

STRONGLY GENERALIZED RADICAL SUPPLEMENTED MODULES

SOUMITRA DAS

AND

ARDELIN M. BUHPHANG

Department of Mathematics
North-Eastern Hill University
Shillong-793022, Meghalaya, India

e-mail: soumitrad330@gmail.com
ardeline17@gmail.com

Abstract

We introduce and study strongly generalized radical-supplemented modules (or briefly sgrs-modules). With the notation $Rad_g(R) := \cap\{K : K \leq R_R, K \text{ is both essential and maximal}\}$, we prove that (under some mild conditions on a ring R) every right R -module is a sgrs-module if and only if $\frac{R}{Soc(R)}$ is right perfect and the idempotents lift module $Rad_g(R)$.

Keywords: essential submodules, supplemented modules, strongly radical-supplemented modules, (semi-) perfect rings.

2010 Mathematics Subject Classification: Primary 16D10; Secondary 16D40, 16D70, 16D60.

1. INTRODUCTION

Throughout this article, all rings are associative with unity, and all modules are unital right modules. Let R be a ring. If M_R and N_R are modules, we use the following notations: if $N \subseteq M$, then $N \leq M$ denote N is a submodule of M . A submodule $S \leq M$ is called *small* (in M), denoted by $S \ll M$, if for every submodule $L \leq M$, the equality $S + L = M$ implies $L = M$. By $Rad(M)$ we denote the intersection of all maximal submodules of M , equivalently the sum of all small submodules of M (see [3, 2.7]).

A module direct sum decomposition $A \oplus B = M$ of M is determined by the two conditions (i) $A + B = M$ and (ii) $A \cap B = 0$. In this case A and B are known

as *direct complements* of each other. As a proper generalization of the concept, direct complement a submodule $B \leq M$ is called a *supplement* of a submodule $A \leq M$ if B is minimal subject to $A + B = M$, or equivalently, $M = A + B$ and $A \cap B \ll B$ (see, for instance, [3, 20.1]). If every submodule of M has a supplement in M , then M is called a *supplemented module*. Zöschinger [15] studied the module M such that $Rad(M)$ has a supplement in M and called it *radical supplemented module*. As a proper generalization Büyükaşık and Türkmen [2] called a module M *strongly radical supplemented* (briefly *srs*) if every submodule containing the radical $Rad(M)$ has a supplement. Carrying on in this direction we shall introduce and study strongly generalized radical supplemented modules or briefly *sgrs*-modules. These modules are different from the existing ones in the literature.

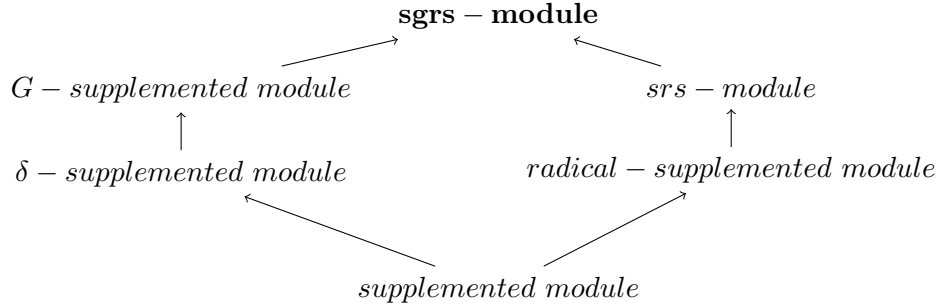
A submodule $T \leq M$ is called an *essential submodule* of M if $T \cap K \neq 0$ for every $K (\neq 0) \leq M$ and it is denoted by $T \trianglelefteq M$. An R -module M is called *singular* if there exists R -modules $A \trianglelefteq B$ such that $M \cong B/A$. Following [6, Definition 2.10] a submodule $L \leq M$ is called δ -*supplement* of a submodule $N \leq M$ if $M = N + L$, and for any proper submodule K of L with $\frac{L}{K}$ singular, $M \neq N + K$. The module M is called δ -*supplemented* if every submodule of M has a δ -supplement in M . A submodule $X \leq M$ is called *generalized small* (briefly, *g-small*) if for every $T \trianglelefteq M$ with $M = X + T$ we have $T = M$, this is written as $X \ll_g M$ (in [14], it is called an e-small submodule of M and denoted by $X \ll_e M$). If T is both essential and maximal submodule of M , then T is called a *generalized maximal submodule* of M . The intersection of all generalized maximal submodules of M is called the *generalized radical* of M and it is denoted by $Rad_g(M)$ (in [14], it is denoted by $Rad_e M$). If M have no generalized maximal submodules, then the generalized radical of M is defined by $Rad_g(M) = M$. Let U and V be submodules of M . If $M = U + V$ and $M = U + T$ with $T \trianglelefteq V$ implies that $T = V$, or equivalently, $M = U + V$ and $U \cap V \ll_g V$, then V is called a *g-supplement* of U in M . M is called a *G-supplemented module*, if every submodule of M has a *g-supplement* in M (see [5] and [9, Definition 2], where it is called e-supplemented). Notice that a δ -supplemented module is G-supplemented. In Definition 2 we called a module M is *strongly generalized radical supplemented* (or briefly *sgrs-module*) if every submodule of M containing the generalized radical $Rad_g(M)$ has a *g-supplement* in M .

