

ON QUASI-P-ALMOST DISTRIBUTIVE LATTICES

RAVI KUMAR BANDARU¹

Department of Mathematics
GITAM (Deemed to be University)
Hyderabad Campus, Telangana, India-502 329

e-mail: ravimaths83@gmail.com

AND

G.C. RAO

Department of Mathematics
Andhra University
Andhra Pradesh, India-530 003

e-mail: gcraomaths@yahoo.co.in

Abstract

In this paper, the concept of quasi pseudo-complementation on an Almost Distributive Lattice (ADL) as a generalization of pseudo-complementation on an ADL is introduced and its properties are studied. Necessary and sufficient conditions for a quasi pseudo-complemented ADL(q -p-ADL) to be a pseudo-complemented ADL(p -ADL) and Stone ADL are derived and the set $S(L) = \{a^* \mid a \in L\}$ is proved to be a Boolean algebra. Also, the notions of $*$ -congruence and kernel ideals are introduced in a quasi-p-ADL and characterized kernel ideals. Finally, some equivalent conditions are given for every ideal of a quasi-p-ADL to be a kernel ideal.

Keywords: pseudo-complementation, quasi pseudo-complementation, Almost Distributive Lattice (ADL), p -ADL, quasi-p-ADL.

2010 Mathematics Subject Classification: 06D99, 06D15.

¹Corresponding author.

1. INTRODUCTION

A pseudo-complemented lattice is a lattice L with 0 such that to each $a \in L$, the largest annihilating element of a exists in L . That is, there exists $a^* \in L$ such that, for all $x \in L$, $a \wedge x = 0$ if and only if $x \leq a^*$. Here a^* is called the pseudo-complement of a . For each element a of a pseudo-complemented lattice L , a^* is uniquely determined by a , so that $*$ can be regarded as a unary operation on L . Moreover, each pseudo-complemented lattice contains the unit element namely 0^* . It follows that every pseudo-complemented lattice L can be regarded as an algebra $(L, \vee, \wedge, *, 0, 1)$ of type $(2, 2, 1, 0, 0)$. The fact that the class of pseudo-complemented distributive lattices is equationally definable was first observed by Ribenboim in 1949. Also, in [5], it was proved that the class of pseudo-complemented distributive lattices is generated by its finite members and a complete description of the lattice of equational classes of pseudo-complemented distributive lattices is given. In [8], Sankappanavar introduced a new class of algebras, called semi-De Morgan algebras, as a common abstraction of De Morgan algebras and distributive pseudocomplemented lattices and studied its properties. Also, he studied several important subvarieties of semi-De Morgan algebras, such as demi-p-lattices, weak Stone algebras and almost p-lattices. In [3], Frink studied about the pseudo-complemented semilattice L and proved that the set $L^* = \{a^* \mid a \in L\}$, where $*$ is a pseudo-complementation on L , becomes a Boolean algebra. In [1], Cornish considered the kernels of $*$ -congruences on distributive pseudo-complemented lattices and studied its important properties. Later these concepts were extended to the case of semi lattices by Blyth in [2] and to the case of ADLs by Rao in [7].

The concept of pseudo-complementation in an ADL and the concept of Stone ADL was given by Swamy, Rao and Nanaji Rao [9, 10]. They have proved that there is a one-to-one correspondence between the pseudo-complementations on an ADL L with 0 and the set of all maximal elements of L . Also, they proved that if $*$ is a pseudo-complementation on an ADL L , then the set $L^* = \{a^* \mid a \in L\}$ is a Boolean algebra and the pseudo-complementation $*$ on L is equationally definable. In [6] Rao *et al.* studied the properties of minimal prime ideals in an ADL.

In this paper, we introduce the concept of quasi pseudo-complementation on an ADL as a generalization of pseudo-complementation on an ADL like the concept of almost p-lattice as a generalization of pseudo-complemented distributive lattice. Here we extend the concept of almost p-lattice to the case of almost distributive lattices and name it quasi-p-ADL. We give necessary and sufficient conditions for a quasi-p-ADL to be a p-ADL and we prove that if $*$ is a quasi pseudo-complementation on an ADL L then the set $S(L) = \{a^* \mid a \in L\}$ becomes a Boolean algebra. It is observed that there exists an induced surjective

correspondence between the set of maximal elements and the set of quasi pseudo-complementations on L , provided there is a quasi pseudo-complementation on L . We introduce the concept of $*$ -congruence, kernel ideals on a quasi-p-ADL and give equivalent conditions under which every ideal of L is a kernel ideal.

2. PRELIMINARIES

In this section, we give the definition and some elementary properties of a pseudo-complemented ADL and Stone ADL [9, 10]. For the concept of ADL refer to [11] and for the concept of minimal prime ideals in an ADL refer to [6].

Definition 2.1. Let (L, \vee, \wedge) be an ADL with 0. Then a unary operation $a \mapsto a^*$ on L is called a pseudo-complementation on L if, for any $a, b \in L$ it satisfies the following conditions:

- (1) $a \wedge b = 0 \Rightarrow a^* \wedge b = b$,
- (2) $a \wedge a^* = 0$,
- (3) $(a \vee b)^* = a^* \wedge b^*$.

L is called a Stone ADL if, for any $x \in L$, $x^* \vee x^{**} = 0^*$.

If (L, \vee, \wedge) is an ADL with 0 and $*$ is a pseudo-complementation on L , then we say that $(L, \vee, \wedge, *, 0)$ is a pseudo-complemented ADL (p-ADL, for brevity).

In the following, we give an example of an ADL with a pseudo-complementation which is not a Lattice.

