

THE CLONE OF $K^*(n, r)$ -FULL TERMS

KHWANCHEEWA WATTANATRIPOP

Department of Mathematics, Faculty of Science
Khon Kaen University, Khon Kaen
40002, Thailand

e-mail: khwancheewa12@gmail.com

AND

THAWHAT CHANGPHAS

Department of Mathematics, Faculty of Science
Khon Kaen University, Khon Kaen
40002, Thailand
Centre of Excellence in Mathematics, CHE, Si Ayuttaya Rd.
Bangkok 10400, Thailand

e-mail: thacha@kku.ac.th

Dedicated to Professor Klaus Denecke on the occasion of 75th birthday

Abstract

Let τ_n be a type of algebras in which all operation symbols have arity n , for a fixed $n \geq 1$. For $0 < r \leq n$, this paper introduces a special kind of n -ary terms of type τ_n called $K^*(n, r)$ -full terms. The set of all $K^*(n, r)$ -full terms of type τ_n is closed under the superposition operation S^n ; hence forms a clone denoted by $\text{clone}_{K^*(n, r)}(\tau_n)$. We prove that $\text{clone}_{K^*(n, r)}(\tau_n)$ is a Menger algebra of rank n . We study $K^*(n, r)$ -full hypersubstitutions and the related $K^*(n, r)$ -full closed identities and $K^*(n, r)$ -full closed varieties. A connection between identities in $\text{clone}_{K^*(n, r)}(\tau_n)$ and $K^*(n, r)$ -full closed identities is established. The results obtained generalize the results of Denecke and Jampachon [K. Denecke and P. Jampachon, *Clones of full terms*, Algebra and Discrete Math. 4 (2004) 1–11].

Keywords: restricted range transformation, full term, clone, Menger algebra, hypersubstitution, hyperidentity, solid variety.

2010 Mathematics Subject Classification: 08B15, 08A62, 08B05.

1. THE CLONE OF $K^*(n, r)$ -FULL TERMS

Let $n \geq 1$ be a fixed. Let $\tau_n := (n_i)_{i \in I}$ be an n -ary type of algebras with operation symbols f_i indexed by some set I , each f_i has arity $n_i = n$.

Let $X_n := \{x_1, \dots, x_n\}$ be an n -elements alphabet of variables, disjoint from the set of operation symbols of type τ_n . The set of all n -ary terms of type τ_n is denoted by $W_{\tau_n}(X_n)$. For each f_i , define $\bar{f}_i : (W_{\tau_n}(X_n))^n \rightarrow W_{\tau_n}(X_n)$ by

$$\bar{f}_i(t_1, \dots, t_n) := f_i(t_1, \dots, t_n) \text{ for all } t_1, \dots, t_n \in W_{\tau_n}(X_n).$$

This forms the absolutely free algebra:

$$\mathcal{F}_{\tau_n}(X_n) := (W_{\tau_n}(X_n), (\bar{f}_i)_{i \in I}).$$

Otherwise, an operation $S^n : (W_{\tau_n}(X_n))^{n+1} \rightarrow W_{\tau_n}(X_n)$ is inductively defined by:

- (i) $S^n(x_i, t_1, \dots, t_n) = t_i$ if $x_i \in X_n$; and
- (ii) $S^n(f_i(s_1, \dots, s_n), t_1, \dots, t_n) = f_i(S^n(s_1, t_1, \dots, t_n), \dots, S^n(s_n, t_1, \dots, t_n))$.

Thus

$$(W_{\tau_n}(X_n), S^n)$$

is an algebra with $(n + 1)$ -ary operation S^n .

The *full transformation semigroup* T_n consists of the set of all maps $\alpha : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ and the usual composition of mappings. Indeed, T_n is a monoid, an identity map 1_n acts as its identity. For a fixed $0 < r \leq n$, it is well-known that the set

$$K(n, r) := \{\alpha \in T_n : |\text{ran}(\alpha)| \leq r\}$$

of all *restricted range transformations* is a subsemigroup of T_n . Hence,

$$K^*(n, r) := K(n, r) \cup \{1_n\}$$

is a submonoid of T_n . It is observed that $K^*(n, r) = K(n, r) = T_n$ if $r = n$. We inductively define n -ary $K^*(n, r)$ -full terms of type τ_n as follows:

- (i) $f_i(x_{\alpha(1)}, \dots, x_{\alpha(n)})$ is an n -ary $K^*(n, r)$ -full term of type τ_n if f_i is an n -ary operation symbol and $\alpha \in K^*(n, r)$.
- (ii) $f_i(t_1, \dots, t_n)$ is an n -ary $K^*(n, r)$ -full term of type τ_n if f_i is an n -ary operation symbol, and t_1, \dots, t_n are n -ary $K^*(n, r)$ -full terms of type τ_n .

