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PRINCIPAL INTUITIONISTIC FUZZY IDEALS AND FILTERS ON A LATTICE

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Abstract

In this paper, we generalize the notion of principal ideal (resp. filter) on a lattice to the setting of intuitionistic fuzzy sets and investigate their various characterizations and properties. More specifically, we show that any principal intuitionistic fuzzy ideal (resp. filter) coincides with an intuitionistic fuzzy down-set (resp. up-set) generated by an intuitionistic fuzzy singleton. Afterwards, for a given intuitionistic fuzzy set, we introduce two intuitionistic fuzzy sets: its intuitionistic fuzzy down-set and up-set, and we investigate their interesting properties.

Keywords: lattice, intuitionistic fuzzy set, principal intuitionistic fuzzy ideal, principal intuitionistic fuzzy filter.

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1. Introduction

Ideals and filters are among the oldest notions in ordered set theory [9]. These notions play a prominent role in the mathematical branches such as topology and its analysis approaches, they appear to enhance the concept of convergence and to express completeness and compactness in metric spaces [12, 34]. Also, they are closely related to congruence relations and quotient algebras [24, 32]. In lattice theory, they are used to provide a simpler representation theory of distributive lattices [14, 16, 27].

In fuzzy setting, for the same purposes, some kinds of fuzzy ideals (resp. filters) introduced and were studied in different ways and on different structures. The first approach considered fuzzy ideal and fuzzy filter as fuzzy sets on crisp structures, like on lattices or on residuated lattices [11, 15, 18, 19, 26, 31], on BL-algebras [21] and on ordered ternary semigroups [2, 8, 13]. The second approach proposed similar notions on fuzzy structures, see eg. Mezzomo *et al.* [22] for the approaches in fuzzy ordered lattices.

An extension of fuzzy sets was introduced by Atanassov [3, 4] as intuitionistic fuzzy sets which considered the degree of hesitation (1 minus the sum of membership and non-membership degrees respectively). Intuitionistic fuzzy sets theory is also related to the different area of mathematics and it has showed meaningful applications (see; e.g., [5, 28]). Inspired by the theory and the applications of (fuzzy) crisp ideals and filters in the different algebraic structures, several authors extended and studied these concepts in intuitionistic fuzzy setting. In particular, Kim and Jun [17] introduced the notion of intuitionistic fuzzy interior ideals of semigroups. On pseudo-BL-algebras, Wojciechowska-Rysiawa [35] established some characterizations of intuitionistic fuzzy filters. Banerjee and Basnet [7] studied the notion of intuitionistic fuzzy ideals on a ring. In Lie algebras, Akram and Dudek [1] introduced and studied a concept of intuitionistic fuzzy Lie ideals. In [36], Xu introduced the notion of interval valued intuitionistic (T, S)-fuzzy filter on a lattice implication algebra and showed its basic properties. Qin and Liu [20, 25] introduced and investigated the properties of intuitionistic fuzzy filters on a residuated lattice. Thomas and Nair [29, 30] studied intuitionistic fuzzy sublattices, intuitionistic fuzzy ideals and intuitionistic fuzzy filters on a lattice. Very recently, two of the present authors and an other colleague [23] characterized these notions of intuitionistic fuzzy ideal and filter on a lattice in terms of the lattice meet and join operations.

In this paper we continue the further by generalized notion of principal ideal (resp. principal filter) to the intuitionistic fuzzy setting. More specifically, a principal intuitionistic fuzzy ideal (resp. filter) is an intuitionistic fuzzy ideal (resp. filter) generated by an intuitionistic fuzzy singleton. It is the smallest intuitionistic fuzzy ideal (resp. filter) containing a given intuitionistic fuzzy singleton. As the crisp case and based on the characterization theorems showed in [23], we prove that these notions coincide with the notion of intuitionistic fuzzy down-set (resp. up-set) generated by a singleton. Moreover, we introduce the notion of intuitionistic fuzzy down-set (resp. up-set) associated with an intuitionistic fuzzy set and discuss its interesting properties.