Example 2.2. Let $(X, \vee, \wedge, 0)$ be a discrete ADL. Fix $x_0 \neq 0$ in X and define $*$ on X as follows

$$a^* = \begin{cases} 0, & \text{if } a \neq 0; \\ x_0, & \text{if } a = 0. \end{cases}$$

Then $*$ is a pseudo-complementation on X .

Now we give some elementary properties of pseudo-complementation.

Theorem 2.3. Let L be an ADL with 0 and $*$ a pseudo-complementation on L and $a, b \in L$. Then we have the following:

- (1) 0^* is maximal element,
- (2) $0^{**} = 0$,
- (3) $a^{**} \wedge a = a$,
- (4) $a^{***} = a^*$,
- (5) $a^* \wedge b^* = b^* \wedge a^*$,
- (6) $a \leq b \Rightarrow b^* \leq a^*$,

- (7) $a^* \leq (a \wedge b)^*$ and $b^* \leq (a \wedge b)^*$,
(8) $a \wedge b = 0 \Leftrightarrow a^{**} \wedge b = 0$,
(9) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$.

Definition 2.4. For any non-empty subset A of an ADL L with 0, define

$$A^* = \{x \in L \mid x \wedge a = 0, \text{ for all } a \in A\}.$$

This A^* is an ideal of L and is called the annihilator ideal of A .

For any $a \in L$, we write $[a]^*$ for $\{a\}^*$ and is called annulet of L .

It can be easily observed that, for any subset A of L , $A \cap A^* = \{0\}$.

Lemma 2.5. *Let L be an ADL with 0 and $a \in L$. Then $[a] = L$ if and only if a is a maximal element.*

Theorem 2.6. *Let L be an ADL with 0. Then for any $a \in L$, the annulet $[a]^*$ is a principal ideal if and only if L has a pseudocomplementation.*

Theorem 2.7. *Let L be an ADL with 0 and $*$ a pseudo-complementation on L . For any $a^*, b^* \in L^*$, define $a^* \leq b^*$ if and only if $a^* \wedge b^* = a^*$. Then (L^*, \leq) is a Boolean algebra.*

Corollary 2.8. *Let L be an ADL with 0 and $*$ a pseudo-complementation on L . Then the map $f : L \mapsto L^*$ defined by $f(a) = a^{**}$ is an epimorphism.*

Theorem 2.9. *Let I be an ideal of L and F a filter of L such that $I \cap F = \emptyset$. Then there exists a prime ideal (filter) P of L such that $I \subseteq P$ and $P \cap F = \emptyset$ ($F \subseteq P$ and $P \cap I = \emptyset$).*

3. QUASI PSEUDO-COMPLEMENTATION ON AN ADL

In this section, we give the definition of a quasi pseudo-complementation on an ADL with 0 and study some elementary properties of quasi pseudo-complementation.

Definition 3.1. Let (L, \vee, \wedge) be an ADL with 0. Then a unary operation $a \mapsto a^*$ on L is called a quasi pseudo-complementation on L if, for any $a, b \in L$, the following are satisfied

- (1) 0^* is a maximal element,
- (2) $(a \vee b)^* = a^* \wedge b^*$,
- (3) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$,
- (4) $a^{***} = a^*$,
- (5) $a \wedge a^* = 0$.

If (L, \vee, \wedge) is an ADL with 0 and $*$ a quasi pseudo-complementation on L then we say that $(L, \vee, \wedge, *, 0)$ is a quasi pseudo-complemented ADL. For brevity, we will call quasi pseudo-complemented ADL as q-p-ADL.

Note that every p-ADL is a q-p-ADL but converse need not be true which we show in the following example.

Example 3.2. (i) Let $L = \{0, a, b, c\}$. Define two binary operations \vee and \wedge on L as follows:

\vee	0	a	b	c
0	0	a	b	c
a	a	a	a	a
b	b	a	b	a
c	c	a	a	c

\wedge	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	b	b	0
c	0	c	0	c

and define $x^* = 0$ for all $x \neq 0$ and $0^* = a$. Then $(L, \vee, \wedge, 0)$ is a distributive lattice and hence an ADL and $*$ is a quasi pseudo-complementation on L but not a pseudo-complementation on L . We can observe that $b \wedge c = 0$ but $b^* \wedge c = 0 \wedge c = 0 \neq c$.

(ii) Let $D = \{0', a', b'\}$ be a discrete ADL and $L = \{0, a, b, c\}$ a distributive lattice given in Example 3.2(i). Then

$$R = D \times L = \{(0', 0), (0', a), (0', b), (0', c), (a', 0), (a', a), (a', b), (a', c), (b', 0), (b', a), (b', b), (b', c)\}$$

and hence $(R, \vee, \wedge, 0^\diamond)$ is an ADL which is not a lattice, where $0^\diamond = (0', 0)$, under point-wise operation. Define $(x, y)^* = (0', 0)$ for all $(x, y) \neq (0', 0)$ and $(0', 0)^* = (a', a)$. Then $*$ is a quasi pseudo-complementation on R . But it is not a pseudo-complementation on R because $(0', b) \wedge (0', c) = (0', b \wedge c) = (0', 0)$ implies that $(0', b)^* \wedge (0', c) = (0', 0) \wedge (0', c) = (0', 0) \neq (0', c)$.

Example 3.3. Let $(L, +, \cdot, 0)$ be a commutative regular ring. To each $a \in L$, let a° be the unique idempotent element in L such that $aL = a^\circ L$. Define, for any $a, b \in L$,

- (i) $a \wedge b = a^\circ b$,
- (ii) $a \vee b = a + (1 - a^\circ)b$,
- (iii) $a^* = 1 - a^\circ$,

then $(L, \vee, \wedge, 0)$ is an almost distributive lattice with 0 and $*$ is a quasi pseudo-complementation on L .

Now we give some elementary properties of a quasi pseudo-complementation.

