^{s_2} \dots m_r^{s_r} (m_1 + m_2 + \dots + m_r)^{s_{r+1}}}.$$

He proved that the series (1.4) is convergent absolutely when $\operatorname{Re}(s_i) > 1$ ($1 \leq i \leq r$), $\operatorname{Re}(s_{r+1}) > 0$ and can be continued meromorphically to the whole space \mathbb{C}^{r+1} .

Let $\{F_n\}_{n \geq 0}$ be the sequence of Fibonacci numbers and is recursively defined by

$$F_0 = 0, F_1 = 1, F_{n+1} = F_n + F_{n-1}, \text{ for } n \geq 1.$$

The closed form expression for $\{F_n\}_{n \geq 0}$ is $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$, where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are the zeros of Fibonacci characteristic equation $x^2 - x - 1 = 0$. In [7], Navas presented Fibonacci Dirichlet series $\zeta_F(s) = \sum_{m=1}^{\infty} F_m^{-s}$, $\operatorname{Re}(s) > 0$ for $s \in \mathbb{C}$, where F_m denotes the m -th Fibonacci number and obtained that $\zeta_F(s)$ is analytically continued to a meromorphic function on the complex plane \mathbb{C} . Rout and Panda [9] considered balancing zeta function $\zeta_B(s) = \sum_{m=1}^{\infty} B_m^{-s}$, $\operatorname{Re}(s) > 0$ for $s \in \mathbb{C}$, where B_m denotes the m -th balancing number and gave that $\zeta_B(s)$ is meromorphically continued all over \mathbb{C} . They also obtained that $\zeta_B(-m)$ is an irrational number when m is an odd natural number. In a consequent paper, Behera *et al.* [2] proved the analytical continuation of $\zeta_C(s) = \sum_{m=1}^{\infty} C_m^{-s}$, $\operatorname{Re}(s) > 0$ for $s \in \mathbb{C}$, where C_m denotes the m -th Lucas-balancing number and $\zeta_C(-m)$ is a rational number for any odd natural number m . In [8], Rout and Meher defined the multiple Fibonacci zeta function of depth k as:

$$(1.5) \quad \zeta_F(s_1, \dots, s_k) = \sum_{1 \leq m_1 < m_2 < \dots < m_k < \infty} \frac{1}{F_{m_1}^{s_1} F_{m_2}^{s_2} \dots F_{m_k}^{s_k}}.$$

They derived the analytic continuation of $\zeta_F(s_1, \dots, s_k)$ of depth 2 and found a complete list of poles and their corresponding residues. Recently, Meher and Rout [5] proved the meromorphic continuation of multiple Lucas zeta functions of depth d :

$$\zeta_U(s_1, \dots, s_d) = \sum_{0 < n_1 < n_2 < \dots < n_d} \frac{1}{U_{n_1}^{s_1} \dots U_{n_d}^{s_d}},$$

where (U_n) is the Lucas sequence of first kind.

In this note, we present Apostol-Vu multiple Fibonacci zeta function and explore its analytic continuation. We also estimate the residues corresponding to their respective poles.

2. ANALYTIC CONTINUATION OF APOSTOL-VU MULTIPLE FIBONACCI ZETA FUNCTIONS

In this section, the Apostol-Vu multiple Fibonacci zeta function is introduced and its analytic continuation is examined.

Definition 1. We define Apostol-Vu multiple Fibonacci zeta functions $\zeta_{AVF,k}$ as

$$(2.1) \quad \zeta_{AVF,k}(s_1, \dots, s_k; s_{k+1}) = \sum_{1 \leq m_1 < \dots < m_k < \infty} \frac{1}{F_{m_1}^{s_1} F_{m_2}^{s_2} \dots F_{m_k}^{s_k} F_{m_1+m_2+\dots+m_k}^{s_{k+1}}},$$

where s_1, \dots, s_{k+1} are complex variables.

The sum $s_1 + \dots + s_{k+1}$ is called the weight of $\zeta_{AVF,k}(s_1, \dots, s_k; s_{k+1})$ and k is called its depth. One can observe that $\zeta_{AVF,k}(s_1, \dots, s_k; 0)$ is the multiple Fibonacci zeta function (1.5).