The contents of the paper are organized as follows. In Section 2, we recall basic concepts and properties of intuitionistic fuzzy sets, intuitionistic fuzzy lattice and intuitionistic fuzzy ideal and filter that will be needed throughout this paper. In Section 3, we introduce analogously to the crisp down-set (resp. up-set) on a lattice the intuitionistic fuzzy down-set (resp. the intuitionistic fuzzy up-set) on a lattice, and their interesting properties are given. In Section 4, we introduce the notions of principal intuitionistic fuzzy ideal and principal intuitionistic fuzzy filter on lattice, and we show their various characterizations. The last section includes some concluding remarks and future research.

2. Basic concepts

This section contains the basic definitions and properties of intuitionistic fuzzy sets, intuitionistic fuzzy lattices, intuitionistic fuzzy ideals (resp. filters) and some related notions that will be needed in this paper.

Next, unless otherwise stated, L always denotes a lattice $(L, \leqslant, \sqcap, \sqcup)$. It is well known that the order-dual of the lattice $(L, \leqslant, \sqcap, \sqcup)$ is the lattice $(L, \geqslant, \sqcup, \sqcap)$. We denote by L^d the order-dual lattice of the lattice L. To avoid any confusion or misunderstanding of some equations, we use the notation $(\leqslant, \sqcap, \sqcup)$ to refer the (order, min, max) on the lattice L and $(\leq, \wedge, \vee)$ to refer the (usual order, min, max) on the real interval $[0, 1]$.

2.1. Atanassov's intuitionistic fuzzy sets

The notion of fuzzy set introduced by Zadeh in [38] is characterized by a membership function which associates with each element x on a universe X a real number $\mu_A(x) \in [0, 1]$, where $\mu_A(x)$ represents the membership degree of x in the fuzzy set A.

In 1983, Atanassov [3] proposed a generalization of Zadeh membership degree and introduced the notion of the intuitionistic fuzzy set.

Definition 2.1 [3]**.** Let X be a nonempty set (alternative terms include universe of discourse). An intuitionistic fuzzy set (IFS, for short) A on X is an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$ characterized by a membership function $\mu_A : X \to [0, 1]$ and a non-membership function $\nu_A : X \to [0, 1]$ satisfying the condition:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1, \text{ for any } x \in X.$$

For a given nonempty set X, we denote by $IFS(X)$ the set of all intuitionistic fuzzy sets on X. In fuzzy set theory, the non-membership degree of an element x can be viewed as $\nu_A(x) = 1 - \mu_A(x)$ (using the standard strong negation on the real interval $[0, 1]$), which is fixed. While, in intuitionistic fuzzy setting, the non-membership degree is a more-or-less independent degree: the only condition is that $\nu_A(x) \leq 1 - \mu_A(x)$. Certainly, fuzzy sets are Atanassov's intuitionistic fuzzy sets by setting $\nu_A(x) = 1 - \mu_A(x)$.

For any two IFSs A and B on a set X, several operations are defined (see, e.g., Atanassov [4, 5, 6], Biswas [10], Gy [33]) and Yager [37]. Here we will present only those which are related to the present paper.

Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle \mid x \in X\}$ be two IFSs on a set X, then

(i) $A \subseteq B$ if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$, for all $x \in X$;

(ii) $A = B$ if $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$, for all $x \in X$;

(iii) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)\rangle \mid x \in X\}$;

(iv) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\rangle \mid x \in X\}$;

(v) $Supp(A) = \{x \in X \mid \mu_A(x) > 0$ or $(\mu_A(x) = 0$ and $\nu_A(x) < 1)\}$;

(vi) $Ker(A) = \{x \in X \mid \mu_A(x) = 1$ or $\nu_A(x) = 0\}$.

2.2. Intuitionistic fuzzy ideals and filters

The notion of intuitionistic fuzzy ideal (resp. filter) on a lattice was first introduced by Thomas and Nair [30].