Lemma 3.4. *Let L be an ADL with 0 and $*$ a quasi pseudo-complementation on L . Then, for $a, b \in L$, we have the following:*

- (1) $a^* \wedge a = 0$,
- (2) $0^{**} = 0$,
- (3) $a^* \wedge b^* = b^* \wedge a^*$,
- (4) $a^* \wedge a^{**} = 0$,
- (5) $a \leq b \Rightarrow b^* \leq a^*$,
- (6) $a \wedge b^* \leq a \wedge (a \wedge b)^*$,
- (7) $(a \vee b)^* = (b \vee a)^*$,
- (8) $(a \wedge b)^* = (b \wedge a)^*$,
- (9) $a^* \wedge (a^* \wedge b)^* = a^* \wedge b^*$,
- (10) $a \wedge b^* = 0 \Rightarrow a^* \wedge b^* = b^*$ and $a^{**} \wedge b^{**} = a^{**}$.

Proof. (1) $a^* \wedge a = a \wedge a^* \wedge a = 0 \wedge a = 0$.

- (2) Since 0^* is a maximal element, we have $0^* \vee 0 = 0^*$. So that $0^{**} = (0^* \vee 0)^* = 0^{**} \wedge 0^* = 0$.
- (3) We know that for any $a, b \in L$, $a \vee 0 = a$ and $b \vee 0 = b$. Therefore $a^* \wedge 0^* = a^*$ and $b^* \wedge 0^* = b^*$. Then $a^* \leq 0^*$ and $b^* \leq 0^*$ and hence $a^* \wedge b^* = b^* \wedge a^*$.
- (4) Since $a \wedge a^* = 0$, we have $(a \wedge a^*)^{**} = 0^{**} = 0$. Hence, by Definition 3.1(3, 4), $a^{**} \wedge a^* = 0$. Thus $a^* \wedge a^{**} = a^{**} \wedge a^* \wedge a^{**} = 0$.
- (5) Suppose $a \leq b$. Then $a \vee b = b$. So that $b^* = (a \vee b)^* = a^* \wedge b^* = b^* \wedge a^*$ by (3). Hence $b^* \leq a^*$.
- (6) Since $a \wedge b \leq b$, by (4), we get $b^* \leq (a \wedge b)^*$ and hence $a \wedge b^* \leq a \wedge (a \wedge b)^*$.
- (7) $(a \vee b)^* = a^* \wedge b^* = b^* \wedge a^* = (b \vee a)^*$.
- (8) $(a \wedge b)^* = (a \wedge b)^{***} = (a^{**} \wedge b^{**})^* = (b^{**} \wedge a^{**})^* = (b \wedge a)^{***} = (b \wedge a)^*$.
- (9) $a^* \wedge (a^* \wedge b)^* = [a \vee (a^* \wedge b)]^* = [(a \vee a^*) \wedge (a \vee b)]^{***} = [(a \vee a^*)^{**} \wedge (a \vee b)^{**}]^* = [0^* \wedge (a \vee b)^{**}]^* = (a \vee b)^{***} = (a \vee b)^* = a^* \wedge b^*$.
- (10) Suppose $a \wedge b^* = 0$. Then $b^* = 0^* \wedge b^* = (a \wedge b^*)^* \wedge b^* = b^* \wedge (b^* \wedge a)^* = b^* \wedge a^*$. So that $b^* \leq a^*$ and hence $a^{**} \leq b^{**}$. Therefore $a^{**} \wedge b^{**} = a^{**}$. ■

Now we prove that quasi-pseudo-complementation on an ADL is equationally definable.

Theorem 3.5. *Let L be an ADL with 0 . Then $*$ is a quasi pseudo-complementation on L if and only if*

- (1) $(a \wedge b)^* = (a \wedge b^{**})^*$
- (2) 0^* is a maximal element
- (3) $(a \vee b)^* = a^* \wedge b^*$
- (4) $(a \wedge b)^* = (b \wedge a)^*$
- (5) $a \wedge a^* = 0$.

Proof. Suppose $*$ is a quasi pseudo-complementation on L and $a, b \in L$. Then (2), (3), (4) and (5) follow from Definition 3.1 and Lemma 3.4. Now

$$\begin{aligned} (a \wedge b)^* &= (a \wedge b)^{***} \\ &= (a^{**} \wedge b^{**})^* \\ &= (a^{**} \wedge b^{****})^* \\ &= (a \wedge b^{**})^{***} \\ &= (a \wedge b^{**})^*. \end{aligned}$$

Conversely, assume that the conditions hold. Let $a, b \in L$. Then

$$a^* = (0^* \wedge a)^* = (0^* \wedge a^{**})^* = a^{***}$$

and

$$\begin{aligned} (a \wedge b)^{**} &= ((a \wedge b)^*)^* \\ &= ((a \wedge b^{**})^*)^* \\ &= ((a^{**} \wedge b^{**})^*)^* \\ &= (a^* \vee b^*)^{***} \\ &= (a^* \vee b^*)^* \\ &= a^{**} \wedge b^{**}. \end{aligned} \quad \blacksquare$$

Now we give necessary and sufficient conditions for a q-p-ADL to be a p-ADL.

Theorem 3.6. *Let L be an ADL with 0 and $*$ is a quasi pseudo-complementation on L . Then, for $a, b \in L$, the following are equivalent*

- (1) $*$ is a pseudo-complementation on L
- (2) $a^{**} \wedge a = a$
- (3) $a^* \wedge b = (a \wedge b)^* \wedge b$
- (4) $[a]^* \subseteq (a^*)$.

Proof. (1) \Rightarrow (2) is clear.