The set of all n -ary $K^*(n, r)$ -full terms of type τ_n , closed under finite applications of (ii), is denoted by $W_{\tau_n}^{K^*(n, r)}(X_n)$.

Example 1. Let us consider a type τ_3 with I is a singleton set. Then

$$\begin{aligned} &f(x_1, x_2, x_3), f(x_1, x_1, x_1), f(x_2, x_2, x_2), f(x_3, x_3, x_3), f(x_1, x_1, x_2), f(x_1, x_2, x_1), \\ &f(x_2, x_1, x_1), f(x_1, x_2, x_2), f(x_2, x_1, x_2), f(x_2, x_2, x_1), f(x_1, x_1, x_3), f(x_1, x_3, x_1), \\ &f(x_3, x_1, x_1), f(x_1, x_3, x_3), f(x_3, x_1, x_3), f(x_3, x_3, x_1), f(x_2, x_2, x_3), f(x_2, x_3, x_2), \\ &f(x_2, x_2, x_3), f(x_2, x_3, x_3), f(x_3, x_2, x_3), f(x_3, x_3, x_2), \\ &f(f(x_1, x_1, x_1), f(x_1, x_1, x_1), f(x_3, x_3, x_2)) \in W_{\tau_3}^{K^*(3,2)}(X_3). \end{aligned}$$

But, $x_1, x_2, x_3, f(x_3, x_1, x_2)$ and $f(f(x_2, x_2, x_2), f(x_2, x_3, x_1), f(x_2, x_2, x_2))$ do not belong to $W_{\tau_3}^{K^*(3,2)}(X_3)$.

It is observed that $W_{\tau_n}^{K^*(n,r)}(X_n)$ contains no variables x_1, \dots, x_n . In [2], Denecke and Jampachon inductively defined n -ary full terms of type τ_n by:

- (i) $f_i(x_{\alpha(1)}, \dots, x_{\alpha(n)})$ is an n -ary full term of type τ_n if f_i is an n -ary operation symbol and $\alpha \in T_n$; and
- (ii) $f_i(t_1, \dots, t_n)$ is an n -ary full term of type τ_n if t_1, \dots, t_n are n -ary full terms of type τ_n and f_i is an n -ary operation symbol.

The set of all n -ary full terms of type τ_n , closed under finite application of (ii), is denoted by $W_{\tau_n}^F(X_n)$. If T_n is replaced by a submonoid $\{1_n\}$ then $W_{\tau_n}^F(X_n)$ is denoted by $W_{\tau_n}^{SF}(X_n)$, called the set of all n -ary strongly full terms of type τ_n [3]. It is observed that if $r < n$ then

$$W_{\tau_n}^{K^*(n,r)}(X_n) \subset W_{\tau_n}^F(X_n).$$

And, if $r = n$ then

$$W_{\tau_n}^{SF}(X_n) \subset W_{\tau_n}^{K^*(n,r)}(X_n) = W_{\tau_n}^F(X_n).$$

Generalizations of full terms and strongly full terms of type τ_n have been introduced and studied [7, 8]. By the definition of $K^*(n, r)$ -full terms of type τ_n we have that $(W_{\tau_n}^{K^*(n,r)}(X_n), (\bar{f}_i)_{i \in I})$ is a subalgebra of $(W_{\tau_n}(X_n), (\bar{f}_i)_{i \in I})$.

We inductively define

$$S^n : \left(W_{\tau_n}^{K^*(n,r)}(X_n) \right)^{n+1} \rightarrow W_{\tau_n}^{K^*(n,r)}(X_n)$$

by:

- (i) $S^n(f_i(x_{\alpha(1)}, \dots, x_{\alpha(n)}), t_1, \dots, t_n) := f_i(t_{\alpha(1)}, \dots, t_{\alpha(n)});$ and
- (ii) $S^n(f_i(s_1, \dots, s_n), t_1, \dots, t_n) := f_i(S^n(s_1, t_1, \dots, t_n), \dots, S^n(s_n, t_1, \dots, t_n)).$

Then we have an algebra

$$\text{clone}_{K^*(n,r)}(\tau_n) := \left(W_{\tau_n}^{K^*(n,r)}(X_n), S^n \right).$$

The following theorem follows from the fact that the algebra $(W_{\tau_n}(X_n), S^n)$ satisfies the super associative law (SASS):

$$\begin{aligned} & \tilde{S}^n(\tilde{S}^n(X_0, Y_1, \dots, Y_n), X_1, \dots, X_n) \\ & \approx \tilde{S}^n(X_0, \tilde{S}^n(Y_1, X_1, \dots, X_n), \dots, \tilde{S}^n(Y_n, X_1, \dots, X_n)). \end{aligned}$$

(Here, \tilde{S}^n is an $(n+1)$ -ary operation symbol and X_i, Y_j are variables). However, we will prove the result directly.