Proposition 2. The series (2.1) is absolutely convergent in

$$D_{k+1} = \left\{ (s_1, \dots, s_k; s_{k+1}) \in \mathbb{C}^{k+1} \mid \sum_{j=d}^k Re(s_j) + (k+1-d)Re(s_{k+1}) > 0, 1 \leq d \leq k \right\}.$$

Proof. Let $s_j = \sigma_j + it_j \in \mathbb{C}$ and $\sigma_j = Re(s_j) > 0, j = 1, 2, \dots, k + 1$. From (2.1), we can write

$$(2.2) \quad \begin{aligned} & \sum_{1 \leq m_1 < \dots < m_k} \frac{1}{F_{m_1}^{s_1} F_{m_2}^{s_2} \dots F_{m_k}^{s_k} F_{m_1+m_2+\dots+m_k}^{s_{k+1}}} \\ &= \sum_{m_1=1}^{\infty} \frac{1}{F_{m_1}^{s_1}} \sum_{m_2=1}^{\infty} \frac{1}{F_{m_1+m_2}^{s_2}} \dots \\ & \sum_{m_{k-1}=1}^{\infty} \frac{1}{F_{m_1+m_2+\dots+m_{k-1}}^{s_{k-1}}} \sum_{m_k=1}^{\infty} \frac{1}{F_{m_1+m_2+\dots+m_k}^{s_k} F_{m_1+(k-1)m_2+\dots+m_k}^{s_{k+1}}}. \end{aligned}$$

Now,

$$\begin{aligned}
(2.3) \quad \sum_{m_1=1}^{\infty} \left| \frac{1}{F_{m_1}^{s_1}} \right| &= (\alpha - \beta)^{\sigma_1} \sum_{m_1=1}^{\infty} \left| \frac{1}{(\alpha^{m_1} - \beta^{m_1})^{s_1}} \right| \\
&\leq (\alpha - \beta)^{\sigma_1} \sum_{m_1=1}^{\infty} \frac{1}{\alpha^{m_1 \sigma_1} (1 - (|\frac{\beta}{\alpha}|)^{m_1})^{s_1}} \\
&\leq \left(\frac{\alpha - \beta}{1 - |\beta/\alpha|} \right)^{\sigma_1} \sum_{m_1=1}^{\infty} \frac{1}{\alpha^{\sigma_1 m_1}} = \Lambda_{\sigma_1} (\alpha - \beta)^{\sigma_1} \frac{1}{\alpha^{\sigma_1} - 1}
\end{aligned}$$

and similarly,

$$(2.4) \quad \left| \frac{1}{F_{m_1+\dots+m_d}^{s_d}} \right| \leq \Lambda_{\sigma_d} (\alpha - \beta)^{\sigma_d} \frac{1}{\alpha^{\sigma_d (m_1+\dots+m_d)}}, \quad 2 \leq d \leq k,$$

and

$$(2.5) \quad \left| \frac{1}{F_{km_1+(k-1)m_2+\dots+m_k}^{s_{k+1}}} \right| \leq \Lambda_{\sigma_{k+1}} (\alpha - \beta)^{\sigma_{k+1}} \frac{1}{\alpha^{\sigma_{k+1} (km_1+(k-1)m_2+\dots+m_k)}},$$

where $\Lambda_{\sigma_j} = \frac{1}{(1-|\beta/\alpha|)^{\sigma_j}}$. By virtue of (2.2), (2.3), (2.4) and (2.5),

$$\begin{aligned}
&\sum_{1 \leq m_1 < \dots < m_k} \left| \frac{1}{F_{m_1}^{s_1} \dots F_{m_k}^{s_k} F_{m_1+\dots+m_k}^{s_{k+1}}} \right| \\
&\leq \sum_{m_1=1}^{\infty} \left| \frac{1}{F_{m_1}^{s_1}} \right| \sum_{m_2=1}^{\infty} \left| \frac{1}{F_{m_1+m_2}^{s_2}} \right| \dots \\
&\quad \sum_{m_{k-1}=1}^{\infty} \left| \frac{1}{F_{m_1+\dots+m_{k-1}}^{s_{k-1}}} \right| \sum_{m_k=1}^{\infty} \left| \frac{1}{F_{m_1+\dots+m_k}^{s_k} F_{km_1+(k-1)m_2+\dots+m_k}^{s_{k+1}}} \right| \\
&\leq \Lambda_{\sigma_1} \Lambda_{\sigma_2} \dots \Lambda_{\sigma_k} \Lambda_{\sigma_{k+1}} (\alpha - \beta)^{\sigma_1} \dots (\alpha - \beta)^{\sigma_{k+1}} \sum_{m_1=1}^{\infty} \frac{1}{\alpha^{\sigma_1 m_1}} \sum_{m_2=1}^{\infty} \frac{1}{\alpha^{\sigma_2 (m_1+m_2)}} \\
&\quad \times \dots \times \sum_{m_{k-1}=1}^{\infty} \frac{1}{\alpha^{\sigma_{k-1} (m_1+m_2+\dots+m_{k-1})}} \sum_{m_k=1}^{\infty} \frac{1}{\alpha^{\sigma_k (m_1+\dots+m_k)}} \frac{1}{\alpha^{\sigma_{k+1} (km_1+(k-1)m_2+\dots+m_k)}} \\
&= \Lambda_{\sigma} (\alpha - \beta)^{\sum_{j=1}^{k+1} \sigma_j} \sum_{m_1=1}^{\infty} \frac{1}{\alpha^{(\sigma_1+\dots+\sigma_k+k\sigma_{k+1})m_1}} \sum_{m_2=1}^{\infty} \frac{1}{\alpha^{(\sigma_2+\dots+\sigma_k+(k-1)\sigma_{k+1})m_2}} \\
&\quad \times \dots \times \sum_{m_k=1}^{\infty} \frac{1}{\alpha^{(\sigma_k+\sigma_{k+1})m_k}}
\end{aligned}$$

