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MAXIMAL B_p-SUBALGEBRAS OF B-ALGEBRAS

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Abstract

We provide some properties of maximal B_p-subalgebras of B-algebras. In particular, we show that for each prime p, a finite B-algebra has a maximal B_p-subalgebra. We also show that for a finite B-algebra of order $p^r m$, where $(p, m) = 1$, any two maximal B_p-subalgebras are conjugate and the number of maximal B_p-subalgebras is $kp + 1$ for some $k \in \mathbb{Z}^+$.

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1. Introduction and preliminaries

In [12], Neggers and Kim introduced and established the notion of B-algebras. A *B-algebra* is an algebra $(X; *, 0)$ of type (2, 0) satisfying:

(I) $x * x = 0$,

(II) $x * 0 = x$,

(III) $(x * y) * z = x * (z * (0 * y))$, for any $x, y, z \in X$.

The following are some of the basic properties of B-algebras. We have
(P1) $0 * (0 * x) = x$ [12],
(P2) $x * y = 0 * (y * x)$ [15],
(P3) $x * (y * z) = (x * (0 * z)) * y$ [12],
(P4) $0 * x = 0 * y$ implies $x = y$ [12],
(P5) $x * y = 0$ implies $x = y$ [12],
(P6) $(y * x) * (z * x) = y * z$ [15],
(P7) $x * y = x * z$ implies $y = z$ [4], for any $x, y, z \in X$.

From now on, let X stand for a B-algebra $(X; *, 0)$. A *subalgebra* of X is a nonempty subset N of X such that $x * y \in N$ for any $x, y \in N$. N is *normal* in X if $x * y, a * b \in N$ implies $(x * a) * (y * b) \in N$.

Theorem 1 [16]**.** *A subalgebra N is normal in X if and only if $x * (x * y) \in N$ for any $x \in X, y \in N$.*

From [5], the subset HK is defined by $HK = \{h * (0 * k) : h \in H,\ k \in K\}$, where H and K are subalgebras of X.

Lemma 2 [5]**.** *If K is normal in X, then HK is a subalgebra of X.*

In [13], one constructs a quotient B-algebra via normal subalgebra. Let N be normal in X. Define a relation $\sim_N$ on X by $x \sim_N y$ if and only if $x * y \in N$. Then $\sim_N$ is an equivalence relation on X. Denote the equivalence class containing x by xN, that is, $xN = \{y \in X : x \sim_N y\}$. Let $X/N = \{xN : x \in X\}$. Then X/N is a B-algebra, where $xN * yN = (x * y)N$. A map $\varphi : X \to Y$ is called a *B-homomorphism* if $\varphi(x * y) = \varphi(x) * \varphi(y)$. The subset $\{x \in X : \varphi(x) = 0_Y\}$ of X is called the *kernel* of the B-homomorphism φ, denoted by $Ker\ \varphi$.

Lemma 3 [5]**.** *Let $\varphi : X \to Y$ be a B-homomorphism from X into Y. Suppose that H is a subalgebra of X and K is a subalgebra of Y. Then* (i) *$\varphi(H)$ is a subalgebra of Y and* (ii) *$\varphi^{-1}(K)$ is a subalgebra of X containing $Ker\ \varphi$.*

In [6], the *centralizer C(x) of x in X* is defined by $C(x) = \{y \in X : y*(0*x) = x * (0 * y)\}$. Let H be a nonempty subset of X. The *centralizer C(H) of H in X* is defined by $C(H) = \{y \in X : y * (0 * x) = x * (0 * y)$ for all $x \in H\}$. Then $C(H)$ is a subalgebra of X. Let K be a nonempty subset of X. We define H_x as the set $H_x = \{x * (x * h) : h \in H\}$. The *normalizer of H in K*, denoted by $N_K(H)$, is defined by $N_K(H) = \{x \in K : H_x = H\}$. If $K = X$, then $N_X(H)$ is called the *normalizer of H*, denoted by $N(H)$. If $H = \{x\}$, then we write $N(x)$ in place of $N(\{x\})$. Then $N_K(H)$ is a subalgebra of X.