Theorem 2. *The algebra clone $_{K^*(n,r)}(\tau_n)$ satisfies the previous super associative law (SASS).*

Proof. We will proceed by induction on the complexity of the $K^*(n, r)$ -full term which is substituted for X_0 . Firstly, we have

$$\begin{aligned} & S^n(S^n(f_i(x_{\alpha(1)}, \dots, x_{\alpha(n)}), s_1, \dots, s_n), t_1, \dots, t_n) \\ & = S^n(f_i(s_{\alpha(1)}, \dots, s_{\alpha(n)}), t_1, \dots, t_n) \\ & = f_i(S^n(s_{\alpha(1)}, t_1, \dots, t_n), \dots, S^n(s_{\alpha(n)}, t_1, \dots, t_n)) \\ & = S^n(f_i(x_{\alpha(1)}, \dots, x_{\alpha(n)}), S^n(s_1, t_1, \dots, t_n), \dots, S^n(s_n, t_1, \dots, t_n)). \end{aligned}$$

Consider $f_i(r_1, \dots, r_n) \in W_{\tau_n}^{K^*(n,r)}(X_n)$ such that r_1, \dots, r_n satisfy (SASS). Then

$$\begin{aligned} & S^n(S^n(f_i(r_1, \dots, r_n), s_1, \dots, s_n), t_1, \dots, t_n) \\ & = S^n(f_i(S^n(r_1, s_1, \dots, s_n), \dots, S^n(r_n, s_1, \dots, s_n)), t_1, \dots, t_n) \\ & = f_i(S^n(S^n(r_1, s_1, \dots, s_n), t_1, \dots, t_n), \dots, S^n(S^n(r_n, s_1, \dots, s_n), t_1, \dots, t_n)) \\ & = f_i(S^n(r_1, S^n(s_1, t_1, \dots, t_n), \dots, S^n(s_n, t_1, \dots, t_n)), \dots, \\ & \quad S^n(r_n, S^n(s_1, t_1, \dots, t_n), \dots, S^n(s_n, t_1, \dots, t_n))) \\ & = S^n(f_i(r_1, \dots, r_n), S^n(s_1, t_1, \dots, t_n), \dots, S^n(s_n, t_1, \dots, t_n)). \end{aligned}$$

Hence the proof is completed. ■

An algebra $\mathcal{M} := (M, S^n)$ of type $\tau = (n+1)$ is called a *Menger algebra* of rank n if \mathcal{M} satisfies the condition (SASS) [4]. Then, by Theorem 2, $clone_{K^*(n,r)}(\tau_n)$ is an example of a Menger algebra of rank n . It is observed that $clone_{K^*(n,r)}(\tau_n)$ is generated by

$$F_{W_{\tau_n}^{K^*(n,r)}(X_n)} := \{f_i(x_{\alpha(1)}, \dots, x_{\alpha(n)}) \mid i \in I, \alpha \in K^*(n, r)\}.$$

Let V_{Menger} be the variety of all Menger algebras of type $(n+1)$, and let $\mathcal{F}_{V_{Menger}}(Y)$ be the free algebra with respect to V_{Menger} , freely generated by $Y := \{y_j \mid j \in J\}$ where Y is an alphabet of variables indexed by the set $J := \{(i, \alpha) \mid i \in I, \alpha \in K^*(n, r)\}$. The operation of $\mathcal{F}_{V_{Menger}}(Y)$ will be denoted by \tilde{S}^n . We can state and prove the following theorem.

Theorem 3. *The algebra clone $clone_{K^*(n,r)}(\tau_n)$ is a free algebra of the variety V_{Menger} of Menger algebras of rank n , freely generated by the set $Y = \{y_{(i,\alpha)} \mid i \in I, \alpha \in K^*(n, r)\}$.*

Proof. We claim that $clone_{K^*(n,r)}(\tau_n)$ is isomorphic to $\mathcal{F}_{V_{Menger}}(Y)$. Define

$$\varphi : W_{\tau_n}^{K^*(n,r)}(X_n) \rightarrow F_{V_{Menger}}(Y)$$

by:

- (i) $\varphi(f_i(x_{\alpha(1)}, \dots, x_{\alpha(n)})) := y_{(i,\alpha)}$.
- (ii) $\varphi(S^n(f_i(x_{\alpha(1)}, \dots, x_{\alpha(n)}), t_1, \dots, t_n)) := \tilde{S}^n(y_{(i,\alpha)}, \varphi(t_1), \dots, \varphi(t_n))$.