Thus we have the following summarized picture of all the above mentioned modules.

We shall see in Example 1 (below) that all the arrows are strict-inclusions in the above situation.

For the other definitions in this note, we refer to [1, 3] and [12].

We note that there are some important properties of *g-small* submodules in [5, 9] and [14].



Lemma 1 (see [14] and [8]). *For an R -module M and for $K, N \leq M$, the following conditions hold.*

- (i) *If $K \leq N$ and $N \ll_g M$, then $K \ll_g M$.*
- (ii) *If $K \ll_g N$, then K is a g -small submodule of every submodule of M which contains N .*
- (iii) *If $f : M \rightarrow N$ is an R -module homomorphism and $K \ll_g M$, then $f(K) \ll_g N$.*
- (iv) *If $K \ll_g L$ and $N \ll_g T$ for $L, T \leq M$, then $K + N \ll_g L + T$.*

Corollary 1. (i) *Let M be an R -module and $K \leq N \leq M$. If $N \ll_g M$, then $\frac{N}{K} \ll_g \frac{M}{K}$.*

(ii) *Let M be an R -module, $K \ll_g M$ and $L \leq M$. Then $\frac{K+L}{L} \ll_g \frac{M}{L}$.*

Lemma 2 [5, Lemma 5]. *Let M be an R -module. Then $Rad_g(M) = \sum_{L \ll_g M} L$.*

Lemma 3. *The following assertions are hold for an R -module M .*

- (i) *If M is an R -module, then $mR \ll_g M$ for every $m \in Rad_g(M)$.*
- (ii) *If $N \leq M$, then $Rad_g N \leq Rad_g(M)$.*
- (iii) *If $K, L \leq M$, then $Rad_g(K) + Rad_g(L) \leq Rad_g(K + L)$.*
- (iv) *If $f : M \rightarrow N$ is an R -module homomorphism, then $f(Rad_g M) \leq Rad_g(N)$.*
- (v) *If $L \leq M$, then $\frac{Rad_g(M+L)}{L} \leq Rad_g(\frac{M}{L})$.*
- (vi) *Let $M = \oplus_{i \in I} M_i$. Then $Rad_g(M) = \oplus_{i \in I} Rad_g(M_i)$.*

Proof. (i), (ii), (iii), (iv), (v) follows from Lemma 1 and Lemma 2 (we use [1, Lemma 5.19] as essential criteria for a module), where (vi) follows from (i) and (ii) (see [4, Lemma 4]). ■

2. STRONGLY GENERALIZED RADICAL-SUPPLEMENTED MODULES

Definition. We call a module M *strongly generalized radical supplemented module* (or briefly *sgrs-module*) if every submodule N of M with $Rad_g(M) \leq N$ has

a g -supplement in M . In other words for any $U \leq M$ with $Rad_g(M) \leq U$, there exists $V \leq M$ such that $U + V = M$ and $U \cap V \ll_g V$.

Example 1. (i) [13, Example 4.3] Let F be a field, consider $I = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ and $R = \{(x_1, \dots, x_n, x, x, \dots) : n \in \mathbb{N}, x_i \in M_2(F), x \in I\}$. Notice that R is a ring under component-wise operations. Here, $Rad(R) = Rad_g(R) = \{(x_1, \dots, x_n, x, x, \dots) : n \in \mathbb{N}, x_i \in M_2(F), x \in J\}$, where $J = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. R is not semiregular. Hence R_R is not supplemented but R_R is δ -supplemented and hence R_R is a sgrs-module.