$$\begin{aligned}
 &= \Lambda_\sigma (\alpha - \beta)^{\frac{\sum_{j=1}^{k+1} \sigma_j}{2}} \frac{1}{(\alpha^{\sigma_1 + \dots + \sigma_k + k\sigma_{k+1}} - 1)} \frac{1}{(\alpha^{\sigma_2 + \dots + \sigma_k + (k-1)\sigma_{k+1}} - 1)} \\
 &\quad \times \dots \times \frac{1}{(\alpha^{\sigma_k + \sigma_{k+1}} - 1)} < \infty,
 \end{aligned}$$

as $\alpha > 1$, where $\Lambda_\sigma = \Lambda_{\sigma_1} \dots \Lambda_{\sigma_k} \Lambda_{\sigma_{k+1}}$ and $\Lambda_{\sigma_1}, \dots, \Lambda_{\sigma_k}$ and $\Lambda_{\sigma_{k+1}}$ are the positive constants depending on $\sigma_1, \dots, \sigma_k$ and σ_{k+1} respectively. Therefore, the series (2.1) converges absolutely in the domain D_{k+1} . This completes the proof. ■

Theorem 3. *The Apostol-Vu multiple Fibonacci zeta function $\zeta_{AVF,k}(s_1, \dots, s_k; s_{k+1})$ can be analytically continued to a meromorphic function on \mathbb{C}^{k+1} . It has possible simple poles on the hyperplanes*

$$\begin{aligned}
 s_d + \dots + s_k + (k+1-d)s_{k+1} &= -2(r_d + \dots + r_k + (k+1-d)r_{k+1}) \\
 &\quad + \frac{\pi i (r_d + \dots + r_k + (k+1-d)r_{k+1} + 2a)}{\log \alpha}, \\
 1 \leq d \leq k,
 \end{aligned}$$

with $r_1, \dots, r_{k+1} \in \mathbb{Z}_{\geq 0}$ and $a \in \mathbb{Z}$.

Proof. For any $s = \sigma + it \in \mathbb{C}$, we have

$$\begin{aligned}
 (2.6) \quad F_m^s &= \left(\frac{\alpha^m - \beta^m}{\sqrt{5}} \right)^s = (\sqrt{5})^{-s} \alpha^{ms} \left(1 - \left(\frac{\beta}{\alpha} \right)^m \right)^s \\
 &= 5^{-\frac{s}{2}} \alpha^{ms} \sum_{k=0}^{\infty} \binom{s}{k} (-1)^k \left(\frac{\beta}{\alpha} \right)^{mk} \\
 &= 5^{-\frac{s}{2}} \sum_{k=0}^{\infty} \binom{s}{k} (-1)^{(m+1)k} \alpha^{m(s-2k)}.
 \end{aligned}$$

As $\alpha > 1$, the series (2.6) converges. Using the above identity in (2.2), we get

$$\begin{aligned}
 &\zeta_{AVF,k}(s_1, \dots, s_k; s_{k+1}) \\
 &= \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \dots \sum_{m_k=1}^{\infty} \frac{1}{F_{m_1}^{s_1}} \frac{1}{F_{m_1+m_2}^{s_2}} \dots \frac{1}{F_{m_1+m_2+\dots+m_k}^{s_k}} \frac{1}{F_{k m_1 + (k-1)m_2 + \dots + m_k}^{s_{k+1}}} \\
 &= \sum_{m_1, m_2, \dots, m_k=1}^{\infty} 5^{\frac{s_1}{2}} \sum_{r_1=0}^{\infty} \binom{-s_1}{r_1} (-1)^{(m_1+1)r_1} \alpha^{-m_1(s_1+2r_1)} \\
 &\quad \times 5^{\frac{s_2}{2}} \sum_{r_2=0}^{\infty} \binom{-s_2}{r_2} (-1)^{(m_1+m_2+1)r_2} \alpha^{-(m_1+m_2)(s_2+2r_2)} \times \dots
 \end{aligned}$$

$$\begin{aligned}
& \times 5^{\frac{s_k}{2}} \sum_{r_k=0}^{\infty} \binom{-s_k}{r_k} (-1)^{(1+\sum_{d=1}^k m_d)r_k} \alpha^{-\left(\sum_{d=1}^k m_d\right)(s_k+2r_k)} \\
(2.7) \quad & \times 5^{\frac{s_{k+1}}{2}} \sum_{r_{k+1}=0}^{\infty} \binom{-s_{k+1}}{r_{k+1}} (-1)^{(1+\sum_{d=1}^k (k+1-d)m_d)r_{k+1}} \\
& \times \alpha^{-\left(\sum_{d=1}^k (k+1-d)m_d\right)(s_{k+1}+2r_{k+1})}.
\end{aligned}$$