Definition 2.2 [30]**.** Let $(L, \leqslant, \sqcap, \sqcup)$ be a lattice and $I = \{\langle x, \mu_I(x), \nu_I(x)\rangle \mid x \in L\}$ be an IFS on L. Then I is called an intuitionistic fuzzy ideal on L (IF-ideal, for short) if for all $x, y \in L$, the following conditions hold:

(i) $\mu_I(x \sqcup y) \geq \mu_I(x) \wedge \mu_I(y)$;

(ii) $\mu_I(x \sqcap y) \geq \mu_I(x) \vee \mu_I(y)$;

(iii) $\nu_I(x \sqcup y) \leq \nu_I(x) \vee \nu_I(y)$;

(iv) $\nu_I(x \sqcap y) \leq \nu_I(x) \wedge \nu_I(y)$.

Definition 2.3 [30]**.** Let $(L, \leqslant, \sqcap, \sqcup)$ be a lattice and $F = \{\langle x, \mu_F(x), \nu_F(x)\rangle \mid x \in L\}$ be an IFS on L. Then F is called an intuitionistic fuzzy filter on L (IF-filter, for short) if for all $x, y \in L$, the following conditions hold:

(i) $\mu_F(x \sqcup y) \geq \mu_F(x) \vee \mu_F(y)$;

(ii) $\mu_F(x \sqcap y) \geq \mu_F(x) \wedge \mu_F(y)$;

(iii) $\nu_F(x \sqcup y) \leq \nu_F(x) \wedge \nu_F(y)$;

(iv) $\nu_F(x \sqcap y) \leq \nu_F(x) \vee \nu_F(y)$.

For further details on intuitionistic fuzzy ideals and filters, we refer to [23, 29, 30].

Example 2.1. Let L be the lattice given by the Hasse diagram in Figure 1. The intuitionistic fuzzy set I on L defined by: $I = \{< 0, 0.5, 0.1 >, < a, 0.4, 0.3 >, < b, 0.1, 0.2 >, < c, 0.1, 0.7 >, < 1, 0.1, 0.7 >\}$ is an IF-ideal.

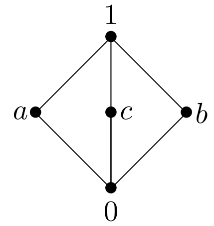

Figure 1. Hasse diagram of a lattice $(L, \sqcap, \sqcup)$ with $L = \{0, a, b, c, 1\}$.

Example 2.2. *Let L be the lattice given by the Hasse diagram in Figure 1. The intuitionistic fuzzy set F on L defined by: $F = \{< 0, 0.1, 0.5 >, < a, 0.1, 0.6 >, < b, 0.1, 0.5 >, < c, 0.3, 0.4 >, < 1, 0.4, 0.3 >\}$ is an IF-filter.*

The following proposition is straightforward.

Proposition 2.1. *Let L be a lattice, L^d be its order-dual lattice and $S \in IFS(L)$. The following equivalences hold:*

(i) *S is an IF-ideal on L if and only if S is an IF-filter on L^d;*

(ii) *S is an IF-filter on L if and only if S is an IF-ideal on L^d.*

3. Intuitionistic fuzzy down-sets and up-sets

In this section, we introduce the notion of intuitionistic fuzzy down-set (resp. intuitionistic fuzzy up-set) on a lattice analogously to that crisp down-set (resp. up-set). We define the smallest down-set (resp. the smallest up-set) containing an intuitionistic fuzzy set on a given lattice by using the operators of closure of intuitionistic fuzzy set introduced by Atanassov [4]. For more details on the notion of closure of an intuitionistic fuzzy set on a given universe and its properties, we refer to [5, 6].

3.1. Definitions

Let L be a lattice and S be a subset on L. S is called a down-set (alternative terms include lower-set) if $y \in S$ implies $x \in S$ for all $x \leqslant y$. Dually, S is called an up-set (alternative terms include upper-set) if $y \in S$ implies $x \in S$ for all $y \leqslant x$. For a given subset S on L, we denote by $\downarrow S$ the set of all elements smaller than or equal to some element of S, i.e.,

$$\downarrow S = \{x \in L \mid x \leqslant y, \text{ for some } y \in S\},$$

and $\uparrow S$ the set of all elements bigger than or equal to some element of S, i.e.,

$$\uparrow S = \{x \in L \mid y \leqslant x, \text{ for some } y \in S\}.$$

It is easily to check that $\downarrow S$ (resp. $\uparrow S$) is the smallest down-set (resp. the smallest up-set) containing S. $\downarrow S$ (resp. $\uparrow S$) is called the down-set (resp. the up-set) of S. Similarly, for a given element x on a lattice L, the down-set $\downarrow \{x\}$ ($\downarrow x$, for short) and the up-set $\uparrow \{x\}$ ($\uparrow x$, for short) are defined as

$$\downarrow x = \{y \in L \mid y \leqslant x\} \text{ (resp. } \uparrow x = \{y \in L \mid x \leqslant y\}).$$

Note that if S is a down-set (resp. an up-set), then $\downarrow S$ (resp. $\uparrow S$) coincides with S.