Theorem 4 [6]**.** *Let H be a subalgebra of X. Then* (i) *H is normal in X if and only if $N(H) = X$ and* (ii) *H is normal in $N(H)$.*

The *left* and *right B-cosets* of H in X is given by $xH = \{x * (0 * h) : h \in H\}$ and $Hx = \{h * (0 * x) : h \in H\}$, respectively.

Lemma 5 [3]. *Let H be a subalgebra of X. Then $aH = H$ if and only if $a \in H$.*

Theorem 6 [3]. *Let H be a subalgebra of X. Then* (i) *$aH = bH$ if and only if $(0 * b) * (0 * a) \in H$ and* (ii) *$Ha = Hb$ if and only if $a * b \in H$.*

The *index* of H in X, denoted by $[X : H]_B$, is the number of distinct left (or right) B-cosets of H in X.

Theorem 7 [3]. *Let H be a subalgebra of a finite B-algebra X. Then $|X|_B = [X : H]_B|H|_B$.*

Theorem 8 [3]. *If H and K are finite subalgebras of X, then $|HK|_B = \frac{|H|_B|K|_B}{|H \cap K|_B}$.*

Theorem 9 [9]. *Let X be a finite B-algebra with $|X|_B = n$ such that n is divisible by a prime p. Then X contains an element of order p and hence a subalgebra of order p.*

In [7], a *B-action* of X on a set S is a map $*' : X \times S \to S$, written $x *' s$ for all $(x, s) \in X \times S$, satisfying:

(B1) $0 *' s = s$

(B2) $x_1 *' (x_2 *' s) = (x_1 * (0 * x_2)) *' s$, for any $x_1, x_2 \in X$ and $s \in S$.

In this case, we say that X acts on S.

Example 10 [7]. Let H and K be subalgebras of X.

1. Define $*' : H \times X \to X$ by $(h, x) \to h * (0 * x)$ for all $(h, x) \in H \times X$. Then $*'$ is a B-action and is called the *left B-translation* of H on X.
2. Let $\mathcal{L}$ be the set of all left B-cosets of K in X. Define $*' : H \times \mathcal{L} \to \mathcal{L}$ by $(h, xK) \to (h * (0 * x))K$. Then H acts on $\mathcal{L}$ by left B-translation.
3. Define $*' : H \times X \to X$ by $(h, x) \to h * (h * x)$ for all $(h, x) \in H \times X$. Then $*'$ is a B-action and is called the *B-conjugation*.

Let $*'$ be a B-action of X on S. Define $\sim$ on S by $s \sim s'$ if and only if $x *' s = s'$ for some $x \in X$. Then $\sim$ is an equivalence relation on S and for each $s \in S$, $X_s = \{x \in X : x *' s = s\}$ is a subalgebra of X. The equivalence classes are called the *B-orbits of X on S* and the B-orbit of $s \in S$ is denoted by $[s]_B$. The subalgebra X_s is called the *B-stabilizer of s*.

Theorem 11 [7]. *Let $*'$ be a B-action of X on S. Then $|[s]_B|_B = [X : X_s]_B$ for any $s \in S$.*

Let $S_0 = \{s \in S : x *' s = s$ for all $x \in X\}$.

Theorem 12 [7]. *Let $*'$ be a B-action of X on a finite set S. If $|X|_B = p^n$ for some prime p, then $|S| \equiv |S_0|$ mod p.*

Theorem 13 [7]. *Let H be a subalgebra of a finite B-algebra X, where $|H|_B = p^k$ for some prime p and $k \in \mathbb{Z}^+$. Then $[X : H]_B \equiv [N(H) : H]_B$ mod p. Moreover, if p divides $[X : H]_B$, then $N(H) \neq H$.*

2. MAXIMAL B_p-SUBALGEBRAS

Let p be a prime number. A B-algebra X is called a *B_p-algebra* [9] if the order of each element of X is a power of p. A subalgebra H of X is called *B_p-subalgebra* if H is a B_p-algebra.