(ii) [3, Example 20.12] Consider \mathbb{Q} as a \mathbb{Z} -module. Since $Rad_g(\mathbb{Q}) = Rad(\mathbb{Q}) = \mathbb{Q}$, \mathbb{Q} is a sgrs-module. But, since \mathbb{Q} is not supplemented and every non-zero submodule of \mathbb{Q} is essential in \mathbb{Q} , \mathbb{Q} is not G-supplemented as a \mathbb{Z} -module.

(iii) (see [6, Example 2.14] and [3, Example 17.10]) Let $R = \mathbb{Z}$ and $M = \frac{\mathbb{Q}}{\mathbb{Z}} = \bigoplus_{i=1}^{\infty} M_i$ with each $M_i = \mathbb{Z}_{p^\infty} := \{r \in \mathbb{Q} : p^{n_i} r \in \mathbb{Z} \text{ for some } n_i\}$, where p is a prime number. Then $Rad_g(M) = \bigoplus_i Rad_g(M_i) = \bigoplus_i M_i = M$ is essential in M . But since the p -component of M is M that is not artinian, M is not supplemented by [12, p. 370]. Since M is singular, M is not G-supplemented.

(iv) [13, Example 4.1] Let F be a field and $F_i = F$ for all $i \in \mathbb{N}$. Consider $R = \langle \bigoplus_{i=1}^{\infty} F_i, 1_{\prod_{i=1}^{\infty} F_i} \rangle$, which is an F -subalgebra of $\prod_{i=1}^{\infty} F_i$ generated by $\bigoplus_{i=1}^{\infty} F_i$ and $1_{\prod_{i=1}^{\infty} F_i}$. Note that R is not semisimple and the Jacobson radical, $J(R) = 0$. Therefore, R is not semilocal and hence R_R is not a srs-module. However R_R is a sgrs-module (see Theorem 8 below).

We now discuss some properties of sgrs-modules.

Proposition 1. *Every factor module and homomorphic image of a sgrs-module are sgrs-modules.*

Proof. Let $L \leq N \leq M$ with $Rad_g(\frac{M}{L}) \leq \frac{N}{L}$. Since, $\frac{Rad_g(M+L)}{L} \leq Rad_g(\frac{M}{L})$, we have $Rad_g(M) \leq N$. By, assumption, N has g -supplement K (say) in M . So we have $N + K = M$ and $N \cap K \ll_g K$. Now it is easy to see that $\frac{N}{L} + \frac{K+L}{L} = \frac{M}{L}$ and $\frac{N}{L} \cap \frac{K+L}{L} = \frac{(N \cap K) + L}{L} \ll_g \frac{K+L}{L}$. Therefore, $\frac{K+L}{L}$ is a g -supplement of $\frac{N}{L}$ in $\frac{M}{L}$. The remain is clear. \blacksquare

We now aim to show that any finite sum of sgrs-submodules is a sgrs-module. For that we need the following lemma.

Lemma 4. *Let M be an R -module and let M_1 and N be submodules of M with $Rad_g(M) \leq N$. If M_1 is a sgrs-module and $M_1 + N$ has a g -supplement in M , then N has a g -supplement.*

Proof. Let L be a g -supplement of $M_1 + N$ in M . Then $L + (M_1 + N) = M$ with $L \cap (M_1 + N) \ll_g L$. Since, $Rad_g(M_1) \leq Rad_g(M) \leq N$, we have $Rad_g(M_1) \leq (L + N) \cap M_1$. Then $(L + N) \cap M_1$ has a g -supplement (say) K in M_1 , because M_1 is an sgrs-module. Therefore, $M = ((L + N) \cap M_1 + K) + N + L = ((L + N) \cap M_1) + K + (N + L) = K + (N + L) = N + (K + L)$. Since, $N + K \leq N + M_1$, $L \cap (N + K) \leq L \cap (M_1 + N) \ll_g L$, hence $N \cap (K + L) \leq (N + L) \cap K + (N + K) \cap L \ll_g K + L$. This shows that $K + L$ is a g -supplement of N . ■

Proposition 2. *Let $M = M_1 + M_2$, where M_1 and M_2 are sgrs-modules. Then M is a sgrs-module.*

Proof. Suppose that $N \leq M$ with $Rad_g(M) \leq N$. Clearly, $M_1 + M_2 + N$ has the trivial g -supplement 0 in M , and so by Lemma 4, $M_1 + N$ has g -supplement in M . Applying the lemma once again, we obtain a g -supplement for N in M . ■

Corollary 2. *Every finite sum of sgrs-modules is a sgrs-module.*

Let M be an R -module. Recall that an R -module N is said to be M -generated if N is a homomorphic image of a direct sum of copies of M .