Analogously to the notion of crisp down-set (resp. up-set) on a lattice L, we introduce the notion of an intuitionistic fuzzy down-set (resp. an intuitionistic fuzzy up-set).

Definition 3.1. Let L be a lattice and $S \in IFS(L)$.

(i) S is called an intuitionistic fuzzy down-set (IF-down-set, for short) if $\mu_S(x) \geq \mu_S(y)$ and $\nu_S(x) \leq \nu_S(y)$ for all $x \leqslant y$.

(ii) Dually, S is called an intuitionistic fuzzy up-set (IF-up-set, for short) if $\mu_S(x) \leq \mu_S(y)$ and $\nu_S(x) \geq \nu_S(y)$ for all $x \leqslant y$.

Using the notion of closure of intuitionistic fuzzy set introduced by Atanassov [4], we define the following two sets $\Downarrow S$ and $\Uparrow S$ associated to a given intuitionistic fuzzy set S on a lattice L.

Definition 3.2. For a given intuitionistic fuzzy set S on a lattice L we denote by:

(i) $\Downarrow S$ the intuitionistic fuzzy set associated to S defined as

$$\mu_{\Downarrow S}(x) = \sup_{y \in \uparrow x} \mu_S(y),$$

$$\nu_{\Downarrow S}(x) = \inf_{y \in \uparrow x} \nu_S(y).$$

(ii) $\Uparrow S$ the intuitionistic fuzzy set associated to S defined as

$$\mu_{\Uparrow S}(x) = \sup_{y \in \downarrow x} \mu_S(y),$$

$$\nu_{\Uparrow S}(x) = \inf_{y \in \downarrow x} \nu_S(y).$$

One can easily verify that $\Downarrow S$ is the smallest IF-down-set (resp. $\Uparrow S$ is the smallest IF-up-set) containing S.

Remark 3.1. For any crisp set S on a given lattice L, it holds that

(i) $\Downarrow S = \downarrow S$;

(ii) $\Uparrow S = \uparrow S$.

For a given lattice L and $S \in IFS(L)$, it is clear that

(i) $\mu_{\Downarrow S}$ is an antitone mapping and $\nu_{\Downarrow S}$ is a monotone mapping;

(ii) $\mu_{\Uparrow S}$ is a monotone mapping and $\nu_{\Uparrow S}$ is an antitone mapping.

The following properties are immediate.

Proposition 3.1. *Let L be a lattice, L^d be its order-dual lattice and $S \in IFS(L)$. The following statements hold:*

(i) *S is an IF-down-set on L if and only if S is an IF-up-set on L^d;*

(ii) *S is an IF-up-set on L if and only if S is an IF-down-set on L^d;*

(iii) *$\Downarrow S$ on L coincides with $\Uparrow S$ on L^d ;*

(iv) *$\Uparrow S$ on L coincides with $\Downarrow S$ on L^d.*

3.2. Properties of IF-down-sets and IF-up-sets

In this subsection, we show some properties of intuitionistic fuzzy down-sets and up-sets on a lattice. First, we recall those which are analogous to the properties of the closure of intuitionistic fuzzy set.

The following proposition shows that IF-down-sets (resp. IF-up-sets) on a lattice are closed under the union and intersection of intuitionistic fuzzy sets.

Proposition 3.2 [5, 6]**.** *Let L be a lattice and $R, S \in IFS(L)$. It holds that*

(i) *If R and S are IF-down-sets, then $R \cup S$ and $R \cap S$ are IF-down-sets;*

(ii) *If R and S are IF-up-sets, then $R \cup S$ and $R \cap S$ are IF-up-sets.*

The following proposition list some properties of IF-down sets.