Theorem 14 [9]. *Let X be a nontrivial B-algebra. Then X is a finite B_p-algebra if and only if $|X|_B = p^k$ for some $k \in \mathbb{Z}^+$.*

Theorem 15. *Let f be a B-homomorphism of X onto Y. Then f induces a one-to-one preserving correspondence between the subalgebras of X containing Ker f and the subalgebras of Y. Moreover, if H and K are corresponding subalgebras of X and Y, respectively, then H is normal in X if and only if K is normal in Y.*

Proof. Let $\mathcal{H} = \{H\colon H$ is a subalgebra of X such that $Ker\ f \subseteq H\}$ and $\mathcal{K} = \{K : KisasubalgebraofY\}$. Define $f^* : \mathcal{H} \to \mathcal{K}$ by $f^*(H) = \{f(h) : h \in H\}$ for all $H \in \mathcal{H}$. By Lemma 3(i), $f^*(H) \in \mathcal{K}$. Moreover, f^* is well-defined since f is well-defined. Let $K \in \mathcal{K}$. Denote $f^{-1}(K) = H$. By Lemma 3(ii), $H \in \mathcal{H}$ and $f^*(H) = K$. Thus, f^* maps $\mathcal{H}$ onto $\mathcal{K}$. Let $H_1, H_2 \in \mathcal{H}$. Suppose that $f^*(H_1) = f^*(H_2)$. Let $h_1 \in H_1$. Then there exists $h_2 \in H_2$ such that $f(h_1) = f(h_2)$. By (I), $f(h_1 * h_2) = f(h_1) * f(h_2) = 0$ and so $h_1 * h_2 \in Kerf \subseteq H_2$. Thus, by (P6) and (II), $h_1 = (h_1 * h_2) * (0 * h_2) \in H_2$. Therefore, $H_1 \subseteq H_2$. Similarly, $H_2 \subseteq H_1$. Thus, $H_1 = H_2$ and so f^* is one-to-one. Now, $H_1 \subseteq H_2$ if and only if $f^*(H_1) \subseteq f^*(H_2)$. Moreover, since f^* is one-to-one, $H_1 \subset H_2$ if and only if $f^*(H_1) \subset f^*(H_2)$. Suppose that H is normal in X such that $Kerf \subseteq H$. Let $K = f^*(H)$. Let $f(a) \in Y$ and $f(h) \in K$, where $a \in X, h \in H$. By Theorem 1, $a * (a * h) \in H$. Thus, $f(a) * (f(a) * f(h)) = f(a * (a * h)) \in K$. Hence, K is normal in Y. Let J be normal in Y and $L \in \mathcal{H}$ be such that $f^*(L) = J$. Let $a \in X$ and $h \in L$. Then by Theorem 1, $f(a * (a * h)) = f(a) * (f(a) * f(h)) \in J$ and so $a * (a * h) \in L$. Therefore, L is normal in X. ■

Corollary 16. *Let N be normal in X. Then every subalgebra of X/N is of the form K/N, where K is a subalgebra of X that contains N. Moreover, K/N is normal in X/N if and only if K is normal in X.*

Theorem 17. *Let X be a finite B-algebra of order $p^r m$, where p is a prime and $(p, m) = 1$. Then X has a subalgebra of order p^k for all k, where $0 \leq k \leq r$.*

Proof. If $r = 0$, then $\{0\}$ is the required subalgebra of order p^r. Suppose that $r \geq 1$. Since $p||X|_B$, X has a subalgebra of order p by Theorem 9. We show that if X has a subalgebra of order p^i, then X has a subalgebra of order p^{i+1}, where $1 \leq i < r$. Suppose that X has a subalgebra H of order p^i, $1 \leq i < r$. Then H is a proper subalgebra of X. By Theorem 13, $[X : H]_B \equiv [N(H) : H]_B \; mod \; p$. Since $p|[X : H]_B$, $N(H) \neq H$ and so $p||N(H)/H|_B$. By Theorem 9 and Corollary 16, $N(H)/H$ has a subalgebra K/H of order p. Now, $|K|_B = |K/H|_B|H|_B = pp^i = p^{i+1}$. Therefore, K is a subalgebra of X of order p^{i+1}. The result follows by induction. ■

The following theorem shows the existence of maximal B_p-subalgebras in a finite B-algebra.

Theorem 18. *For each prime p, a finite B-algebra X has a maximal B_p-subalgebra.*

Proof. If $|X|_B = 1$ or p does not divide $|X|_B$, then $\{0\}$ is the required maximal B_p-subalgebra of X. If $p||X|_B$, then there exists at least one subalgebra H of X of order p by Theorem 9. Since X is finite, there are a finite number of subalgebras of X which contain H. Hence, one of these subalgebras is a maximal B_p-subalgebra of X. ■

The following example shows that maximal B_p-subalgebra need not be unique.