We show that φ is a homomorphism:

$$\varphi(S^n(t, t_1, \dots, t_n)) = \tilde{S}^n(\varphi(t), \varphi(t_1), \dots, \varphi(t_n))$$

for all $t, t_1, \dots, t_n \in W_{\tau_n}^{K^*(n,r)}(X_n)$. Indeed: if $t = f_i(x_{\alpha(1)}, \dots, x_{\alpha(n)})$, then

$$\begin{aligned} \varphi(S^n(t, t_1, \dots, t_n)) &= \varphi(S^n(f_i(x_{\alpha(1)}, \dots, x_{\alpha(n)}), t_1, \dots, t_n)) \\ &= \varphi(f_i(t_{\alpha(1)}, \dots, t_{\alpha(n)})) \\ &= \tilde{S}^n(y_{(i,\alpha)}, \varphi(t_1), \dots, \varphi(t_n)) \\ &= \tilde{S}^n(\varphi(f_i(x_{\alpha(1)}, \dots, x_{\alpha(n)})), \varphi(t_1), \dots, \varphi(t_n)) \\ &= \tilde{S}^n(\varphi(t), \varphi(t_1), \dots, \varphi(t_n)). \end{aligned}$$

Consider $t = f_i(r_1, \dots, r_n)$ and assume that, for $1 \leq k \leq n$,

$$\varphi(S^n(r_k, t_1, \dots, t_n)) = \tilde{S}^n(\varphi(r_k), \varphi(t_1), \dots, \varphi(t_n)).$$

Using the fact that

$$\varphi(f_i(t'_1, \dots, t'_n)) = \tilde{S}^n(y_{(i,1_n)}, \varphi(t'_1), \dots, \varphi(t'_n))$$

for all $t'_1, \dots, t'_n \in W_{\tau_n}^{K^*(n,r)}(X_n)$, we then have

$$\begin{aligned} &\varphi(S^n(t, t_1, \dots, t_n)) \\ &= \varphi(S^n(f_i(r_1, \dots, r_n), t_1, \dots, t_n)) \\ &= \varphi(f_i(S^n(r_1, t_1, \dots, t_n), \dots, S^n(r_n, t_1, \dots, t_n))) \\ &= \tilde{S}^n(y_{(i,1_n)}, \varphi(S^n(r_1, t_1, \dots, t_n)), \dots, \varphi(S^n(r_n, t_1, \dots, t_n))) \\ &= \tilde{S}^n(y_{(i,1_n)}, \tilde{S}^n(\varphi(r_1), \varphi(t_1), \dots, \varphi(t_n)), \dots, \tilde{S}^n(\varphi(r_n), \varphi(t_1), \dots, \varphi(t_n))) \\ &= \tilde{S}^n(\tilde{S}^n(y_{(i,1_n)}, \varphi(r_1), \dots, \varphi(r_n)), \varphi(t_1), \dots, \varphi(t_n)) \\ &= \tilde{S}^n(\varphi(t), \varphi(t_1), \dots, \varphi(t_n)). \end{aligned}$$

Indeed, φ is a bijection. To see this:

$$y_{(i,\alpha)} = y_{(j,\beta)} \Rightarrow (i, \alpha) = (j, \beta) \Rightarrow f_i(x_{\alpha(1)}, \dots, x_{\alpha(n)}) = f_j(x_{\beta(1)}, \dots, x_{\beta(n)}).$$

And,

$$y_{(i,\alpha)} \in Y \Rightarrow \varphi(f_i(x_{\alpha(1)}, \dots, x_{\alpha(n)})) = y_{(i,\alpha)}.$$

Hence φ is an isomorphism. ■

2. $K^*(n, r)$ -FULL HYPERSUBSTITUTIONS

A *hypersubstitution* of type τ is a map $\sigma : \{f_i \mid i \in I\} \rightarrow W_\tau(X)$ which maps each operation symbol f_i to an n_i -ary term $\sigma(f_i)$ of type τ . Any hypersubstitution $\sigma : \{f_i \mid i \in I\} \rightarrow W_\tau(X)$ can be uniquely extended to a map $\widehat{\sigma} : W_\tau(X) \rightarrow W_\tau(X)$ as follows:

- (i) $\widehat{\sigma}[t] = t$ if $t \in X$; and
- (ii) $\widehat{\sigma}[t] = \sigma(f_i)(\widehat{\sigma}[t_1], \dots, \widehat{\sigma}[t_{n_i}])$ if $t = f_i(t_1, \dots, t_{n_i})$.