Using the fact $\left| \binom{-s_i}{r_i} \right| \leq (-1)^{r_i} \binom{-|s_i|}{r_i}$ for $1 \leq i \leq k+1$, we have

$$\begin{aligned}
& \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \cdots \sum_{m_k=1}^{\infty} \left| \frac{1}{F_{m_1}^{s_1}} \frac{1}{F_{m_1+m_2}^{s_2}} \cdots \frac{1}{F_{m_1+m_2+\cdots+m_k}^{s_k}} \frac{1}{F_{km_1+(k-1)m_2+\cdots+m_k}^{s_{k+1}}} \right| \\
& = \sum_{m_1, m_2, \dots, m_k=1}^{\infty} \left| 5^{\frac{s_1}{2}} \sum_{r_1=0}^{\infty} \binom{-s_1}{r_1} (-1)^{(m_1+1)r_1} \alpha^{-m_1(s_1+2r_1)} \right. \\
& \quad \times 5^{\frac{s_2}{2}} \sum_{r_2=0}^{\infty} \binom{-s_2}{r_2} (-1)^{(m_1+m_2+1)r_2} \alpha^{-(m_1+m_2)(s_2+2r_2)} \times \dots \\
& \quad \times 5^{\frac{s_k}{2}} \sum_{r_k=0}^{\infty} \binom{-s_k}{r_k} (-1)^{(1+\sum_{d=1}^k m_d)r_k} \alpha^{-\left(\sum_{d=1}^k m_d\right)(s_k+2r_k)} \\
& \quad \left. \times 5^{\frac{s_{k+1}}{2}} \sum_{r_{k+1}=0}^{\infty} \binom{-s_{k+1}}{r_{k+1}} (-1)^{(1+\sum_{d=1}^k (k+1-d)m_d)r_{k+1}} \alpha^{-\left(\sum_{d=1}^k (k+1-d)m_d\right)(s_{k+1}+2r_{k+1})} \right| \\
& \leq \sum_{m_1, m_2, \dots, m_k=1}^{\infty} 5^{\frac{\sum_{n=1}^{k+1} \sigma_n}{2}} \alpha^{-m_1 \sigma_1} \sum_{r_1=0}^{\infty} \binom{-|s_1|}{r_1} (-1)^{r_1} \alpha^{-2m_1 r_1} \\
& \quad \times \alpha^{-(m_1+m_2)\sigma_2} \sum_{r_2=0}^{\infty} \binom{-|s_2|}{r_2} (-1)^{r_2} \alpha^{-2(m_1+m_2)r_2} \times \dots \\
& \quad \times \alpha^{-\left(\sum_{d=1}^k m_d\right)\sigma_k} \sum_{r_k=0}^{\infty} \binom{-|s_k|}{r_k} (-1)^{r_k} \alpha^{-2\left(\sum_{d=1}^k m_d\right)r_k} \\
& \quad \times \alpha^{\left(\sum_{d=1}^k (k+1-d)m_d\right)\sigma_{k+1}} \sum_{r_{k+1}=0}^{\infty} \binom{-|s_{k+1}|}{r_{k+1}} (-1)^{r_{k+1}} \alpha^{-2\left(\sum_{d=1}^k (k+1-d)m_d\right)r_{k+1}} \\
& = \sum_{m_1, m_2, \dots, m_k=1}^{\infty} 5^{\frac{\sum_{n=1}^{k+1} \sigma_n}{2}} \alpha^{-m_1 \sigma_1} (1 - \alpha^{-2m_1})^{-|s_1|} \alpha^{-(m_1+m_2)\sigma_2} \\
& \quad \times (1 - \alpha^{-2(m_1+m_2)})^{-|s_2|} \times \dots \times \alpha^{\left(\sum_{d=1}^k m_d\right)\sigma_k} (1 - \alpha^{-2\left(\sum_{d=1}^k m_d\right)})^{-|s_k|}
\end{aligned}$$