Lemma 5. *Let M be a sgrs-module. Then every finitely M -generated module is sgrs-module.*

Proof. Clear from Proposition 1 and Corollary 2. ■

Corollary 3. *Let R be a ring. Then R_R is a sgrs-module if and only if every finitely generated R -module is a sgrs-module.*

Following [8, Definition 2] a module M is called g -semilocal if $\frac{M}{Rad_g(M)}$ is a semisimple module.

Proposition 3. *Every sgrs-module is g -semilocal.*

Proof. Let $\frac{U}{Rad_g(M)}$ be a submodule of $\frac{M}{Rad_g(M)}$. Since M is a sgrs-module, there exists a submodule V of M such that $M = U + V$ and $U \cap V \ll_g V$. Since, $U \cap V \ll_g V$, by Lemma 1(iv), $U \cap V \leq Rad_g(M)$. Hence we have, $\frac{M}{Rad_g(M)} = \frac{U+V}{Rad_g(M)} = \frac{U}{Rad_g(M)} + \frac{V+Rad_g(M)}{Rad_g(M)}$ and $\frac{U}{Rad_g(M)} \cap \frac{V+Rad_g(M)}{Rad_g(M)} = \frac{Rad_g(M) + (U \cap V)}{Rad_g(M)} = \frac{Rad_g(M)}{Rad_g(M)} = 0$. Thus, M is g -semilocal. ■

Corollary 4. *Let M be a sgrs-module. Then $M = M_1 \oplus M_2$, where M_1 is semisimple, $Rad_g(M) \trianglelefteq M_2$ and $\frac{M_2}{Rad_g(M)}$ is semisimple.*

Proof. Follows from Proposition 3 and [7, Proposition 2.1]. ■

Recall that a submodule $V \leq M$ is called a *weak g -supplement* of $U \leq M$ if $M = U + V$ and $U \cap V \ll_g M$. The module M is called *weakly g -supplemented* if every submodule of M has a weak g -supplement in M [8, Definition 1].

Example 2. (i) [11, Example 2.1] Let R be a DVR (that is, a local Dedekind domain) and K be the quotient field of R . Then the left R -module K is injective (see [1, Exercise 18. (2)]). Let $M = \bigoplus_I K$, where I is an infinite index set, be a left R -module. Since R is noetherian, M is injective and $\text{Rad}_g(M) = \text{Rad}(M) = M$. Therefore M is a sgrs-module but it is not weakly g -supplemented.

(ii) [8, Example 1] Let p and q be prime numbers and consider the ring $R = \mathbb{Z}_{p,q} = \{\frac{a}{b} \in \mathbb{Q} : p \nmid b, q \nmid b\}$. Then R is a commutative domain with exactly two maximal ideals pR and qR and every non-zero ideal is essential in R . Here, R_R is weakly g -supplemented but is not a sgrs-module.

We have noticed above that the concept of weakly g -supplemented modules and sgrs-modules are quite independent from each other. However we have the following result.

Proposition 4. *Assume M is a sgrs-module with $\text{Rad}_g(M) \ll_g M$. Then M is weakly g -supplemented.*

Proof. Follows from Proposition 3 and [8, Lemma 13]. ■

Observe that the \mathbb{Z} -module $M = \mathbb{Q} \oplus \frac{\mathbb{Z}}{p^2\mathbb{Z}}$ for any prime p , is sgrs-module by Proposition 2 but not a G -supplemented module. So, we try to explore conditions for which a sgrs-module will be a G -supplemented module. Clearly if $\text{Rad}_g(M)$ is semisimple (see [10, Lemma 2.4]) then any sgrs-module M is G -supplemented. In fact we have the following:

Proposition 5. *Assume M be a sgrs-module with $\text{Rad}_g(M)$ a G -supplemented submodule. Then M is G -supplemented.*

Proof. Let U be a submodule of M . By assumption, $\text{Rad}_g(M + U)$ has a g -supplement X , (say) in M . Again $\text{Rad}_g(M)$ is G -supplemented, hence $(X + U) \cap \text{Rad}_g(M)$ has a g -supplement Y (say) in $\text{Rad}_g(M)$. Then $X + Y$ is the required g -supplement of U in M . ■

The following results which appeared for amply g -supplemented modules in [9, Theorem 5] generalizes to sgrs-modules.