Example 19. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with the following table:

$*$	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

Then $(X; *, 0)$ is a B-algebra [12]. Let $H_1 = \{0, 3\}$, $H_2 = \{0, 4\}$, and $H_3 = \{0, 5\}$. Then H_1, H_2, and H_3 are maximal B_2-subalgebras of X.

Lemma 20. *Let X be a finite B-algebra of order $p^r m$, where p is a prime and $(p, m) = 1$.*

(i) *Let H be a subalgebra of X of order p^i, $1 \leq i < r$. Then there exists a subalgebra K of X such that $|K|_B = p^{i+1}$ and H is normal in K.*

(ii) *Let H be a subalgebra of X. Then H is a maximal B_p-subalgebra of X if and only if $|H|_B = p^r$.*

Proof. (i) By Theorem 13, $[X : H]_B \equiv [N(H) : H]_B \ mod\ p$. Since $p|[X : H]_B$, $p||N(H)/H|_B$. Thus, $N(H)/H$ has a subalgebra K/H of order p by Theorem 9. Now, $|K|_B = |H|_B|K/H|_B = p^{i+1}$. By Theorem 4(ii), H is normal in $N(H)$. Since $K \subseteq N(H)$, H is normal in K. Thus, K is the desired subalgebra of X.

(ii) Suppose that H is a maximal B_p-subalgebra of X. Then H is a B_p-subalgebra of X. By Theorem 14, $|H|_B = p^k$ for some positive integer k. Suppose that $k \neq r$. By (i), there exists a subalgebra K of X such that $H \subset K$ and $|K|_B = p^{k+1}$. Thus, H is not a maximal B_p-subalgebra of X, a contradiction. Hence, $k = r$. Conversely, suppose that $|H|_B = p^r$. Since $|X|_B = p^r m$ and $(p, m) = 1$, it follows that H is a maximal B_p-subalgebra of X. Hence, H is a maximal B_p-subalgebra of X. ■

Proposition 21. *Let H be a maximal B_p-subalgebra of a finite B-algebra X. If K is a subalgebra of X such that $H \subseteq K$, then H is a maximal B_p-subalgebra of K.*

Proof. By Lemma 20(ii), $|H|_B = p^r$, where p^r is the highest power dividing $|X|_B$. Thus, $|X|_B = p^r m$, where $(p, m) = 1$ for some positive integer m. By Theorem 7, $|K|_B = p^r t$ for some $t \leq m$ and $(p, t) = 1$. Therefore, by Lemma 20(ii), H is a maximal B_p-subalgebra of K. ■

3. CONJUGATE OF MAXIMAL B_p-SUBALGEBRAS

For every $x \in X$, we recall that $H_x = \{x * (x * h) : h \in H\}$.

Lemma 22. *Let H and K be subalgebras of X.*

(i) *If $H \subseteq K$, then $H_x \subseteq K_x$ for all $x \in X$.*

(ii) *For all $x \in X$, $(H \cap K)_x = H_x \cap K_x$.*

(iii) *For all $x, y \in X$, $(H_x)_y = H_{y*(0*x)}$.*

Example 23. Let X be the B-algebra in Example 19 and $H = \{0, 3\}$. We have $H_0 = H_3 = H$, $H_1 = H_4 = \{0, 5\}$, $H_2 = H_5 = \{0, 4\}$. This means that H need not be equal to H_x for all $x \in X$.

Theorem 24. *Let H be a subalgebra of a B-algebra X and $x \in X$. Then H_x is a subalgebra of X. Moreover, $H \cong H_x$.*