$$\begin{aligned}
& \times \alpha^{\left(\sum_{d=1}^k (k+1-d)m_d\right)\sigma_{k+1}} \left(1 - \alpha^{-2\left(\sum_{d=1}^k (k+1-d)m_d\right)}\right)^{-|s_{k+1}|} \\
\leq & \sum_{m_1, m_2, \dots, m_k=1}^{\infty} 5^{\frac{\sum_{n=1}^{k+1} \sigma_n}{2}} \alpha^{-m_1 \sigma_1} \left(1 - \alpha^{-2}\right)^{-|s_1|} \alpha^{-(m_1+m_2)\sigma_2} \left(1 - \alpha^{-2}\right)^{-|s_2|} \\
& \times \dots \times \alpha^{\left(\sum_{d=1}^k m_d\right)\sigma_k} \left(1 - \alpha^{-2}\right)^{-|s_k|} \times \alpha^{\left(\sum_{d=1}^k (k+1-d)m_d\right)\sigma_{k+1}} \left(1 - \alpha^{-2}\right)^{-|s_{k+1}|} \\
= & 5^{\frac{\sum_{n=1}^{k+1} \sigma_n}{2}} \left(1 - \alpha^{-2}\right)^{-\left(\sum_{n=1}^{k+1} |s_n|\right)} \sum_{m_1=1}^{\infty} \alpha^{-(\sigma_1 + \dots + \sigma_k + k\sigma_{k+1})m_1} \\
& \sum_{m_2=1}^{\infty} \alpha^{-(\sigma_2 + \dots + \sigma_k + (k-1)\sigma_{k+1})m_2} \dots \sum_{m_k=1}^{\infty} \alpha^{-(\sigma_k + \sigma_{k+1})m_k} < \infty.
\end{aligned}$$

Thus, the above series absolutely converges for a fixed point $(s_1, \dots, s_k; s_{k+1})$ in D_{k+1} . Then, by interchanging the order of summation in (2.7), we get

$$\begin{aligned}
& \zeta_{AVF,k}(s_1, \dots, s_k; s_{k+1}) \\
= & 5^{\frac{\sum_{n=1}^{k+1} \sigma_n}{2}} \sum_{r_1=0}^{\infty} \binom{-s_1}{r_1} (-1)^{r_1} \sum_{r_2=0}^{\infty} \binom{-s_2}{r_2} (-1)^{r_2} \dots \sum_{r_{k+1}=0}^{\infty} \binom{-s_{k+1}}{r_{k+1}} (-1)^{r_{k+1}} \\
& \times \sum_{m_1, m_2, \dots, m_k=1}^{\infty} (-1)^{m_1 r_1} \alpha^{-m_1 (s_1 + 2r_1)} (-1)^{(m_1+m_2)r_2} \alpha^{-(m_1+m_2)(s_2+2r_2)} \\
& \dots (-1)^{\left(\sum_{d=1}^k m_d\right)r_k} \\
& \times \alpha^{-\left(\sum_{d=1}^k m_d\right)(s_k+2r_k)} (-1)^{\left(\sum_{d=1}^k (k+1-d)m_d\right)r_{k+1}} \alpha^{-\left(\sum_{d=1}^k (k+1-d)m_d\right)(s_{k+1}+2r_{k+1})} \\
= & 5^{\frac{\sum_{n=1}^{k+1} \sigma_n}{2}} \sum_{r_1=0}^{\infty} \binom{-s_1}{r_1} (-1)^{r_1} \sum_{r_2=0}^{\infty} \binom{-s_2}{r_2} (-1)^{r_2} \dots \sum_{r_{k+1}=0}^{\infty} \binom{-s_{k+1}}{r_{k+1}} (-1)^{r_{k+1}} \\
& \times \sum_{m_1, m_2, \dots, m_k=1}^{\infty} \left((-1)^{r_1 + \dots + r_k + k r_{k+1}} \alpha^{-(s_1 + \dots + s_k + k s_{k+1} + 2(r_1 + \dots + r_k + k r_{k+1}))} \right)^{m_1} \\
& \times \left((-1)^{r_2 + \dots + r_k + (k-1)r_{k+1}} \alpha^{-(s_2 + \dots + s_k + (k-1)s_{k+1} + 2(r_2 + \dots + r_k + (k-1)r_{k+1}))} \right)^{m_2} \\
(2.8) \quad & \times \dots \times \left((-1)^{r_k + r_{k+1}} \alpha^{-(s_k + s_{k+1} + 2(r_k + r_{k+1}))} \right)^{m_k}.
\end{aligned}$$