Proposition 6. *Let M be a module. Then M is Artinian if and only if M is a sgrs-module and satisfies DCC on g -supplement submodules and on g -small submodules.*

Corollary 5. *Let M be finitely generated. Then M is Artinian if and only if M is a sgrs-module satisfying DCC on g -small submodules.*

Now using the same technique as in proof (1) \Rightarrow (2) of [13, Lemma 1.2] we have the following.

Lemma 6. *Let A and B be two submodules of a module M with $M = A + B$. Then $A \oplus N$ is essential in M for some submodule N of B .*

Proof. By Zorn's Lemma, there always exists a submodule N of B maximal with respect to the property $A \cap N = 0$. Let $0 \neq m \in M$. We may assume that $m \notin N$. By the maximality of N , we have $A \cap (N + mR) \neq 0$. Take, $0 \neq a = n + mr \in A$, where $n \in N$ and $r \in R$. Then $mr = a - n \in A + N$. Since $A \cap N = 0$, we have $mr \neq 0$. Therefore, $(A \oplus N) \cap mR \neq 0$. ■

Notice that $Rad_g(R) = \delta(R) :=$ the intersection of all essential maximal right ideals of R (see [13, Theorem 1.6]). Following [13, Definition 3.1 and Theorem 3.6]), a ring R is called δ -semiperfect if $\frac{R}{Rad_g(R)}$ is a semisimple ring and idempotents lift modulo $Rad_g(R)$.

Before stating the next theorem, we insert a remark here.

Remark 7. For any two right ideals I and J of a ring R with $I \leq J$ such that $\frac{J}{I}$ is a singular module, then I need not be essential in J .

For instance, consider $R = \mathbb{Z} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}$. Then $I = 0 \oplus 0$ and $J = 0 \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}$ are right ideals of R with $I \leq J$ and $\frac{J}{I}$ is singular R -module but I is not essential in J .

Theorem 8. *Let R be a ring with $Rad_g(R) \ll_g R$ and such that whenever any two right ideals $I \leq J$ of R satisfy the property that if $\frac{J}{I}$ singular then $I \trianglelefteq J$. Then R_R is a sgrs-module if and only if R is a δ -semiperfect ring.*

Proof. By using [6, Theorem 3.3], we only show that every right ideal of R has a g -supplement in R_R . Let I be a right ideal of R . Since R_R is a sgrs-module, we have $I + \delta(R) + K = R$ with $(I + \delta(R)) \cap K \ll_g K$ for some right ideal K of R . Now by Lemma 6 we can find a submodule N of $\delta(R)$ such that $(I + K) \cap N = 0$ and $(I + K) \oplus N$ essential in R . Thus, $R = I + (K \oplus N) + \delta(R)$ implies that $R = I + (K \oplus N)$ (since, $\delta(R) \ll_g R$) and $I \cap (K \oplus N) \ll_g (K \oplus N)$. Therefore, $K + N$ is the required g -supplement of I in R . The other direction is clear (see [13, Theorem 3.6]). ■

Remark 9. Consider the ring, $R = \mathbb{Z}_{(6)} = \{\frac{a}{b} \in \mathbb{Q} : a, b \in \mathbb{Z}, gcd(b, 6) = 1\}$ consisting of integers localized away from the ideal $6\mathbb{Z}$ (of \mathbb{Z}) (see [1, Exercise 27.(4)]). This ring is a classic example of a ring where idempotents do not lift modulo the Jacobson radical (which is denoted by $J(R)$), since $\frac{R}{J(R)} \cong \frac{\mathbb{Z}}{6\mathbb{Z}}$ has four idempotents and R has only the trivial idempotents. Observe, here that $Rad_g(R) = \delta(R) = J(R) = 6R$, $Rad_g(R) \ll_g R$ and $\frac{R}{Rad_g(R)}$ is semisimple but R_R is not a sgrs-module.