Proof. By (I) and (II), $0 = x*(x*0) \in H_x$ and so $H_x \neq \varnothing$. Let $a, b \in H_x$. Then $a = x*(x*h_1)$ and $b = x*(x*h_2)$ for some $h_1, h_2 \in H$. Thus, by (III), (P2), and (P6), $a*b = x*(x*(h_1*h_2))$. Since H is a subalgebra, $h_1*h_2 \in H$. Thus, $a*b \in H_x$ and so H_x is a subalgebra of X. Define $f : H \to H_x$ by $f(h) = x*(x*h)$ for all $h \in H$. Let $h_1, h_2 \in H$. If $h_1 = h_2$, then $f(h_1) = x*(x*h_1) = x*(x*h_2) = f(h_2)$ and so f is well-defined. Suppose that $f(h_1) = f(h_2)$. Then $x*(x*h_1) = x*(x*h_2)$. By (P7), $h_1 = h_2$. Thus, f is one-to-one. Let $a \in H_x$. Then $a = x*(x*h)$ for some $h \in H$. Hence, f is onto. By (P6), (P2), and (III), f is a B-homomorphism. Therefore, $H \cong H_x$. ■

The subalgebra H_x of X in Theorem 24 is called a *conjugate* of H.

Example 25. Let $X = \{0, 1, 2, 3\}$ be a set with the following table:

$*$	0	1	2	3
0	0	2	1	3
1	1	0	3	2
2	2	3	0	1
3	3	1	2	0

Then $(X; *, 0)$ is a B-algebra [13]. Let $H = \{0, 3\}$. Then $H_x = H$ for all $x \in X$.

Corollary 26. *If H is normal in X, then $H_x = H$.*

Corollary 27. *Let H and K be subalgebras of X such that K is normal in X. Then $(HK)_x = H_xK_x$.*

Proof. By Lemma 22(i), $H_x \subseteq (HK)_x$ and $K_x \subseteq (HK)_x$. By Lemma 2, HK is a subalgebra of X. By Theorem 24, $(HK)_x$ is a subalgebra of X. Thus, $H_xK_x \subseteq (HK)_x$. Let $y \in (HK)_x$. Then $y = x*(x*i)$ for some $i \in HK$. Thus, $y = x*(x*(h*(0*k)))$ for some $h \in H, k \in K$. By (III), (P6), and (P2), $y = (x*(x*h))*[0*(x*(x*k))] \in H_xK_x$. Hence, $(HK)_x \subseteq H_xK_x$. Therefore, $(HK)_x = H_xK_x$. ■

The following lemma shows that any conjugate of a B_p-subalgebra is also a B_p-subalgebra. Moreover, any conjugate of a maximal B_p-subalgebra is also a maximal B_p-subalgebra.

Lemma 28. *Let X be a finite B-algebra of order $p^r m$, where p is a prime and $(p, m) = 1$. Suppose that H is a subalgebra of X.*

(i) *If H is a B_p-subalgebra of X, then H_x is a B_p-subalgebra of X.*
(ii) *If H is a maximal B_p-subalgebra of X, then H_x is a maximal B_p-subalgebra of X for all $x \in X$.*
(iii) *If H is the only maximal B_p-subalgebra of X, then H is normal in X.*

Proof. (i) By Theorem 24, $|H|_B = |H_x|$ and H_x is a subalgebra of X. Therefore, by Theorem 14, H_x is a B_p-subalgebra.

(ii) By Lemma 20(ii), $|H|_B = p^r$. Hence, $|H_x|_B = p^r$. Thus, by Lemma 20(ii), H_x is a maximal B_p-subalgebra.

(iii) By (ii), $H_x = H$. By Theorem 1, H is normal in X. ■

Let H be normal in X. For any positive integer k, $(xH)^k = x^kH$.

Lemma 29. *Let H be normal in X. If H and X/H are both B_p-algebras, then X is a B_p-algebra.*

Proof. Let $x \in X$. Then $xH \in X/H$. Since X/H is a B_p-algebra, xH has order some power of p, say p^k. Thus, $(xH)^{p^k} = x^{p^k}H = H$. By Lemma 5, $x^{p^k} \in H$. Since H is a B_p-algebra, x^{p^k} has order p^m. Hence, $(x^{p^k})^{p^m} = 0$, that is, $x^{p^{k+m}} = 0$. This means that x has order some power of p. Since x is arbitrary in X, X is a B_p-algebra. ■

Lemma 30. *Let H be a maximal B_p-subalgebra of a finite B-algebra X. Suppose that $x \in X$ such that the order of x is a power of p. If $H_x = H$, then $x \in H$.*