(2) \Rightarrow (1): Assume (2). Let $a, b \in L$ and $a \wedge b = 0$. Then

$$\begin{aligned} b &= b^{**} \wedge b \quad (\text{by (2)}) \\ &= 0^* \wedge b^{**} \wedge b \\ &= (a^* \wedge a^{**})^* \wedge b^{**} \wedge b \\ &= (a \vee a^*)^{**} \wedge b^{**} \wedge b \\ &= b^{**} \wedge (a \vee a^*)^{**} \wedge b \\ &= (b \wedge (a \vee a^*))^{**} \wedge b \\ &= [(b \wedge a) \vee (b \wedge a^*)]^{**} \wedge b \\ &= [0 \vee (b \wedge a^*)]^{**} \wedge b \\ &= (b \wedge a^*)^{**} \wedge b \\ &= b^{**} \wedge a^{***} \wedge b \\ &= a^{***} \wedge b^{**} \wedge b \\ &= a^* \wedge b. \end{aligned}$$

Therefore $*$ is a pseudo-complementation on L . Similarly, we can prove (1) \Leftrightarrow (3) and (1) \Leftrightarrow (4). \blacksquare

Now, we prove that if $*$ is a quasi pseudo-complementation on an ADL L , the set $S(L) = \{a^* \mid a \in L\} = \{a \in L \mid a = a^{**}\}$ becomes a Boolean algebra.

Theorem 3.7. *Let L be an ADL with 0 and $*$ a quasi pseudo-complementation on L . For any $a^*, b^* \in S(L)$, define $a^* \leq b^*$ if and only if $a^* \wedge b^* = a^*$. Then $(S(L), \leq)$ is a Boolean algebra.*

Proof. Clearly \leq is a partial ordering on $S(L)$. Let $a^*, b^* \in S(L)$. Since $(a \vee b)^* = a^* \wedge b^*$, we have $a^* \wedge b^* \in S(L)$ and $a^* \wedge b^* = b^* \wedge a^*$. So that $a^* \wedge b^*$ is the greatest lower bound of $\{a^*, b^*\}$ in $S(L)$. Now we prove that $(a^{**} \wedge b^{**})^*$ is the lub of a^*, b^* in the poset $(S(L), \leq)$. We have $a^{**} \wedge b^{**} \leq b^{**}$ and hence $b^* = b^{***} \leq (a^{**} \wedge b^{**})^*$. Similarly, we get that $a^* \leq (a^{**} \wedge b^{**})^*$. Therefore $(a^{**} \wedge b^{**})^*$ is an upper bound of $\{a^*, b^*\}$ in $S(L)$. Let $c^* \in S(L)$ and $a^* \leq c^*, b^* \leq c^*$. Then $c^{**} \leq a^{**}$ and $c^{**} \leq b^{**}$. Hence $c^{**} \leq a^{**} \wedge b^{**}$. Therefore $(a^{**} \wedge b^{**})^* \leq c^{***} = c^*$. Thus $(a^{**} \wedge b^{**})^*$ is the least upper bound of $\{a^*, b^*\}$ in $S(L)$ and we denote this by $a^* \underline{\vee} b^*$. Hence $(S(L), \leq)$ is a lattice.

It can be easily seen that $(S(L), \leq)$ is a bounded lattice in which 0^* is the greatest element and 0 is the least element. Let $a^* \in S(L)$. Then $a^{**} \in S(L)$, $a^* \underline{\vee} a^{**} = (a^{**} \wedge a^{***})^* = 0^*$ and $a^* \wedge a^{**} = 0$. Hence a^{**} is the complement of a^* in $S(L)$. Finally we prove that $S(L)$ is distributive. Let a^*, b^* and $c^* \in S(L)$. Then,

$$\begin{aligned} a^* \underline{\vee} (b^* \wedge c^*) &= [a^{**} \wedge (b^* \wedge c^*)^*]^* \\ &= [a^{****} \wedge (b^{**} \vee c^{**})^{**}]^* \text{ by definition 3.1} \\ &= [a^{**} \wedge (b^{**} \vee c^{**})]^{***} \text{ by definition 3.1} \\ &= [a^{**} \wedge (b^{**} \vee c^{**})]^* \text{ by definition 3.1} \\ &= [(a^{**} \wedge b^{**}) \vee (a^{**} \wedge c^{**})]^* \\ &= (a^{**} \wedge b^{**})^* \vee (a^{**} \wedge c^{**})^* \\ &= (a^* \underline{\vee} b^*) \wedge (a^* \underline{\vee} c^*). \end{aligned}$$

Therefore $a^* \underline{\vee} (b^* \wedge c^*) = (a^* \underline{\vee} b^*) \wedge (a^* \underline{\vee} c^*)$. Thus $(S(L), \leq)$ is a Boolean algebra. \blacksquare

Corollary 3.8. *Let L be an ADL with 0 and $*$ a quasi pseudo-complementation on L . Then the map $f : L \mapsto S(L)$ defined by $f(a) = a^{**}$ is an epimorphism.*

Definition 3.9. Two quasi pseudo-complementations $*$ and \perp on an ADL L are said to be equivalent, denoted by $*$ \approx \perp , if $0^* = 0^\perp$. Then clearly \approx is an equivalence relation on the set $\mathcal{QPC}(L)$, of all quasi pseudo-complementations on L .