Recall that for a ring R the *right socle* of R , denoted by $Soc(R)$, is defined as the sum of all its minimal right ideals and can be shown to coincide with the intersection of all the essential right ideals of R . Moreover $Soc(R)$ is a two sided-ideal of R (see [1, Proposition 9.7]). Following [13, Definition 3.1 and Theorem 3.8]), a ring R is called δ -perfect if $\frac{R}{Soc(R)}$ is right perfect and idempotents lift modulo $Rad_g(R)$.

Theorem 10. *Let Λ be a countable set, R a ring such that $Rad_g(\bigoplus_{i \in \Lambda} R) \ll_g \bigoplus_{i \in \Lambda} R$ and such that for any two right ideals $I \leq J$ of R if $\frac{J}{I}$ singular then $I \leq J$. Then the following statements are equivalent:*

- (i) R is a δ -perfect ring.
- (ii) Every right R -module is δ -supplemented.
- (iii) Every right R -module is G -supplemented.
- (iv) Every right R -module is strongly-generalized-radical-supplemented (sgrs-module).

Proof. (i) \Leftrightarrow (ii) follows from [6, Theorem 3.4].

(ii) \Rightarrow (iii) is clear from the fact that if N is a δ -small submodule of M , then N is a g -small submodule of M .

(iii) \Rightarrow (iv) is clear. So, it remains to see (iv) \Rightarrow (i). By Theorem 8, R is δ -semiperfect. By [13, Theorem 3.7 and Theorem 3.8] we only need to show that $Rad(\frac{R}{Soc(R)}) (= \frac{\delta(R)}{Soc(R)}$ by [13, Corollary 1.7]) is right T -nilpotent. For this we shall use the technique of [1, Lemma 28.1]. Let $F = \bigoplus_{\mathbb{N}} R$ be a free right R -module with basis $x_1, x_2, \dots, x_i, \dots, i \in \mathbb{N}$, and G the submodule of F spanned by $y_i = x_i - x_{i+1}a_i, i \in \mathbb{N}$, where a_1, a_2, a_3, \dots , is a sequence of elements from $\delta(R) = Rad_g(R)$. Then, $F = G + \delta(F)$. By hypothesis, $\delta(F) \ll_g F$ and hence by Lemma 6, $F = G \oplus B$ for some submodule B of $\delta(F)$. By [1, Lemma 28.2], there exists $n \in \mathbb{N}$ such that $Ra_{n+1}a_n \cdots a_1 = Ra_n a_{n-1} \cdots a_1$. Therefore, $ra_{n+1}a_n a_{n-1} \cdots a_1 = a_n a_{n-1} \cdots a_1$ for some r in R , and thus $(1 - ra_{n+1})a_n a_{n-1} \cdots a_1 = 0$. Therefore, $a_n a_{n-1} \cdots a_1 \in Soc(R)$. Hence, $Rad(\frac{R}{Soc(R)})$ is right T -nilpotent and R is right δ -perfect. \blacksquare

3. SGRS-MODULES OVER DEDEKIND DOMAINS

Throughout this section, unless otherwise stated, all rings that we consider are assumed to be commutative.

If R is an integral domain the *torsion submodule* of M is defined as

$$T(M) = \{m \in M : mr = 0 \text{ for some non-zero } r \in R\}.$$

A module M (over an integral domain) is called a *torsion module* if $T(M) = M$.

The following example shows that over a nonlocal domain every torsion module need not be sgrs-module.

Example 3. Let \mathbb{Z} be the ring of integers and let p be a prime in \mathbb{Z} : Consider the \mathbb{Z} -module $M = \bigoplus_{n \geq 1} \mathbb{Z}_{p^n}$ where $\mathbb{Z}_{p^n} = \frac{\mathbb{Z}}{p^n \mathbb{Z}}$. Then M is a torsion module. To see that M is not a sgrs-module, consider the submodule pM of M . Since $\frac{M}{pM}$ is a semisimple module, we have $Rad(M) \leq pM$. Now, using the same technique as in [2, Example 2.2], it can be proved that pM does not have a g -supplement in M , i.e., M is not a sgrs-module.