Proof. If $H_x = H$, then $x \in N(H)$. Note that $H \subseteq N(H)$. We show that no element of $N(H) \setminus H$ has order a power of p. Suppose that there exists $y \in N(H) \setminus H$ such that the order of y is a power of p. By Theorem 4(ii), H is normal in $N(H)$. Thus, $yH \in N(H)/H$. The order of yH as an element of $N(H)/H$ divides the order of y. Hence, yH has order a power of p in $N(H)/H$. Thus, the cyclic subalgebra $\langle yH \rangle_B$ of $N(H)/H$ has order a power of p and so $\langle yH \rangle_B$ is a B_p-algebra. By Corollary 16, there is a subalgebra K of $N(H)$ such that $H \subseteq K$ and $K/H = \langle yH \rangle_B$. Since $y \notin H$, $H \subset K$. By Lemma 29, K is a B_p-algebra. This contradicts that H is a maximal B_p-subalgebra of X. Therefore, no element of $N(H) \setminus H$ has order a power of p. Consequently, $x \in H$. ■

Theorem 31. *Let X be of order p^rm, where p is a prime and $(p, m) = 1$. Then any two maximal B_p-subalgebras of X are conjugate.*

Proof. Let H and K be maximal B_p-subalgebras of X and $\mathcal{S}$ be the set of all left B-cosets of H in X. Then $|\mathcal{S}|_B = [X : H]_B$. Let K act on $\mathcal{S}$ by left B-translation, that is, $k *' xH = (k * (0 * x))H$ for all $k \in K$, $xH \in \mathcal{S}$. Let $S_0 = \{xH \in \mathcal{S} : k *' xH = xH \text{ for all } k \in K\}$. By Theorem 12, $|\mathcal{S}|_B \equiv |S_0|_B$ *mod* p. Since H is a maximal B_p-subalgebra of X, $|\mathcal{S}|_B = [X : H]_B$ is not divisible by p. Thus, $|S_0|_B \neq 0$. Let $xH \in S_0$. Then $k *' xH = xH$ for all $k \in K$. Thus, $(k * (0 * x))H = xH$. By Theorem 6(i), $(0 * x) * [0 * (k * (0 * x))] \in H$ for all $k \in K$. By (P2), $(0 * x) * ((0 * x) * k) \in H$ for all $k \in K$. Hence, $K_{0*x} \subseteq H$. Since $|K_{0*x}|_B = |K|_B = |H|_B$, $K_{0*x} = H$. Therefore, H and K are conjugate. ■

Corollary 32. *Let H be a maximal B_p-subalgebra of a finite B-algebra X. Then H is a unique maximal B_p-subalgebra of X if and only if H is normal in X.*

4. Number of maximal B_p-subalgebras

Theorem 33. *Let X be of order $p^r m$, where p is a prime and $(p,m)=1$. Then the number n_p of maximal B_p-subalgebras of X is $kp+1$ for $k \in \mathbb{Z}^+$ and $n_p | p^r m$.*

Proof. Let S be the set of all maximal B_p-subalgebras of X and $H \in S$. Let H act on S by B-conjugation. Note that $Q_h \in S$ by Lemma 28(ii). Let $S_0 = \{Q \in S : h *' Q = Q \text{ for all } h \in H\}$. Then $S_0 = \{Q \in S : Q_h = Q \text{ for all } h \in H\}$. By Theorem 12, $|S|_B \equiv |S_0|_B \ mod\ p$. Since $H \in S_0$, $S_0 \neq \varnothing$. Let $Q \in S_0$. Then $Q_h = Q$ for all $h \in H$. Hence, $H \subseteq N(Q)$ and so H and Q are maximal B_p-subalgebras of $N(Q)$. By Theorem 31, $Q_h = H$ for some $h \in N(Q)$. But then $H = Q$. Thus, $S_0 = \{H\}$ and so $|S_0|_B = 1$. Hence, $|S|_B \equiv 1 \ mod\ p$ and so $|S|_B = 1 + kp$ for some integer k. Let X act on S by B-conjugation. By Theorem 31, any two maximal B_p-subalgebras are conjugate. Thus, there is only one B-orbit of S under X. Let $H \in S$. Then $X_H = \{x \in X : x *' H = H\} = \{x \in X : H_x = H\} = N(H)$. Thus, by Theorem 11, $|S|_B =$ the number of elements in the B-orbit of $H = [X : X_H]_B$. But $[X : X_H]_B$ divides $|X|_B$. Therefore, the number of maximal B_p-subalgebras of X divides $|X|_B$. ∎