The set $Hyp(\tau)$ of all hypersubstitutions of type τ forms a monoid under the associative operation \circ_h :

$$(\sigma_1 \circ_h \sigma_2)(f_i) := \widehat{\sigma}_1[\sigma_2(f_i)].$$

The identity is $\sigma : \{f_i \mid i \in I\} \rightarrow W_\tau(X)$ such that $\sigma(f_i) = f_i(x_1, \dots, x_{n_i})$, see [5].

Now, we call mappings

$$\sigma : \{f_i \mid i \in I\} \rightarrow W_{\tau_n}^{K^*(n,r)}(X_n)$$

$K^*(n, r)$ -full hypersubstitutions of type τ_n .

To define an extension of a given $K^*(n, r)$ -full hypersubstitution of type τ_n , we need the following. For $t \in W_{\tau_n}^{K^*(n,r)}(X_n)$ and $\beta \in K^*(n, r)$, define

- (i) $t_\beta := f_i(x_{\beta(\alpha(1))}, \dots, x_{\beta(\alpha(n))})$ if $t = f_i(x_{\alpha(1)}, \dots, x_{\alpha(n)})$.
- (ii) $t_\beta := f_i((t_1)_\beta, \dots, (t_n)_\beta)$ if $t = f_i(t_1, \dots, t_n)$.

It is observed that if t is an $K^*(n, r)$ -full term of type τ_n , then t_β is an $K^*(n, r)$ -full term of type τ_n for all $\beta \in K^*(n, r)$. Then an $K^*(n, r)$ -full hypersubstitution $\sigma : \{f_i \mid i \in I\} \rightarrow W_{\tau_n}^{K^*(n,r)}(X_n)$ of type τ_n can be extended to a mapping

$$\widehat{\sigma} : W_{\tau_n}^{K^*(n,r)}(X_n) \rightarrow W_{\tau_n}^{K^*(n,r)}(X_n)$$

as follows:

- (i) $\widehat{\sigma}[f_i(x_{\alpha(1)}, \dots, x_{\alpha(n)})] := (\sigma(f_i))_\alpha$.
- (ii) $\widehat{\sigma}[f_i(t_1, \dots, t_n)] := S^n(\sigma(f_i), \widehat{\sigma}[t_1], \dots, \widehat{\sigma}[t_n])$.

The set of all $K^*(n, r)$ -full hypersubstitutions of type τ_n will be denoted by $Hyp^{K^*(n, r)}(\tau_n)$. It is easy to see that $(Hyp^{K^*(n, r)}(\tau_n), \circ_h)$ is a submonoid of $(Hyp(\tau_n), \circ_h)$.

Lemma 4. *Let $t, t_1, \dots, t_n \in W_{\tau_n}^{K^*(n, r)}(X_n)$. Then*

$$S^n(t, \widehat{\sigma}[t_{\alpha(1)}], \dots, \widehat{\sigma}[t_{\alpha(n)}]) = S^n(t_\alpha, \widehat{\sigma}[t_1], \dots, \widehat{\sigma}[t_n])$$

for all $\alpha \in K^*(n, r)$.

Proof. If $t = f_i(x_{\beta(1)}, \dots, x_{\beta(n)})$ where $\beta \in K^*(n, r)$, then, for $\alpha \in K^*(n, r)$, we have

$$\begin{aligned} S^n(t, \widehat{\sigma}[t_{\alpha(1)}], \dots, \widehat{\sigma}[t_{\alpha(n)}]) &= S^n(f_i(x_{\beta(1)}, \dots, x_{\beta(n)}), \widehat{\sigma}[t_{\alpha(1)}], \dots, \widehat{\sigma}[t_{\alpha(n)}]) \\ &= f_i(\widehat{\sigma}[t_{\alpha(\beta(1))}], \dots, \widehat{\sigma}[t_{\alpha(\beta(n))}]) \\ &= S^n(f_i(x_{\alpha(\beta(1))}, \dots, x_{\alpha(\beta(n))}), \widehat{\sigma}[t_1], \dots, \widehat{\sigma}[t_n]) \\ &= S^n(t_\alpha, \widehat{\sigma}[t_1], \dots, \widehat{\sigma}[t_n]). \end{aligned}$$