Proposition 3.3 [5, 6]**.** *Let L be a lattice and $R, S \in IFS(L)$. The following statements hold:*

(i) *If $S \subseteq R$, then $\Downarrow S \subseteq \Downarrow R$;*

(ii) $\Downarrow (\Downarrow S) = \Downarrow S$;

(iii) $\Downarrow (S \cup R) = \Downarrow S \cup \Downarrow R$;

(iv) $\Downarrow (S \cap R) \subseteq \Downarrow S \cap \Downarrow R$.

In the same direction, a dual version of Proposition 3.3 can also obtained for intuitionistic fuzzy up-sets. Its proof follows from Propositions 3.1 and 3.3.

Proposition 3.4. *Let L be a lattice and $R, S \in IFS(L)$. The following statements hold:*

(i) *If* $S \subseteq R$, *then* $\Uparrow S \subseteq \Uparrow R$;
(ii) $\Uparrow (\Uparrow S) = \Uparrow S$;
(iii) $\Uparrow (S \cup R) = \Uparrow S \cup \Uparrow R$;
(iv) $\Uparrow (S \cap R) \subseteq \Uparrow S \cap \Uparrow R$.

The following result follows immediately from Propositions 3.3 and 3.4.

Proposition 3.5. *Let* L *be a lattice. Then the mappings* $\Downarrow$ *and* $\Uparrow$ *define topological closures on the set* $IFS(L)$ *of intuitionistic fuzzy sets on* L.

The following proposition shows the interaction of the *Support* and the *Kernel* with the notions of IF-down-set and IF-up-set.

Proposition 3.6. *Let* L *be a lattice and* $S \in IFS(L)$. *It holds that*
(i) $Supp(\Downarrow S) = \downarrow Supp(S)$ *and* $Ker(\Downarrow S) = \downarrow Ker(S)$;
(ii) $Supp(\Uparrow S) = \uparrow Supp(S)$ *and* $Ker(\Uparrow S) = \uparrow Ker(S)$.

Proof. (i) First, we prove that $Supp(\Downarrow S) = \downarrow Supp(S)$. On the one hand, let $x \in Supp(\Downarrow S)$. Then it holds that

$$\mu_{\Downarrow S}(x) > 0 \text{ or } (\mu_{\Downarrow S}(x) = 0 \text{ and } \nu_{\Downarrow S}(x) < 1).$$

Two cases to consider:

(a) If $\mu_{\Downarrow S}(x) > 0$, then $\sup_{y \in \uparrow x} \mu_S(y) > 0$. This implies that there exists $t \in \uparrow x$ such that $\mu_S(t) > 0$. Hence, $t \in Supp(S)$. Since $t \in \uparrow x$, it follows that $x \in \downarrow Supp(S)$.
(b) If $\mu_{\Downarrow S}(x) = 0$ and $\nu_{\Downarrow S}(x) < 1$, then $\mu_S(y) = 0$ for any $y \in \uparrow x$ and $\inf_{y \in \uparrow x} \nu_S(y) < 1$. This implies that there exists $t \in \uparrow x$ such that $\mu_S(t) = 0$ and $\nu_S(t) < 1$. Hence, $t \in Supp(S)$. Since $t \in \uparrow x$, it follows that $x \in \downarrow Supp(S)$.

Thus, $Supp(\Downarrow S) \subseteq \downarrow Supp(S)$. On the other hand, let $x \in \downarrow Supp(S)$. Then it holds that $\mu_S(x) > 0$ or ($\mu_S(x) = 0$ and $\nu_S(x) < 1$). Two cases to consider:

(c) If $\mu_S(x) > 0$, then $\sup_{y \in \uparrow x} \mu_S(y) = \mu_{\Downarrow S}(x) > 0$. Hence, $x \in Supp(\Downarrow S)$.
(d) If $\mu_S(x) = 0$ and $\nu_S(x) < 1$, then $\mu_{\Downarrow S}(x) > 0$ or ($\mu_{\Downarrow S}(x) = 0$ and $\inf_{y \in \uparrow x} \nu_S(y) = \nu_{\Downarrow S}(x) < 1$). Hence, $x \in Supp(\Downarrow S)$.