Proposition 34. *Let X be of order $p^m k$, where p is prime and $(p,k)=1$. Suppose that H is a subalgebra of X of order p^m. Then H is the only maximal B_p-subalgebra of order p^m lying in $N(H)$.*

Proof. By Theorem 4(ii) and Lemma 20(ii), $H \subseteq N(H)$ and H is a maximal B_p-subalgebra of X. Thus, $|N(H)|_B = p^m r$ for some $r \leq k$ and $(p,r)=1$. Let H' be any other maximal B_p-subalgebra of X such that $H' \subseteq N(H)$. By Proposition 21, H and H' are maximal B_p-subalgebras of $N(H)$. By Theorem 31, there exists $x \in N(H)$ such that $H' = H_x$. By Theorem 4(ii) and Corollary 26, $H = H_x$. Therefore, $H' = H$ and so H is the only maximal B_p-subalgebra of order p^m lying in $N(H)$. ∎

Proposition 35. *Let X be a finite B-algebra and p a prime such that p divides $|X|_B$.*

(i) *Let K be normal in X. Then for any maximal B_p-subalgebra H of X, $H \cap K$ is a maximal B_p-subalgebra of K. Conversely, if B is any maximal B_p-subalgebra of K, then there exists a maximal B_p-subalgebra H of X such that $B = H \cap K$.*

(ii) *Let K be normal in X. If H is a maximal B_p-subalgebra of X, then HK/K is a maximal B_p-subalgebra of X/K. Conversely, any maximal B_p-subalgebra of X/K is of the form HK/K, where H is a maximal B_p-subalgebra of X.*

(iii) *Let H be normal in X. If $[X : H]_B$ and p are relatively prime, then H contains all maximal B_p-subalgebras of X.*

Proof. Since a prime p divides $|X|_B$, we may assume that $|X|_B = p^m k$, where $(p, k) = 1$.

(i) Let H be a maximal B_p-subalgebra of X. Then by Lemma 20(ii), $|H|_B = p^m$. By Theorem 7, $|H \cap K|_B$ divides $|H|_B$. Thus, $|H \cap K|_B = p^i$ for some $i \leq m$. Hence, by Theorem 14, $H \cap K$ is a B_p-algebra. Let $|K|_B = p^s t$, where $(p, t) = 1$ and $s \geq i$. Suppose that $s > i$. By Lemma 2, HK is a subalgebra of X. Thus, by Theorem 8, $|HK|_B = \frac{|H|_B|K|_B}{|H \cap K|_B} = \frac{p^m p^s t}{p^i} = p^{m+s-i}t$, where $s - i \geq 1$, a contradiction since $|X|_b = p^m k$. Hence, $s = i$ and so $|H \cap K|_B = p^s$. Therefore, by Lemma 20(ii), $H \cap K$ is a maximal B_p-subalgebra of K. Conversely, suppose that B is a maximal B_p-subalgebra of K. Let $|K|_B = p^s t$, where $(p, t) = 1$. Then by Lemma 20(ii), $|B|_B = p^s$. Now, $H \cap K$ is a maximal B_p-subalgebra of K for any maximal B_p-subalgebra H of X. By Theorem 31, Lemma 22(ii), and Corollary 26, there exists $a \in K$ such that $B = (H \cap K)_a = H_a \cap K_a = H_a \cap K$. By Lemma 28(ii), H_a is a maximal B_p-subalgebra of X.