Let $t = f_i(s_1, \dots, s_n)$ and assume that

$$S^n(s_k, \widehat{\sigma}[t_{\alpha(1)}], \dots, \widehat{\sigma}[t_{\alpha(n)}]) = S^n((s_k)_\alpha, \widehat{\sigma}[t_1], \dots, \widehat{\sigma}[t_n])$$

for all $\alpha \in K^*(n, r)$. Then, for $\alpha \in K^*(n, r)$, we have

$$\begin{aligned} &S^n(t, \widehat{\sigma}[t_{\alpha(1)}], \dots, \widehat{\sigma}[t_{\alpha(n)}]) \\ &= S^n(f_i(s_1, \dots, s_n), \widehat{\sigma}[t_{\alpha(1)}], \dots, \widehat{\sigma}[t_{\alpha(n)}]) \\ &= f_i(S^n(s_1, \widehat{\sigma}[t_{\alpha(1)}], \dots, \widehat{\sigma}[t_{\alpha(n)}]), \dots, S^n(s_n, \widehat{\sigma}[t_{\alpha(1)}], \dots, \widehat{\sigma}[t_{\alpha(n)}])) \\ &= f_i(S^n((s_1)_\alpha, \widehat{\sigma}[t_1], \dots, \widehat{\sigma}[t_n]), \dots, S^n((s_n)_\alpha, \widehat{\sigma}[t_1], \dots, \widehat{\sigma}[t_n])) \\ &= S^n(f_i((s_1)_\alpha, \dots, (s_n)_\alpha), \widehat{\sigma}[t_1], \dots, \widehat{\sigma}[t_n]) \\ &= S^n(t_\alpha, \widehat{\sigma}[t_1], \dots, \widehat{\sigma}[t_n]). \end{aligned}$$

■

Using Lemma 4 we can prove the following.

Theorem 5. *For $\sigma \in Hyp^{K^*(n, r)}(\tau_n)$, the extension*

$$\widehat{\sigma} : W_{\tau_n}^{K^*(n, r)}(X_n) \rightarrow W_{\tau_n}^{K^*(n, r)}(X_n)$$

is an endomorphism on the algebra clone $_{K^*(n, r)}(\tau_n)$.

Proof. It is clear that $\widehat{\sigma} : W_{\tau_n}^{K^*(n,r)}(X_n) \rightarrow W_{\tau_n}^{K^*(n,r)}(X_n)$. Let $t_0, t_1, \dots, t_n \in W_{\tau_n}^{K^*(n,r)}(X_n)$. We will show by induction on the complexity of t_0 that

$$\widehat{\sigma}(S^n(t_0, t_1, \dots, t_n)) = S^n(\widehat{\sigma}[t_0], \widehat{\sigma}[t_1], \dots, \widehat{\sigma}[t_n]).$$

If $t_0 = f_i(x_{\alpha(1)}, \dots, x_{\alpha(n)})$, then

$$\begin{aligned} \widehat{\sigma}(S^n(t_0, t_1, \dots, t_n)) &= \widehat{\sigma}(S^n(f_i(x_{\alpha(1)}, \dots, x_{\alpha(n)}), t_1, \dots, t_n)) \\ &= \widehat{\sigma}(f_i(t_{\alpha(1)}, \dots, t_{\alpha(n)})) \\ &= S^n(\sigma(f_i), \widehat{\sigma}[t_{\alpha(1)}], \dots, \widehat{\sigma}[t_{\alpha(n)}]) \\ &= S^n(\widehat{\sigma}[t_0], \widehat{\sigma}[t_1], \dots, \widehat{\sigma}[t_n]). \end{aligned}$$

Assume $t_0 = f_i(r_{\alpha(1)}, \dots, r_{\alpha(n)})$ such that

$$\widehat{\sigma}(S^n(r_{\alpha(k)}, t_1, \dots, t_n)) = S^n(\widehat{\sigma}[r_{\alpha(k)}], \widehat{\sigma}[t_1], \dots, \widehat{\sigma}[t_n])$$

for all $1 \leq k \leq n$. Then

$$\begin{aligned} &\widehat{\sigma}(S^n(t_0, t_1, \dots, t_n)) \\ &= \widehat{\sigma}(S^n(f_i(r_{\alpha(1)}, \dots, r_{\alpha(n)}), t_1, \dots, t_n)) \\ &= \widehat{\sigma}(f_i(S^n(r_{\alpha(1)}, t_1, \dots, t_n), \dots, S^n(r_{\alpha(n)}, t_1, \dots, t_n))) \\ &= S^n(\sigma(f_i), \widehat{\sigma}[S^n(r_{\alpha(1)}, t_1, \dots, t_n)], \dots, \widehat{\sigma}[S^n(r_{\alpha(n)}, t_1, \dots, t_n)]) \\ &= S^n(\sigma(f_i), S^n(\widehat{\sigma}[r_{\alpha(1)}], \widehat{\sigma}[t_1], \dots, \widehat{\sigma}[t_n]), \dots, S^n(\widehat{\sigma}[r_{\alpha(n)}], \widehat{\sigma}[t_1], \dots, \widehat{\sigma}[t_n])) \\ &= S^n(S^n(\sigma(f_i), \widehat{\sigma}[r_{\alpha(1)}], \dots, \widehat{\sigma}[r_{\alpha(n)}]), \widehat{\sigma}[t_1], \dots, \widehat{\sigma}[t_n]) \\ &= S^n(\widehat{\sigma}[t_0], \widehat{\sigma}[t_1], \dots, \widehat{\sigma}[t_n]). \end{aligned} \quad \blacksquare$$