Thus, $\downarrow Supp(S) \subseteq Supp(\Downarrow S)$. Therefore, $Supp(\Downarrow S) = \downarrow Supp(S)$.

The proof of $Ker(\Downarrow S) = \downarrow Ker(S)$ is analogous.

(ii) Follows from Proposition 3.1 and (i). ■

In the following result, we show that any IF-ideal (resp. IF-filter) on a lattice L is an IF-down-set (resp. IF-up-set) on L.

Theorem 1. *Let L be a lattice and $S \in IFS(L)$. The following implications hold:*

(i) *If S is an IF-ideal, then S is an IF-down-set;*

(ii) *If S is an IF-filter, then S is an IF-up-set.*

Proof. (i) Let $x, y \in L$ such that $x \leqslant y$. Since S is an IF-ideal, it follows that $\mu_S(x) = \mu_S(x \sqcap y) \geq \mu_S(x) \vee \mu_S(y)$. Hence, $\mu_S(x) \geq \mu_S(y)$. Similarly, we obtain that $\nu_S(x) \leq \nu_S(y)$. Thus, S is an IF-down-set.

(ii) Follows from Proposition 2.1, (i) and Proposition 3.1. ■

Theorem 1 leads to the following corollary.

Corollary 3.1. *Let L be a lattice and $S \in IFS(L)$. The following implications hold:*

(i) *If S is an IF-ideal, then $\Downarrow S = S$;*

(ii) *If S is an IF-filter, then $\Uparrow S = S$.*

Remark 3.2. The converse of the implications in the above Theorem 1 and Corollary 3.1 does not necessarily hold. Indeed, consider L the lattice given by the Hasse diagram in Figure 1 and $S \in IFS(L)$ given by $S = \{< 0, 0.7, 0.1 >, < a, 0.4, 0.2 >, < b, 0.3, 0.1 >, < c, 0.2, 0.1 >, < 1, 0.1, 0.3 >\}$. We easily verify that

x	0	a	b	c	1
$\mu_{\Downarrow S}(x)$	0.7	0.4	0.3	0.2	0.1
$\nu_{\Downarrow S}(x)$	0.1	0.2	0.1	0.1	0.3

Then $\Downarrow S = \{< 0, 0.7, 0.1 >, < a, 0.4, 0.2 >, < b, 0.3, 0.1 >, < c, 0.2, 0.1 >, < 1, 0.1, 0.3 >\}$. Hence, $\Downarrow S = S$, i.e., S is an IF-down-set. But, $\mu_S(1) = \mu_S(a \sqcup b) \ngeq \min\{0.4, 0.3\}$, which implies that S is not an intuitionistic fuzzy ideal on L .

4. PRINCIPAL IF-IDEALS AND IF-FILTERS ON A LATTICE

In this section, we introduce the notion of principal IF-ideal (resp. IF-filter) on a lattice. Similarly to the crisp case, we characterize these notions in terms of a down set and an up set generated by intuitionistic fuzzy singletons. First, we need to recall the following definition of crisp principal ideal (resp. filter), and the definition of intuitionistic fuzzy singleton.

Definition 4.1 [14]**.** Let L be a lattice. Then

(i) the principal ideal generated by an element $x \in L$ is the smallest ideal contains x, and is given by

$$\downarrow x = \{y \in L \mid y \leqslant x\};$$

(ii) the principal filter generated by an element $x \in L$ is the smallest filter contains x, and is given by

$$\uparrow x = \{y \in L \mid x \leqslant y\}.$$

Definition 4.2. Let L be a lattice. For any $x \in L$, an intuitionistic fuzzy singleton (IF- singleton, for short) $\tilde{x}$ is an intuitionistic fuzzy set on L given by $\tilde{x} = \{\langle t, \mu_{\tilde{x}}(t), \nu_{\tilde{x}}(t)\rangle \mid t \in L\}$, where

$$\mu_{\tilde{x}}(t) = \begin{cases} 1, & \text{if } x = t, \\ f(t), & \text{otherwise,} \end{cases}$$

and

$$\nu_{\tilde{x}}(t) = \begin{cases} 0, & \text{if } x = t, \\ g(t), & \text{otherwise,} \end{cases}$$

such that f (resp. g) is a monotone (resp. antitone) mapping on $[0, 1[$ and $f(t) + g(t) < 1$, for any $t \in L$.