(ii) Let H be a maximal B_p-subalgebra of X. By Lemma 2, HK is a subalgebra of X. Since K is normal in X, K is normal in HK. Thus, HK/K is well-defined. By Lemma 20(ii), $|H|_B = p^m$. Let $|K|_B = p^s t$, where $(p, t) = 1$. By (i), $H \cap K$ is a maximal B_p-sublagebra of K. Hence, $|H \cap K|_B = p^s$. Now, $|HK/K|_B = \frac{|HK|_B}{|K|_B} = \frac{|H|_B|K|_B}{|K|_B|H \cap K|_B} = \frac{|H|_B}{|H \cap K|_B} = \frac{p^m}{p^s} = p^{m-s}$. Also, $|X/K|_B = \frac{|X|_B}{|K|_B} = \frac{p^m k}{p^s t} = p^{m-s}r$. Hence, HK/K is a maximal B_p-subalgebra of X/K by Lemma 20(ii). Conversely, let B/K be a maximal B_p-subalgebra of X/K. Now, HK/K is a maximal B_p-subalgebra of X/K for any maximal B_p-subalgebra H of X. By Theorem 31, there exists $aK \in X/K$ such that $B/K = (HK/K)_{aK}$. We show that $B = H_a K$. Let $c \in HK$. Then $(a * (a * c))K = aK * (aK * cK) \in (HK/K)_{aK} = B/K$. Thus, $a * (a * c) \in B$, that is, $(HK)_a \subseteq B$. By Lemma 28(ii), H_a is a maximal B_p-subalgebra of X. By Corollaries 26 and 27, $H_a K = H_a K_a = (HK)_a \subseteq B$. Let $b \in B$. Then $bK \in B/K = (HK/K)_{aK}$. Thus, $bK = aK * (aK * iK) = a * (a * i)K$ for some $i \in HK$. Hence, by Theorem 6(i), $(0 * b) * (0 * (a * (a * i))) \in K \subseteq H_a K$. By (III), (P6), and (P2), it follows that $a * (a * i) = a * (a * (h * (0 * k))) = (a * (a * h)) * (0 * (a * (a * k))) \in H_a K$ for some $h \in H, k \in K$. Since $H_a K$ is a subalgebra of X, $0 * b \in H_a K$ and so $b \in H_a K$. Thus, $B \subseteq H_a K$. Therefore, $B = H_a K$.

(iii) Let $[X : H]_B = n$. Then $n|k$. Thus, p^m divides $|H|_B$ since $|X|_B = n|H|_B$. Hence, $|H|_B = p^m r$, where $(p, r) = 1$. Let K be a maximal B_p-subalgebra of H. By Lemma 20(ii), $|K|_B = p^m$. Hence, K is a maximal B_p-subalgebra of X. If Q is any other maximal B_p-subalgebra of X, then there exists $x \in X$ such that $Q = K_x$ by Theorem 31. Therefore, by Lemma 22(i) and Corollary 26, $Q = K_x \subseteq H_x = H$. ■

We observe that Proposition 35(iii) need not be true if H is not normal in X.

Example 36. Consider the B-algebra X in Example 19. Let $H = \{0, 4\}$. Then H is a subalgebra of X, which is not normal in X. Now, $[X : H]_B = 3$, $p = 2$ divides $|X|_B = 6$. But H does not contain all maximal B_2-subalgebras of X. The maximal B_2-subalgebras of X are $H_1 = \{0, 3\}$, $H_2 = \{0, 4\}$, and $H_3 = \{0, 5\}$.

Proposition 37. *If H is normal in a finite B-algebra X and K is a maximal B_p-subalgebra of H, then $X = HN(K)$.*

Proof. Clearly, $HN(K) \subseteq X$. Let $x \in X$. Then by Lemma 22(i) and Corollary 26, $K_x \subseteq H_x = H$. By Lemma 28(ii), K_x is a maximal B_p-subalgebra of H. By Theorem 31, there exists $h \in H$ such that $(K_x)_h = K$. By Lemma 22(iii), $K_{h*(0*x)} = K$. Thus, $h*(0*x) \in N(K)$, that is, $h*(0*x) = y$ for some $y \in N(K)$. Now, by (P1), (I), (P3), and (P2), we have $x = (0 * h) * (0 * y) \in HN(K)$. Therefore, $X = HN(K)$. ■

Corollary 38. *Let K be a maximal B_p-subalgebra of a finite B-algebra X. If H is a subalgebra of X such that $N(K) \subseteq H$, then $N(H) = H$.*

Proposition 39. *Let K be normal in a finite B-algebra X. If K is a B_p-subalgebra of X, then K is contained in every maximal B_p-subalgebra of X.*

Proof. If K is a B_p-subalgebra of X, then there exists a maximal B_p-subalgebra H of X such that $K \subseteq H$. Let Q be a maximal B_p-subalgebra of X. Then $Q = H_x$ for some $x \in X$. Therefore, by Corollary 26 and Lemma 22(i), $K = K_x \subseteq H_x = Q$. ■

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