Now for any $(r_1, \dots, r_k; r_{k+1}) \in \mathbb{Z}_{\geq 0}^{k+1}$, we have

$$\sum_{m_1, m_2, \dots, m_k=1}^{\infty} \left((-1)^{r_1 + \dots + r_k + k r_{k+1}} \alpha^{-(s_1 + \dots + s_k + k s_{k+1} + 2(r_1 + \dots + r_k + k r_{k+1}))} \right)^{m_1}$$

$$\begin{aligned}
& \times \left((-1)^{r_2+\dots+r_k+(k-1)r_{k+1}} \alpha^{-\left(s_2+\dots+s_k+(k-1)s_{k+1}+2(r_2+\dots+r_k+(k-1)r_{k+1})\right)} \right)^{m_2} \\
& \times \dots \times \left((-1)^{r_k+r_{k+1}} \alpha^{-\left(s_k+s_{k+1}+2(r_k+r_{k+1})\right)} \right)^{m_k} \\
= & \frac{(-1)^{r_1+\dots+r_k+kr_{k+1}} \alpha^{-\left(s_1+\dots+s_k+ks_{k+1}+2(r_1+\dots+r_k+kr_{k+1})\right)}}{\left(1 - (-1)^{r_1+\dots+r_k+kr_{k+1}} \alpha^{-\left(s_1+\dots+s_k+ks_{k+1}+2(r_1+\dots+r_k+kr_{k+1})\right)}\right)} \\
& \times \frac{(-1)^{r_2+\dots+r_k+(k-1)r_{k+1}} \alpha^{-\left(s_2+\dots+s_k+(k-1)s_{k+1}+2(r_2+\dots+r_k+(k-1)r_{k+1})\right)}}{\left(1 - (-1)^{r_2+\dots+r_k+(k-1)r_{k+1}} \alpha^{-\left(s_2+\dots+s_k+(k-1)s_{k+1}+2(r_2+\dots+r_k+(k-1)r_{k+1})\right)}\right)} \\
& \times \dots \times \frac{(-1)^{r_k+r_{k+1}} \alpha^{-\left(s_k+s_{k+1}+2(r_k+r_{k+1})\right)}}{\left(1 - (-1)^{r_k+r_{k+1}} \alpha^{-\left(s_k+s_{k+1}+2(r_k+r_{k+1})\right)}\right)} \\
= & \frac{(-1)^{r_1+\dots+r_k+kr_{k+1}}}{\left(\alpha^{s_1+\dots+s_k+ks_{k+1}+2(r_1+\dots+r_k+kr_{k+1})} - (-1)^{r_1+\dots+r_k+kr_{k+1}}\right)} \\
& \times \frac{(-1)^{r_2+\dots+r_k+(k-1)r_{k+1}}}{\left(\alpha^{s_2+\dots+s_k+(k-1)s_{k+1}+2(r_2+\dots+r_k+(k-1)r_{k+1})} - (-1)^{r_2+\dots+r_k+(k-1)r_{k+1}}\right)} \\
(2.9) \quad & \times \dots \times \frac{(-1)^{r_k+r_{k+1}}}{\left(\alpha^{s_k+s_{k+1}+2(r_k+r_{k+1})} - (-1)^{r_k+r_{k+1}}\right)}.
\end{aligned}$$

By virtue of (2.8) and (2.9), we have

$$\begin{aligned}
& \zeta_{AVF,k}(s_1, \dots, s_k; s_{k+1}) \\
= & 5^{\frac{\sum_{n=1}^{k+1} s_n}{2}} \sum_{r_1=0}^{\infty} \binom{-s_1}{r_1} \sum_{r_2=0}^{\infty} \binom{-s_2}{r_2} \dots \sum_{r_{k+1}=0}^{\infty} \binom{-s_{k+1}}{r_{k+1}} (-1)^{\sum_{n=1}^{k+1} r_n} \\
& \times \frac{(-1)^{r_1+\dots+r_k+kr_{k+1}}}{\left(\alpha^{s_1+\dots+s_k+ks_{k+1}+2(r_1+\dots+r_k+kr_{k+1})} - (-1)^{r_1+\dots+r_k+kr_{k+1}}\right)} \\
& \times \frac{(-1)^{r_2+\dots+r_k+(k-1)r_{k+1}}}{\left(\alpha^{s_2+\dots+s_k+(k-1)s_{k+1}+2(r_2+\dots+r_k+(k-1)r_{k+1})} - (-1)^{r_2+\dots+r_k+(k-1)r_{k+1}}\right)} \\
(2.10) \quad & \times \dots \times \frac{(-1)^{r_k+r_{k+1}}}{\left(\alpha^{s_k+s_{k+1}+2(r_k+r_{k+1})} - (-1)^{r_k+r_{k+1}}\right)}.
\end{aligned}$$

Thus, this bound is uniform when $(s_1, \dots, s_k; s_{k+1})$ varies over compact subsets of \mathbb{C}^{k+1} . The infinite series (2.10) determines the holomorphic function on \mathbb{C}^{k+1} except for the poles derived from the functions

$$\begin{aligned}
(2.11) \quad & \alpha^{s_d+\dots+s_k+(k+1-d)s_{k+1}+2(r_d+\dots+r_k+(k+1-d)r_{k+1})} \\
& - (-1)^{r_d+\dots+r_k+(k+1-d)r_{k+1}} = 0
\end{aligned}$$

for $d = 1, \dots, k$.