Definition 4.3. Let L be a lattice, Then

(i) the principal IF-ideal generated by an IF-singleton $\tilde{x}$ is the smallest IF-ideal contains $\tilde{x}$;

(ii) the principal IF-filter generated by an IF-singleton $\tilde{x}$ is the smallest IF-filter contains $\tilde{x}$.

Next, we characterize the principal IF-ideals (resp. IF-filters) on a lattice in terms of the down-sets (resp. up-sets) generated by IF-singletons on that lattice. First, we need to recall the following characterization theorem of IF-ideals and IF-filters on a lattice.

Theorem 2 [23]**.** *Let L be a lattice and $I, F \in IFS(L)$. It holds that*

(1) *I is an IF-ideal on L if and only if the following two conditions are satisfied:*

  (i) $\mu_I(x \sqcup y) = \mu_I(x) \wedge \mu_I(y)$, *for any* $x, y \in L$;

  (ii) $\nu_I(x \sqcup y) = \nu_I(x) \vee \nu_I(y)$, *for any* $x, y \in L$.

(2) *F is an IF-filter on L if and only if the following two conditions are satisfied:*

  (i) $\mu_F(x \sqcap y) = \mu_F(x) \wedge \mu_F(y)$, *for any* $x, y \in L$;

  (ii) $\nu_F(x \sqcap y) = \nu_F(x) \vee \nu_F(y)$, *for any* $x, y \in L$.

The following theorem shows that the IF-down-set (resp. the IF-up-set) generated by an intuitionistic fuzzy singleton on a lattice L is an IF-ideal (resp. is an IF-filter) on L.

Theorem 3. *Let L be a lattice and x be an element on L. Then it holds that*

(i) $\Downarrow \tilde{x}$ *is an IF-ideal on L;*

(ii) $\Uparrow \tilde{x}$ *is an IF-filter on L.*

Proof. (i) From Theorem 2, it suffices to show for any $x, y \in L$ that

$$\mu_{\Downarrow\tilde{x}}(x \sqcup y) = \mu_{\Downarrow\tilde{x}}(x) \wedge \mu_{\Downarrow\tilde{x}}(y) \text{ and } \nu_{\Downarrow\tilde{x}}(x \sqcup y) = \nu_{\Downarrow\tilde{x}}(x) \vee \nu_{\Downarrow\tilde{x}}(y).$$

Let $a, b \in L$. On the one hand, the fact that $\Downarrow \tilde{x}$ is an IF-down-set, implies that $\mu_{\Downarrow\tilde{x}}(a) \geq \mu_{\Downarrow\tilde{x}}(a \sqcup b)$ and $\mu_{\Downarrow\tilde{x}}(b) \geq \mu_{\Downarrow\tilde{x}}(a \sqcup b)$. Hence, $\mu_{\Downarrow\tilde{x}}(a) \wedge \mu_{\Downarrow\tilde{x}}(a) \geq \mu_{\Downarrow\tilde{x}}(a \sqcup b)$. On the other hand, since $\mu_{\tilde{x}}$ is a monotone mapping, it holds that $\mu_{\tilde{x}}(a) \leq \mu_{\tilde{x}}(a \sqcup b)$ and $\mu_{\tilde{x}}(b) \leq \mu_{\tilde{x}}(a \sqcup b)$. This implies that $\sup_{a \leqslant t} \mu_{\tilde{x}}(t) \leq \sup_{a \sqcup b \leqslant t} \mu_{\tilde{x}}(t)$ and $\sup_{b \leqslant t} \mu_{\tilde{x}}(t) \leq \sup_{a \sqcup b \leqslant t} \mu_{\tilde{x}}(t)$. Hence, $\sup_{a \leqslant t} \mu_{\tilde{x}}(t) \wedge \sup_{b \leqslant t} \mu_{\tilde{x}}(t) \leq \sup_{a \sqcup b \leqslant t} \mu_{\tilde{x}}(t)$. Thus, $\mu_{\Downarrow\tilde{x}}(a) \wedge \mu_{\Downarrow\tilde{x}}(b) \leq \mu_{\Downarrow\tilde{x}}(a \sqcup b)$. Therefore, $\mu_{\Downarrow\tilde{x}}(a \sqcup b) = \mu_{\Downarrow\tilde{x}}(a) \wedge \mu_{\Downarrow\tilde{x}}(b)$, for all $a, b \in L$. In analogous way, we easily prove that $\nu_{\Downarrow\tilde{x}}(a \sqcup b) = \nu_{\Downarrow\tilde{x}}(a) \vee \nu_{\Downarrow\tilde{x}}(b)$. Finally, we conclude that $\Downarrow \tilde{x}$ is an IF-ideal on L.