Theorem 3.10. *Let $(L, \vee, \wedge, 0)$ be an ADL with a quasi pseudo-complementation $*$ and M the set of all maximal elements in L . Then, for any $m \in M$, $*_m : L \times L \rightarrow L$ defined by $a^{*m} = a^* \wedge m$ for all $a \in L$ is again a quasi pseudo-complementation on L and the correspondence $m \mapsto *_m$ induces a bijection of M onto $\mathcal{QPC}(L)/\approx$.*

Proof. Let $a, b \in L$ and $m \in M$. Then we can easily show that $*_m$ is a quasi pseudo-complementation on L . Let $m, n \in M$ such that $*_m \approx *_n$. Then $0^{*m} = 0^{*n}$ which implies that $0^* \wedge m = 0^* \wedge n$ and hence $m = n$ since 0^* is maximal in L . Now, let $\perp \in \mathcal{QPC}(L)$. Then $m = 0^\perp \in M$ and $0^\perp = 0^* \wedge 0^\perp = 0^* \wedge m = 0^{*m}$ and hence $*_m \approx \perp$. Thus $m \mapsto *_m$ is a bijection of M onto $\mathcal{QPC}(L)/\approx$. ■

Now we give some equivalent conditions for a q-p-ADL to be a Stone ADL.

Theorem 3.11. *Let L be a q-p-ADL. Then the following are equivalent.*

- (i) L is a Stone ADL.
- (ii) $a^* \vee a^{**} = 0^*$ for all $a \in L$.

Proof. (i) \Rightarrow (ii) is clear. Assume (ii). Let $a \in L$. Then $a^* \vee a^{**} = 0^*$ implies that $(a^* \vee a^{**}) \wedge a = 0^* \wedge a$ which gives $a^{**} \wedge a = a$. Hence, by Theorem 3.6, L is pseudo-complemented and hence L is a Stone ADL. ■

Theorem 3.12. *Let L be a q-p-ADL. Then the following are equivalent.*

- (i) L is a Stone ADL.
- (ii) For any $a, b \in L$, $(a \wedge b)^* = a^* \vee b^*$.

Proof. Assume (i). Suppose $a, b \in L$ and $x = (a \wedge b)^*$. Then $a \wedge b \wedge x = 0$ implies that $a^* \wedge b \wedge x = b \wedge x$ which gives $a^{**} \wedge b \wedge x = 0$. So that $b^* \wedge a^{**} \wedge x = a^{**} \wedge x$ and hence $b^* \vee (a^{**} \wedge x) = b^*$. Now, $a^* \vee b^* = a^* \vee [b^* \vee (a^{**} \wedge x)] = a^* \vee (b^* \vee x)$. Thus $(a^* \vee b^*) \wedge x = [a^* \vee (b^* \vee x)] \wedge x = x$. Now $(a \wedge b)^* = (a^* \vee b^*) \wedge (a \wedge b)^* = [a^* \wedge (a \wedge b)^*] \vee [b^* \wedge (a \wedge b)^*] = a^* \vee b^*$. Conversely, assume (ii). Let $a \in L$. Then $a^* \vee a^{**} = (a \wedge a^*)^* = 0^*$. Hence, by Theorem 3.11, (i) follows. ■

There are no hidden difficulties to prove the following theorem. Hence we omit its proof.

Theorem 3.13. *Let L be a q-p-ADL. Then the following are equivalent.*

- (i) L is a Stone ADL,
- (ii) $S(L)$ is a sublattice of L ,
- (iii) $(a \vee b)^{**} = a^{**} \vee b^{**}$ for all $a, b \in L$,
- (iv) $a \wedge b = 0$ implies $a^* \vee b^* = 0^*$ for all $a, b \in L$.

Definition 3.14. Let L be an ADL with 0 . An element b in L is said to be a semi-complement of the element a in L if $a \wedge b = 0$. We denote the set of all semi-complements of a by $S(a)$.

Lemma 3.15. Let L be an ADL with $a \in L$. Then $S(a)$ is an ideal of L .

Lemma 3.16. Let L be a q - p -ADL. Then the following are equivalent.

- (i) L is a p -ADL.
- (ii) $S(a) = (a^*]$ for all $a \in L$.

Definition 3.17. An ideal I of an ADL L is called a direct factor if there exists an ideal J of L such that $I \cap J = \{0\}$ and $I \vee J = L$.

Now we prove the following.

Theorem 3.18. Let L be a q - p -ADL. Then L is a Stone ADL if and only if, for any $a \in L$, the ideal $S(a) = (a^*]$ is a direct factor of L .

Proof. Suppose L is a Stone ADL and $a \in L$. Then $a^* \vee a^{**} = 0^*$ and $S(a) = (a^*]$. Now $a^* \wedge a^{**} = 0$ and $a^* \vee a^{**} = 0^*$ implies that $(a^*] \wedge (a^{**}] = (0]$ and $(a^*] \vee (a^{**}] = L$. Hence $(a^*]$ is a direct factor of L . Conversely, assume that $S(a) = (a^*]$ is a direct factor of L , for all $a \in L$. Then there exists an ideal J in L such that $(a^*] \cap J = \{0\}$ and $(a^*] \vee J = L$. Write $0^* = b \vee (a^* \wedge x)$ for some $x \in L, b \in J$. Also $a^* \wedge b \in (a^*] \wedge J$ which implies that $a^{**} \wedge b = b$ and $a^{**} \vee b = b$. Now, $0^* = (a^{**} \wedge 0^*) \vee 0^* = (a^{**} \wedge 0^*) \vee ((a^* \vee b) \wedge 0^*) = (a^{**} \vee a^* \vee b) \wedge 0^* = (a^* \vee a^{**}) \wedge 0^*$. Hence $0^* = (a^* \wedge 0^*) \vee (a^{**} \wedge 0^*) = (a \vee 0)^* \vee (a^* \vee 0)^* = a^* \vee a^{**}$. Thus L is a Stone ADL. ■

4. KERNEL IDEALS IN Q-P-ADLS

In this section, we introduce the notions of $*$ -congruences and kernel ideals on a q - p -ADL L . We give a necessary and sufficient condition for a congruence on L to be a $*$ -congruence and we characterize kernel ideals. Finally we give equivalent conditions for every ideal of L to become a kernel ideal. We can recall that a congruence relation on an ADL $(L, \vee, \wedge, 0)$ is an equivalence relation θ , compatible with the operations \vee and \wedge . Throughout this section, L stands for a q - p -ADL $(L, \vee, \wedge, 0)$ with quasi pseudo-complementation $*$, otherwise we specify.