Hence, the series (2.10) converges uniformly and absolutely on compact subsets of \mathbb{C}^{k+1} not containing any poles of the functions (2.11). Therefore, $\zeta_{AVF,k}(s_1, \dots, s_k; s_{k+1})$ can be analytically continued to a meromorphic function on \mathbb{C}^{k+1} and its simple poles are on the hyperplanes

$$(2.12) \quad \begin{aligned} s_d + \dots + s_k + (k+1-d)s_{k+1} &= -2(r_d + \dots + r_k + (k+1-d)r_{k+1}) \\ &+ \frac{\pi i(r_d + \dots + r_k + (k+1-d)r_{k+1} + 2a)}{\log \alpha}, \end{aligned}$$

for $1 \leq d \leq k$, with $r_1, \dots, r_{k+1} \in \mathbb{Z}_{\geq 0}$ and $a \in \mathbb{Z}$. This finishes the proof. \blacksquare

3. RESIDUES OF APOSTOL-VU MULTIPLE FIBONACCI ZETA FUNCTIONS AT POLES

We now calculate the residues of $\zeta_{AVF,k}(s_1, \dots, s_k; s_{k+1})$ at the poles that lie on the hyperplanes derived from Theorem 3. For $1 \leq d \leq k$, let us denote

$$s_{k+1}(d) = s_d + \dots + s_k + (k+1-d)s_{k+1}, r_{k+1}(d) = r_d + \dots + r_k + (k+1-d)r_{k+1},$$

and $r'_{k+1}(d) = r'_d + \dots + r'_k + (k+1-d)r'_{k+1}$, with the assumption that

$$s_{k+1}(d) = 0, r_{k+1}(d) = 0, r'_{k+1}(d) = 0 \text{ for } d \geq k+1.$$

We define the residue of the Apostol-Vu multiple Fibonacci zeta functions $\zeta_{AVF,k}(s_1, \dots, s_k; s_{k+1})$ along the hyperplanes (2.12) to be the restriction of the meromorphic function

$$\left(s_{k+1}(d) + 2r_{k+1}(d) - (r_{k+1}(d) + 2a) \frac{\pi i}{\log \alpha} \right) \zeta_{AVF,k}(s_1, \dots, s_k; s_{k+1})$$

to the hyperplanes (2.12).

Theorem 4. *Let $r'_d, \dots, r'_k, r'_{k+1}$ be non-negative integers for $1 \leq d \leq k$ and $e_{k+1}(d) = -2r'_{k+1}(d) + (r'_{k+1}(d) + 2a) \frac{\pi i}{\log \alpha}$. Then the residue of $\zeta_{AVF,k}(s_1, \dots, s_k; s_{k+1})$ at $s_{k+1}(d) = e_{k+1}(d)$ is*

$$\begin{aligned} &5^{\frac{e_{k+1}(d)}{2}} \frac{(-1)^{r'_{k+1}(d)}}{\log \alpha} \sum_{\substack{r_1, \dots, r_k, r_{k+1} \geq 0 \\ r_{k+1}(d) = r'_{k+1}(d)}} \binom{-s_1}{r_1} \dots \binom{-s_{k+1}}{r_{k+1}} (-1)^{(\sum_{i=1}^{d-1} r_i) + (d-k)r_{k+1}} \\ &\times 5^{\frac{(\sum_{i=1}^{d-1} s_i) + (d-k)s_{k+1}}{2}} \prod_{\substack{l=1 \\ l \neq d}}^k \frac{(-1)^{r_{k+1}(l)}}{(\alpha^{s_{k+1}(l) + 2r_{k+1}(l)} - (-1)^{r_{k+1}(l)})}. \end{aligned}$$

Proof. The functions $\alpha^{s_{k+1}(d)+2r'_{k+1}(d)} - (-1)^{r'_{k+1}(d)}$ are analytic functions which have simple poles at $e_{k+1}(d)$ for $1 \leq d \leq k$.

Now,

$$\begin{aligned} & \lim_{s_{k+1}(d) \rightarrow e_{k+1}(d)} \frac{(-1)^{r_{k+1}(d)}(s_{k+1}(d) - e_{k+1}(d))}{\alpha^{s_{k+1}(d)+2r_{k+1}(d)} - (-1)^{r_{k+1}(d)}} \\ &= (-1)^{r_{k+1}(d)} \operatorname{Res}_{s_{k+1}(d)=e_{k+1}(d)} \frac{1}{\alpha^{s_{k+1}(d)+2r_{k+1}(d)} - (-1)^{r_{k+1}(d)}} \\ &= \frac{(-1)^{r_{k+1}(d)}}{(-1)^{r_{k+1}(d)} \log \alpha} = \frac{1}{\log \alpha}. \end{aligned}$$