(ii) Follows dually by using Proposition 3.1, (i) and Proposition 2.1. ∎

In the following result, we show a characterization of a principal IF-ideal (resp. IF-filter) in terms of a down-set (resp. up-set) generated by an IF-singleton.

Theorem 4. *Let L be a lattice and I (resp. F) be an IF-ideal (resp. IF-filter) on L. Then it holds that*

(i) *I is a principal IF-ideal on L if and only if there exists $x \in L$ such that $I = \Downarrow \tilde{x}$;*

(ii) *F is a principal IF-filter on L if and only if there exists $x \in L$ such that $F = \Uparrow \tilde{x}$.*

Proof. We only prove (i), as (ii) can be proved analogously by using Proposition 3.1 and Proposition 2.1. Suppose that I is a principal IF-ideal on L. Then there exists an IF-singleton $\tilde{x}$ such that I is the smallest IF-ideal contains $\tilde{x}$. Since $\tilde{x} \subseteq I$, it follows from Proposition 3.3 that $\Downarrow \tilde{x} \subseteq \Downarrow I = I$. On the other hand, Theorem 3 guarantees that $\Downarrow \tilde{x}$ is an ideal. Then the fact that I is the smallest ideal contains $\tilde{x}$ implies that $I \subseteq \Downarrow \tilde{x}$. Thus, $I = \Downarrow \tilde{x}$.

Conversely, $I = \Downarrow \tilde{x}$ is an IF-ideal contains $\tilde{x}$. Now, suppose that J is an other IF-ideal contains $\tilde{x}$. From Proposition 3.3, it holds that $\Downarrow \tilde{x} \subseteq \Downarrow J = J$. Hence, $I = \Downarrow \tilde{x}$ is the smallest IF-ideal contains $\tilde{x}$. Thus, I is a principal IF-ideal. ∎

In the following proposition, we show that the kernel of a principal IF-ideal (resp. IF-filter) is a crisp principal ideal (resp. filter).

Proposition 4.1. *Let L be a lattice and x be an element on L. Then it holds that*

(i) $Ker(\Downarrow \tilde{x}) = \downarrow x$;

(ii) $Ker(\Uparrow \tilde{x}) = \uparrow x$.

Proof. (i) From Proposition 3.6, it holds that $Ker(\Downarrow \tilde{x}) = \downarrow Ker(\tilde{x})$. This means that

$$Ker(\Downarrow \tilde{x}) = \downarrow \{t \in L \mid \mu_{\tilde{x}}(t) = 1 \text{ or } \nu_{\tilde{x}}(t) = 0\}.$$

By the definition of IF-singleton, we know that $\mu_{\tilde{x}}(t) = 1$ or $\nu_{\tilde{x}}(t) = 0$ if and only if $t = x$. Hence,

$$Ker(\Downarrow \tilde{x}) = \downarrow \{x\} = \downarrow x.$$

(ii) Follows from Proposition 3.1 and (i). ∎

5. Conclusion

In this work, we have introduced the notion of an intuitionistic fuzzy down-set (resp. up-set) on a lattice, and investigated their interesting properties. We have also used these notions to characterize principal intuitionistic fuzzy ideals and filters on a lattice. The future work will be focused on the characterization of intuitionistic fuzzy ideals and filters on an intuitionistic fuzzy ordered lattice.

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