As mentioned, the algebra $clone_{K^*(n,r)}(\tau_n)$ is generated by the set

$$F_{W_{\tau_n}^{K^*(n,r)}(X_n)} := \{f_i(x_{\alpha(1)}, \dots, x_{\alpha(n)}) \mid i \in I, \alpha \in K^*(n, r)\}.$$

Thus, any mapping

$$\eta : F_{W_{\tau_n}^{K^*(n,r)}(X_n)} \rightarrow W_{\tau_n}^{K^*(n,r)}(X_n)$$

called $K^*(n, r)$ -full clone substitution, can be uniquely extended to an endomorphism

$$\bar{\eta} : W_{\tau_n}^{K^*(n,r)}(X_n) \rightarrow W_{\tau_n}^{K^*(n,r)}(X_n)$$

Denoted by $Subst_{K^*(n,r)}(\tau_n)$ the set of all such $K^*(n, r)$ -full clone substitutions.

For $\eta_1, \eta_2 \in \text{Subst}_{K^*(n,r)}(\tau_n)$, define

$$\eta_1 \odot \eta_2 := \bar{\eta}_1 \circ \eta_2$$

where \circ is the usual composition. Consider $\sigma \in \text{Hyp}^{K^*(n,r)}(\tau_n)$. By Theorem 5, $\widehat{\sigma} : W_{\tau_n}^{K^*(n,r)}(X_n) \rightarrow W_{\tau_n}^{K^*(n,r)}(X_n)$ is an endomorphism. Since $F_{W_{\tau_n}^{K^*(n,r)}(X_n)}$ generates $\text{clone}_{K^*(n,r)}(\tau_n)$, we have $\widehat{\sigma}|_{F_{W_{\tau_n}^{K^*(n,r)}(X_n)}}$ is a $K^*(n, r)$ -full clone substitution with

$$\overline{\widehat{\sigma}|_{F_{W_{\tau_n}^{K^*(n,r)}(X_n)}}} = \widehat{\sigma}.$$

Define a mapping $\psi : \text{Hyp}^{K^*(n,r)}(\tau_n) \rightarrow \text{Subst}_{K^*(n,r)}(\tau_n)$ by

$$\psi(\sigma) = \widehat{\sigma}|_{F_{W_{\tau_n}^{K^*(n,r)}(X_n)}}.$$

We have that ψ is a homomorphism. In fact: Let $\sigma_1, \sigma_2 \in \text{Hyp}^{K^*(n,r)}(\tau_n)$. Then

$$\begin{aligned} \psi(\sigma_1 \circ_h \sigma_2) &= (\widehat{\sigma}_1 \circ \widehat{\sigma}_2)|_{F_{W_{\tau_n}^{K^*(n,r)}(X_n)}} \\ &= \overline{\widehat{\sigma}_1|_{F_{W_{\tau_n}^{K^*(n,r)}(X_n)}}} \circ \widehat{\sigma}_2|_{F_{W_{\tau_n}^{K^*(n,r)}(X_n)}} \\ &= \overline{\psi(\sigma_1)} \circ \psi(\sigma_2) \\ &= \psi(\sigma_1) \odot \psi(\sigma_2). \end{aligned}$$

Clearly, ψ is injection. Hence we have the following corollary.

Corollary 6. $(\text{Hyp}^{K^*(n,r)}(\tau_n), \circ_h)$ can be embedded into $(\text{Subst}_{K^*(n,r)}(\tau_n, \odot))$.

3. $K^*(n, r)$ -FULL CLOSURE

Let V be a variety of type τ_n , and let $\text{Id}V$ be the set of all identities of V . Let $\text{Id}^{K^*(n,r)}V$ be the set of all identities $s \approx t$ of V such that s and t are both $K^*(n, r)$ -full terms of type τ_n ; that is

$$\text{Id}^{K^*(n,r)}V := \left(W_{\tau_n}^{K^*(n,r)}(X_n) \right)^2 \cap \text{Id}V.$$

It is well-known that $\text{Id}V$ is a congruence on the free algebra $\mathcal{F}_{\tau_n}(X_n)$. However, in general, this is not true for $\text{Id}^{K^*(n,r)}V$. The following theorem shows that $\text{Id}^{K^*(n,r)}V$ is a congruence on $\text{clone}_{K^*(n,r)}(\tau_n)$.