Now consider the limit as follows

$$\begin{aligned} & \lim_{s_{k+1}(d) \rightarrow e_{k+1}(d)} (s_{k+1}(d) - e_{k+1}(d)) 5^{\frac{s_{k+1}(d)}{2}} \sum_{r_d, \dots, r_k, r_{k+1}=0} \binom{-s_d}{r_d} (-1)^{r_d} \\ & \dots \binom{-s_{k+1}}{r_{k+1}} (-1)^{r_{k+1}} \frac{(-1)^{r_{k+1}(d)}}{(\alpha^{s_{k+1}(d)+2r_{k+1}(d)} - (-1)^{r_{k+1}(d)})} \\ & \dots \frac{(-1)^{r_{k+1}(k)}}{(\alpha^{s_{k+1}(k)+2r_{k+1}(k)} - (-1)^{r_{k+1}(k)})}. \end{aligned}$$

In the above calculation, after applying the limit, only those terms containing r_d, \dots, r_k, r_{k+1} will survive when $r_{k+1}(d) = r'_{k+1}(d)$ and rest of the terms will vanish. Thus the above limit becomes

$$\begin{aligned} & 5^{\frac{e_{k+1}(d)}{2}} \sum_{\substack{r_d, \dots, r_k, r_{k+1} \geq 0 \\ r_{k+1}(d) = r'_{k+1}(d)}} \binom{-s_d}{r_d} (-1)^{r_d} \dots \binom{-s_{k+1}}{r_{k+1}} (-1)^{r_{k+1}} \\ & \frac{(-1)^{r_{k+1}(d+1)}}{(\alpha^{s_{k+1}(d+1)+2r_{k+1}(d+1)} - (-1)^{r_{k+1}(d+1)})} \dots \frac{(-1)^{r_{k+1}(k)}}{(\alpha^{s_{k+1}(k)+2r_{k+1}(k)} - (-1)^{r_{k+1}(k)})} \frac{1}{\log \alpha}. \end{aligned}$$

Therefore, the residue of $\zeta_{AVF,k}(s_1, \dots, s_k; s_{k+1})$ along $s_{k+1}(d) = e_{k+1}(d)$ is

$$\begin{aligned} & \operatorname{Res}_{s_{k+1}(d)=e_{k+1}(d)} \zeta_{AVF,k}(s_1, \dots, s_k; s_{k+1}) \\ &= \lim_{s_{k+1}(d) \rightarrow e_{k+1}(d)} (s_{k+1}(d) - e_{k+1}(d)) 5^{\frac{s_1 + \dots + s_{k+1}}{2}} \sum_{r_1=0}^{\infty} \binom{-s_1}{r_1} (-1)^{r_1} \\ & \dots \sum_{r_{k+1}=0}^{\infty} \binom{-s_{k+1}}{r_{k+1}} (-1)^{r_{k+1}} \end{aligned}$$

$$\begin{aligned}
 & \times \frac{(-1)^{r_{k+1}(1)}}{(\alpha^{s_{k+1}(1)+2r_{k+1}(1)} - (-1)^{r_{k+1}(1)})} \cdots \frac{(-1)^{r_{k+1}(k)}}{(\alpha^{s_{k+1}(k)+2r_{k+1}(k)} - (-1)^{r_{k+1}(k)})} \\
 = & \sum_{r_1, \dots, r_{k+1}=0}^{\infty} \binom{-s_1}{r_1} \cdots \binom{-s_{k+1}}{r_{k+1}} \lim_{s_{k+1}(d) \rightarrow e_{k+1}(d)} (s_{k+1}(d) - e_{k+1}(d)) 5^{\frac{(\sum_{i=1}^{k+1} s_i)}{2}} (-1)^{(\sum_{i=1}^{k+1} r_i)} \\
 & \times \prod_{d=1}^k \frac{(-1)^{r_{k+1}(d)}}{(\alpha^{s_{k+1}(d)+2r_{k+1}(d)} - (-1)^{r_{k+1}(d)})} \\
 = & 5^{\frac{e_{k+1}(d)}{2}} \frac{(-1)^{r'_{k+1}(d)}}{\log \alpha} \sum_{\substack{r_1, \dots, r_k, r_{k+1} \geq 0 \\ r_{k+1}(d) = r'_{k+1}(d)}} \binom{-s_1}{r_1} \cdots \binom{-s_{k+1}}{r_{k+1}} (-1)^{(\sum_{i=1}^{d-1} r_i) + (d-k)r_{k+1}} \\
 & 5^{\frac{(\sum_{i=1}^{d-1} s_i) + (d-k)s_{k+1}}{2}} \prod_{\substack{l=1 \\ l \neq d}}^k \frac{(-1)^{r_{k+1}(l)}}{(\alpha^{s_{k+1}(l)+2r_{k+1}(l)} - (-1)^{r_{k+1}(l)})}.
 \end{aligned}$$

This ends the proof. ■

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