Recall that a module M over an integral domain R is called *divisible* if $M = Mr$ for all non-zero $r \in R$ (see [12, 16.6]). A module M over an arbitrary ring is *coatomic* if every proper submodule of M is contained in a maximal submodule of M (see [15] for the definition). Note that a module M is coatomic if and only if for every submodule N of M , $Rad(\frac{M}{N}) = \frac{M}{N}$ implies $N = M$. Semisimple modules and finitely generated modules are the examples of coatomic modules.

Lemma 11. *Let R be a Dedekind domain and M an R -module. If N is a g -small submodule of M , then N is coatomic.*

Proof. Let N be a g -small submodule of M and take $L \leq N$ with $Rad(\frac{N}{L}) = \frac{N}{L}$. Then $(\frac{N}{L})P = \frac{N}{L}$ for every maximal ideal P of R . Since R is a Dedekind domain then $\frac{N}{L}$ is divisible and hence an injective R -module. Therefore $\frac{N}{L} \oplus \frac{K}{L} = \frac{M}{L}$ for some $K \leq M$. Then $N + K = M$ which further implies that $N' \oplus K = M$ for some $N' \leq N$ (by Lemma 6) and $N = N' \oplus L$. But, by [14, Proposition 2.3] $N + K = M$ implies that $\frac{M}{K}$ is semisimple and hence $\frac{N}{L} \cong N'$ is semisimple. Therefore $Rad(\frac{N}{L}) = 0$, consequently $N = L$. Thus N is coatomic. ■

Lemma 12. *Let M be a sgrs-module over a Dedekind domain and U be a submodule of M with $Rad_g(M) \leq U$. Then, every g -supplement of U is coatomic.*

Proof. By Proposition 3, $\frac{M}{Rad_g(M)}$ is semisimple. So, $\frac{M}{U}$ is semisimple as a factor module of $\frac{M}{Rad_g(M)}$. Suppose that V is g -supplement of U in M . Then, $M = U + V$ and $U \cap V \ll_g V$. Now in the following exact sequence $0 \rightarrow U \cap V \rightarrow V \rightarrow \frac{V}{U \cap V} \rightarrow 0$ both $U \cap V$ (by Lemma 11) and $\frac{V}{U \cap V} (\cong \frac{M}{U})$ are coatomic. Therefore, V is coatomic by [15, Lemma 1.5.(a)]. ■

Abelian groups (\mathbb{Z} -modules), which do not contain divisible subgroups other than 0 are known as *reduced groups*. Denote

$$P(M) := \sum \{L \leq M : L \text{ has no maximal submodules}\}.$$

The following result is well-known:

Let R be a Dedekind domain. Then an R -module M has no non-zero divisible submodules if and only if $P(M) = 0$.

Following Zöschinger [15, Definition (before Lemma 1.5)] (for any ring R) an R -module M is said to be a *reduced module* if $P(M) = 0$.

The following proposition is an analogue of [2, Proposition 3.2].

Proposition 7. *Let R be a nonlocal domain and let M be a reduced R -module. If M is a sgrs-module, then $M = T(M) + \text{Rad}_g(M)$.*

Proof. Suppose that $T(M) + \text{Rad}_g(M) \neq M$. Since, $\text{Rad}_g(M) \leq T(M) + \text{Rad}_g(M)$, there exist $L \leq M$ such that $T(M) + \text{Rad}_g(M) + L = M$ and $L \cap (T(M) + \text{Rad}_g(M)) \ll_g L$. Now M being reduced we have a maximal submodule K of L such that $K' = T(M) + \text{Rad}_g(M) + K$ is a maximal submodule of M . (To see K' maximal in M , write $X = T(M) + \text{Rad}_g(M)$ and consider $K_0 \leq M$ such that $X + K \leq K_0 \leq M$. Then K being maximal in L , we have either $L \cap K_0 = K$ or $L \cap K_0 = L$. But $L \cap K_0 = K$ implies that $K_0 = X + K$ and $L \cap K_0 = L$ implies that $K_0 = M$, as required). Then K' has a g -supplement V in M . Now K' being maximal, one can find a cyclic submodule V_0 of V such that $K' + V_0 = M$, and so $V_0 \cong \frac{R}{I}$ for some nonzero $I \leq R$. Therefore, V_0 is a torsion submodule of M , and so $V_0 \leq T(M)$. Hence, we have $M = K' + V_0 = T(M) + \text{Rad}_g(M) + K + V_0 = T(M) + \text{Rad}_g(M) + K = K'$, a contradiction. Therefore, $M = T(M) + \text{Rad}_g(M)$. ■

The next three results can be proved in a similar fashion for sgrs-module as they appeared in [2, Proposition 3.3, Proposition 3.4 and Proposition 3.5] for srs-modules, hence we only state them.