Definition 4.1. A congruence relation θ on a q - p -ADL L is called a $*$ -congruence if it satisfies the following condition:

$$(a, b) \in \theta \text{ implies that } (a^*, b^*) \in \theta \text{ for all } a, b \in L.$$

The following example shows that every congruence on a q-p-ADL need not be a $*$ -congruence.

Example 4.2. Let $R = D \times L = \{(0', 0), (0', a), (0', b), (0', c), (a', 0), (a', a), (a', b), (a', c), (b', 0), (b', a), (b', b), (b', c)\}$ be a q-p-ADL as in Example 3.2(ii). Now consider two congruence relations θ_1 and θ_2 on $R = D \times L$ whose partitions A_1 and A_2 are respectively given by

$$A_1 = \left\{ \{(0', 0), (0', a), (0', b), (0', c), (a', a)\}, \{(a', 0), (a', b), (a', c)\}, \{(b', 0), (b', a), (b', b), (b', c)\} \right\}$$

and

$$A_2 = \left\{ \{(0', 0), (0', a), (0', b), (0', c)\}, \{(a', 0), (a', a), (a', b), (a', c)\}, \{(b', 0), (b', a), (b', b), (b', c)\} \right\}.$$

Then clearly θ_1 is a $*$ -congruence on $R = D \times L$. But θ_2 is not a $*$ -congruence on $R = D \times L$, because $((0', b), (0', 0)) \in \theta_2$ and $((0', b)^*, (0', 0)^*) = ((0', 0), (a', a)) \notin \theta_2$.

Now we give an equivalent condition for a congruence relation on q-p-ADL L to be $*$ -congruence.

Theorem 4.3. *A congruence relation θ on L is a $*$ -congruence if and only if $(a, 0) \in \theta$ implies that $(a^*, 0^*) \in \theta$ for any $a \in L$.*

Proof. Let θ be a $*$ -congruence on L and $a \in L$. Then $(a, 0) \in \theta$ implies $(a^*, 0^*) \in \theta$. Conversely, assume that the condition holds and $(a, b) \in \theta$. Then $(b, a) \in \theta$ which implies that $(b \wedge a^*, 0) \in \theta$ and hence $((b \wedge a^*)^*, 0^*) \in \theta$. Therefore $(a^* \wedge b^*, a^*) = (a^* \wedge (a^* \wedge b)^*, a^* \wedge 0^*) \in \theta$. Similarly, we can obtain that $(a^* \wedge b^*, b^*) \in \theta$. Hence $(a^*, b^*) \in \theta$. Thus θ is a $*$ -congruence on L . ■

We proved that $S(L) = \{x \in L \mid x^{**} = x\}$ is a Boolean algebra in which for any $a, b \in S(L)$, $a \vee b = (a^* \wedge b^*)^*$. In a pseudo-complemented distributive lattice, the relation θ defined by $(x, y) \in \theta$ if and only if $x^* = y^*$ is a congruence called the Glivenko congruence. Now, we prove that the same θ is a $*$ -congruence relation on a q-p-ADL L and we show that L/θ is a Boolean algebra under this $*$ -congruence θ on L

Theorem 4.4. *Let L be a q-p-ADL. Then L/θ is a Boolean algebra under the $*$ -congruence relation θ on L defined by $(x, y) \in \theta$ if and only if $x^* = y^*$.*

Proof. Clearly θ is an equivalence relation on L . Suppose $(a, b) \in \theta$ and $c \in L$. Then $a^* = b^*$ and hence $(a \vee c)^* = a^* \wedge c^* = b^* \wedge c^* = (b \vee c)^*$. Again $(a \wedge c)^* = (a^{**} \wedge c^{**})^* = (b^{**} \wedge c^{**})^* = (b \wedge c)^*$. Hence $(a \vee c, b \vee c) \in \theta$ and $(a \wedge c, b \wedge c) \in \theta$. Then θ is a congruence relation on L . Clearly θ is a $*$ -congruence. Now define $\lambda : L/\theta \rightarrow S(L)$ by $\lambda([a]_\theta) = a^{**}$ for all $[a]_\theta \in L/\theta$. Clearly λ is well-defined, one-one and onto. Let $[a]_\theta, [b]_\theta \in L/\theta$. Now $\lambda([a]_\theta \wedge [b]_\theta) = \lambda([a \wedge b]_\theta) = (x \wedge y)^{**} = x^{**} \wedge y^{**} = \lambda([a]_\theta) \wedge \lambda([b]_\theta)$. Again, $\lambda([a]_\theta \vee [b]_\theta) = \lambda([a \vee b]_\theta) = (a \vee b)^{**} = (a^* \wedge b^*)^* = a^{**} \vee b^{**} = \lambda([a]_\theta) \vee \lambda([b]_\theta)$. Therefore λ is an isomorphism. Hence L/θ is a Boolean algebra. ■

For any ideal I of L , we introduce a $*$ -congruence $\psi(I)$ on L corresponding to I .

Theorem 4.5. *Let L be a q - p -ADL and I an ideal of L . Define a binary relation $\psi(I)$ on L by*

$$(a, b) \in \psi(I) \text{ if and only if } a \wedge i^* = b \wedge i^* \text{ for some } i \in I.$$

Then $\psi(I)$ is a $$ -congruence relation on L .*

Proof. Since $(i \vee j)^* = i^* \wedge j^*$ for any $i, j \in L$ and the fact that I is an ideal of L , clearly $\psi(I)$ is an equivalence relation on L . Let $(a, b) \in \psi(I)$ and $(c, d) \in \psi(I)$. Then $a \wedge i^* = b \wedge i^*$ for some $i \in I$ and $c \wedge j^* = d \wedge j^*$ for some $j \in I$. Hence $(a \vee c) \wedge (i \vee j)^* = (a \vee c) \wedge i^* \wedge j^* = (a \wedge i^* \wedge j^*) \vee (b \wedge i^* \wedge j^*) = (c \wedge i^* \wedge j^*) \vee (d \wedge i^* \wedge j^*)$. Therefore $(a \vee c, b \vee d) \in \psi(I)$. Now $(a \wedge c) \wedge (i \vee j)^* = a \wedge c \wedge i^* \wedge j^* = a \wedge i^* \wedge c \wedge j^* = b \wedge i^* \wedge d \wedge j^*$. Hence $(a \wedge c, b \wedge d) \in \psi(I)$. Thus $\psi(I)$ is a congruence on L . Suppose $(a, 0) \in \psi(I)$. Then $a \wedge i^* = 0$ for some $i \in I$. Then $0^* \wedge i^* = (a \wedge i^*)^* \wedge i^* = a^* \wedge i^*$ (by Lemma 3.4(9)). Therefore $(a^*, 0^*) \in \psi(I)$. Thus $\psi(I)$ is a $*$ -congruence relation on L . ■

Definition 4.6. An ideal I of an q - p -ADL is called a kernel ideal if there exists a $*$ -congruence μ on L such that $I = \text{Ker} \mu = \{a \in L : (a, 0) \in \mu\}$.

Theorem 4.7. *If I is a kernel ideal of L then the following conditions hold.*

- (i) $a, b \in I$ implies $(a^* \wedge b^*)^* \in I$.
- (ii) $a, b \in I$ implies that there exists $k \in I$ such that $a^* \wedge b^* = k^*$.

Proof. Let I be kernel ideal of L and $a, b \in I$. Then $I = \text{ker} \theta$ for some $*$ -congruence θ on L . Then $(a, 0) \in \theta$ and $(b, 0) \in \theta$. Hence $(a^*, 0^*) \in \theta$ and $(b^*, 0^*) \in \theta$. So that $(a^* \wedge b^*, 0^*) \in \theta$ and hence $((a^* \wedge b^*)^*, 0) \in \theta$. Thus $(a^* \wedge b^*)^* \in \text{ker} \theta = I$. Hence (i) follows. Put $k = (a^* \wedge b^*)^*$. Then, by (i), $k \in I$ and $k^* = (a^* \wedge b^*)^{**} = a^* \wedge b^*$. Hence (ii) follows. ■

Now we give necessary and sufficient conditions for an ideal to become a kernel ideal.

Theorem 4.8. *For any ideal I of L , the following are equivalent.*

- (i) I is a kernel ideal.
- (ii) For $a, b \in L$, $a^* = b^*$ and $a \in I$ imply $b \in I$.
- (iii) $a \in I$ if and only if $a^{**} \in I$.

Proof. (i) \Rightarrow (ii): Assume (i). Then there exists a $*$ -congruence θ on L such that $\ker\theta = I$. Chose $x, y \in L$ such that $x^* = y^*$ and $x \in I$. Then $(x, 0) \in \theta$ and hence $(y^*, 0^*) = (x^*, 0^*) \in \theta$. Therefore $(0, y) = (y^* \wedge y, 0^* \wedge y) \in \theta$. Thus $y \in \ker\theta = I$. Since $x^* = x^{***}$ for all $x \in L$, (ii) \Rightarrow (iii) follows. Now, assume (iii). We know that $\psi(I)$ is a $*$ -congruence relation on L by Theorem 4.5. If $x \in \ker\psi(I)$. Then $(x, 0) \in \psi(I)$ and hence $x \wedge i^* = 0$ for some $i \in I$. Therefore, by Theorem 3.4(10), $x^{**} = x^{**} \wedge i^{**} \in I$ and hence $x \in I$. Thus I is a kernel ideal. ■

An element $a \in L$ is called a dense element if $a^* = 0$. The set $D(L)$ of all dense elements of L forms a filter of L . The following theorem can be proved easily.

Theorem 4.9. *In L , the following conditions hold.*

- (i) $x \vee x^* \in D(L)$ for all $x \in L$.
- (ii) $D(L)$ is a filter of L .
- (iii) For any ideal I with $I \cap D(L) = \emptyset$, there exists a minimal prime ideal P such that $I \subseteq P$ and $P \cap D(L) = \emptyset$.
- (iv) Every proper kernel ideal is contained in a minimal prime ideal.

Theorem 4.10. *If $(x] = (x^{**}]$ for all $x \in L$, then $(x]$ is a kernel ideal.*

In [11], it is observed that the set $\mathcal{PI}(L)$ of all principal ideals of an ADL L is a distributive lattice with least element $(0]$. Now, we give sufficient condition for $\mathcal{PI}(L)$ to become Boolean algebra.

Theorem 4.11. *If $(x] = (y]$ for all $x, y \in D(L)$ then $\mathcal{PI}(L)$ is a Boolean algebra.*

Proof. Let $(x] = (y]$ for all $x, y \in D(L)$. Then $\{(x] \mid x \in D(L)\} = \{(d]\}$ for some $x \in L$. Clearly $x \vee x^* \in D(L)$. Hence $(x \vee x^*] = (d]$. For any $(x] \in \mathcal{PI}(L)$, $(x] \subseteq (x \vee x^*] = (d]$. Therefore $(d]$ is the greatest element of $\mathcal{PI}(L)$. Also $(x] \cap (x^*] = (0]$ and $(x] \vee (x^*] = (d]$. Hence $\mathcal{PI}(L)$ is a bounded distributive lattice in which every element is complemented. Thus $\mathcal{PI}(L)$ is a Boolean algebra. ■

Now, we give equivalent conditions for every ideal of L to become a kernel ideal.