Proposition 8. *Let R be a domain and M an R -module. Suppose that $M = T(M) + \text{Rad}_g(M)$ and $T(M)$ is G -supplemented. Then M is a sgrs-module.*

Proposition 9. *Let R be a Dedekind domain and M an R -module. Then M is a sgrs-module if and only if the reduced part N of M is a sgrs-module.*

Proposition 10. *Let R be a nonlocal Dedekind domain and M a sgrs-module. Then $M = T(M) + \text{Rad}_g(M)$.*

Acknowledgements

The authors are extremely grateful to the referee and Prof. Joanna Skowronek-Kaziow for a detailed and prompt report of the paper. Also, the authors would like to thank Prof. Kenneth R. Goodearl for providing us with the example mentioned in Remark 7 and the referee for valuable notes and recommendations which greatly improved this paper.

REFERENCES

- [1] F.W. Anderson and K.R. Fuller, *Rings and Categories of Modules* (New York, Springer-Verlag, 1974).
- [2] E. Büyükaşık and E. Türkmen, *Strongly radical supplemented modules*, *Ukrainian Math. J.* **63** (2011) 1306–1313.
doi:10.1007/s11253-012-0579-3
- [3] J. Clark, C. Lomp, N. Vanaja and R. Wisbauer, *Lifting Modules: Supplements and Projectivity in Module Theory* (Basel, Boston, Berlin, Birkhauser Verlag, 2006).
doi:10.1007/3-7643-7573-6
- [4] B. Koşar, C. Nebiyev and A. Pekin, *A generalization of g -supplemented modules*, *Miskolc Math. Notes* **20** (2019) 345–352.
doi:10.18514/MMN.2019.2586
- [5] B. Koşar, C. Nebiyev and N. Sökmez, *G -supplemented modules*, *Ukrainian Math. J.* **67** (2015) 861–864.
doi:10.1007/s11253-015-1127-8
- [6] M.T. Koşan, *δ -lifting and δ -supplemented modules*, *Algebra Colloq.* **14** (2007) 53–60.
doi:10.1142/S1005386707000065
- [7] C. Lomp, *On semilocal modules and rings*, *Comm. Algebra* **27** (1999) 1921–1935.
doi:10.1080/00927879908826539
- [8] C. Nebiyev and H. Hüseyin Ökten, *Weakly G -supplemented modules*, *European J. Pure Appl. Math.* **10** (2017) 521–528.
<https://ejpam.com/index.php/ejpam/article/viewFile/2662/534>
- [9] T.C. Quynh and P.H. Tin, *Some properties of e -supplemented and e -lifting modules*, *Vietnam J. Math.* **41** (2013) 303–312.
doi:10.1007/s10013-013-0022-6
- [10] L.V. Thuyet and P.H. Tin, *Some characterizations of modules via essentially small sub-modules*, *Kyungpook Math. J.* **56** (2016) 1069–1083.
doi:10.5666/kmj.2016.56.4.1069
- [11] B.N. Türkmen and E. Türkmen, *On a generalization of weakly supplemented modules*, *An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)* **63** (2017) 441–448.
doi:10.1515/aicu-2015-0012
- [12] R. Wisbauer, *Foundations of Module and Ring Theory* (Gordon and Breach, Philadelphia, 1991).
- [13] Y. Zhou, *Generalizations of perfect, semiperfect, and semiregular rings*, *Algebra Colloq.* **7** (2000) 305–318.
doi:10.1007/s10011-000-0305-9
- [14] D.X. Zhou and X.R. Zhang, *Small-essential submodules and morita duality*, *South-east Asian Bull. Math.* **35** (2011) 1051–1062.
<http://www.seams-bull-math.ynu.edu.cn/archive.jsp>

- [15] H. Zöschinger, *Komplementierte Moduln über Dedekindringen*, *J. Algebra* **29** (1974) 42–56.
doi:10.1016/0021-8693(74)90109-4

Received 16 May 2019
Revised 31 October 2019
Accepted 27 January 2020