Theorem 4.12. *Let L be a q - p -ADL. Then the following conditions are equivalent.*

- (i) *Every ideal is a kernel ideal.*
- (ii) *Every prime ideal is a kernel ideal.*
- (iii) *For any $a, b \in L$, $a^* = b^*$ implies $(a) = (b)$.*
- (iv) *Every principal ideal is a kernel ideal.*

Proof. (i) \Rightarrow (ii) is clear. Assume (ii) and $a, b \in L$ such that $a^* = b^*$. Suppose $(a) \neq (b)$. Without loss of generality, assume that $(a) \not\subseteq (b)$. Take $\mathfrak{F} = \{J \in \mathcal{I}(L) \mid b \in J \text{ and } a \notin J\}$. Then, by Zorn's lemma, \mathfrak{F} has a maximal element, say P . Chose $r, s \in L$ such that $r \notin P$ and $s \notin P$. Then $P \subset P \vee (r)$ and $P \subset P \vee (s)$. By the maximality of P , we can get $a \in \{P \vee (r)\} \cap \{P \vee (s)\} = P \vee (r \wedge s)$. If $r \wedge s \in P$, then $a \in P$ which is a contradiction. Hence P is prime which is kernel ideal. Now $a^* = b^*$ and $b \in P$ implies that $a \in P$, which is a contradiction. Therefore $(a) = (b)$. Hence (iii) follows. Now, assume (iii) and I is a principal ideal of L . Then $I = (a)$ for some $a \in L$. Let $r, s \in L$ such that $r^* = s^*$ and $r \in (a)$. Then $(r) = (s)$ and $s \in (r) \subseteq (a)$. Hence (iv) follows. Finally, assume (iv) and I is an ideal of L . Let $a \in I$. Then $(a) \subseteq I$ and hence $a^{**} \in I$ since (a) is a kernel ideal. Conversely assume $a^{**} \in I$. Then $(a^{**}) \subseteq I$ and hence $a \in (a^{**}) \subseteq I$ since (a^{**}) is a kernel ideal. Hence I is a kernel ideal of L . ■

CONCLUSION AND FUTURE WORK

In this paper, we have introduced the concept of quasi-pseudo-complementation on an ADL as a generalization of pseudo-complementation on an ADL and studied its properties. We have given necessary and sufficient conditions for a q-p-ADL to be a p-ADL and a stone ADL. We proved that if $*$ is a quasi pseudo-complementation on an ADL L then the set $S(L) = \{a^* \mid a \in L\}$ becomes a Boolean algebra. Also, it is observed that, there exists an induced surjective correspondence between the set of maximal elements and the set of quasi pseudo-complementations on L , provided there is a quasi pseudo-complementation. Also, the concept of $*$ -congruence, kernel ideals on a q-p-ADL is introduced and given equivalent conditions for every ideal of L to become a kernel ideal.

In our future work, we will introduce the concepts of demi-pseudo-complementation on an ADL (for brevity, demi-p-ADL), Weak-Stone ADL and study their properties.

REFERENCES

- [1] W.H. Cornish, *Congruences on distributive pseudo-complemented lattices*, Bull. Aust. Math. Soc. **8** (1973) 161–179.
doi:10.1017/S0004972700042404

- [2] T.S. Blyth, *Ideals and filters of Pseudo-complemented semilattices*, Proc. Edinburgh Math. Soc. **23** (1980) 301–316.
doi:10.1017/S0013091500003850
- [3] O. Frink, *Pseudo-complements in semilattices*, Duke Math. J. **29** (1962) 505–514.
doi:10.1215/S0012-7094-62-02951-4
- [4] G. Gratzer, *General Lattice Theory* (Academic Press, New York, 1978).
- [5] K.B. Lee, *Equational class of distributive pseudo-complemented lattices*, Canad. J. Math. **22** (1970) 881–891.
doi:10.4153/CJM-1970-101-4
- [6] G.C. Rao and S. Ravikumar, *Minimal prime ideals in almost distributive lattices*, Int. J. Contemp. Math. Sci. **4** (10) (2009) 475–484.
<http://m-hikari.com/ijcms-password2009/9-12-2009/raoIJCMS9-12-2009-3.pdf>
- [7] M.S. Rao, *Kernel ideals in Almost Distributive Lattices*, Asian-European J. Math. **5** (2) (2012), 1250024(11 pages).
doi:10.1142/S1793557112500246
- [8] H.P. Sankappanavar, *Semi-de Morgan algebras*, J. Symbolic Logic **52** (3) (1987) 712–724.
doi:10.2307/2274359
- [9] U.M. Swamy, G.C. Rao and G. Nanaji Rao, *Pseudo-complementation on Almost Distributive Lattices*, South. Asian Bull. Math. **24** (2000) 95–104.
doi:10.1007/s10012-000-0095-5
- [10] U.M. Swamy, G.C. Rao and G. Nanaji Rao, *Stone Almost Distributive Lattices*, South. Asian Bull. Math. **27** (3) (2003) 115–119.
<http://www.seams-bull-math.ynu.edu.cn/archive.jsp>
- [11] U.M. Swamy and G.C. Rao, *Almost Distributive Lattices*, J. Australian. Math. Soc. Ser. A **31** (1981) 77–91.
doi:10.1017/S1446788700018498

Received 10 September 2018

Revised 3 September 2019

Accepted 12 January 2020