Theorem 7. Let V be a variety of type τ_n . Then $\text{Id}^{K^*(n,r)}V$ is a congruence on the clone $\text{clone}_{K^*(n,r)}(\tau_n)$.

Proof. Assume that

$$r \approx t, r_1 \approx t_1, \dots, r_n \approx t_n \in Id^{K^*(n,r)}V.$$

We will show that

$$S^n(r, r_1, \dots, r_n) \approx S^n(t, t_1, \dots, t_n) \in Id^{K^*(n,r)}V.$$

To prove the assertion we claim that

$$S^n(r, r_1, \dots, r_n) \approx S^n(r, t_1, \dots, t_n) \in Id^{K^*(n,r)}V.$$

Assume $r = f_i(x_{\alpha(1)}, \dots, x_{\alpha(n)})$ for some $\alpha \in K^*(n, r)$. It is known that IdV is compatible with the operations \bar{f}_i of the absolutely free algebra $\mathcal{F}_{\tau_n}(X_n)$. By the definition of $K^*(n, r)$ -full terms, we have

$$f_i(r_{\alpha(1)}, \dots, r_{\alpha(n)}) \approx f_i(t_{\alpha(1)}, \dots, t_{\alpha(n)}) \in Id^{K^*(n,r)}V.$$

This means

$$\begin{aligned} & S^n(f_i(x_{\alpha(1)}, \dots, x_{\alpha(n)}), r_1, \dots, r_n) \\ & \approx S^n(f_i(x_{\alpha(1)}, \dots, x_{\alpha(n)}), t_1, \dots, t_n) \in Id^{K^*(n,r)}V. \end{aligned}$$

Moreover,

$$S^n(r, r_1, \dots, r_n) \approx S^n(r, t_1, \dots, t_n) \in Id^{K^*(n,r)}V.$$

Assume that $r = f_i(s_1, \dots, s_n)$ such that for all $1 \leq k \leq n$,

$$S^n(s_k, r_1, \dots, r_n) \approx S^n(s_k, t_1, \dots, t_n) \in Id^{K^*(n,r)}V.$$

Thus

$$\begin{aligned} & f_i(S^n(s_1, r_1, \dots, r_n), \dots, S^n(s_n, r_1, \dots, r_n)) \\ & \approx f_i(S^n(s_1, t_1, \dots, t_n), \dots, S^n(s_n, t_1, \dots, t_n)) \in Id^{K^*(n,r)}V. \end{aligned}$$

Further,

$$S^n(f_i(s_1, \dots, s_n), r_1, \dots, r_n) \approx S^n(f_i(s_1, \dots, s_n), t_1, \dots, t_n) \in Id^{K^*(n,r)}V.$$

This means

$$S^n(r, r_1, \dots, r_n) \approx S^n(r, t_1, \dots, t_n) \in Id^{K^*(n,r)}V.$$

So we proved the claim.

By

$$S^n(r, t_1, \dots, t_n) \approx S^n(t, t_1, \dots, t_n) \in Id^{K^*(n,r)}V$$

it follows that

$$S^n(r, r_1, \dots, r_n) \approx S^n(r, t_1, \dots, t_n) \approx S^n(t, t_1, \dots, t_n) \in Id^{K^*(n,r)}V. \quad \blacksquare$$

Let V be a variety of type τ_n . An identity $s \approx t \in Id^{K^*(n,r)}V$ is called a $K^*(n, r)$ -full closed identity of V if

$$\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV \quad \text{for all } \sigma \in Hyp^{K^*(n,r)}(\tau_n).$$

And, V is called $K^*(n, r)$ -full closed if the following hold:

$$\forall s \approx t \in Id^{K^*(n,r)}V \quad \forall \sigma \in Hyp^{K^*(n,r)}(\tau_n), \quad \hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV.$$

Then we can prove the following lemma.

Lemma 8. *Let V be a variety of type τ_n . If $Id^{K^*(n,r)}V$ is a fully invariant congruence on $clone_{K^*(n,r)}(\tau_n)$ then V is $K^*(n, r)$ -full closed.*

Proof. Assume that $Id^{K^*(n,r)}V$ is a fully invariant congruence on $clone_{K^*(n,r)}(\tau_n)$. Let $s \approx t \in Id^{K^*(n,r)}V$ and $\sigma \in Hyp^{K^*(n,r)}(\tau_n)$. By Theorem 5, $\hat{\sigma}$ is an endomorphism of $clone_{K^*(n,r)}(\tau_n)$. Hence $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV$, that is V is $K^*(n, r)$ -full closed. \blacksquare

For a variety V of type τ_n , $Id^{K^*(n,r)}V$ is a congruence on $clone_{K^*(n,r)}(\tau_n)$ by Theorem 7. We then form the quotient algebra

$$clone_{K^*(n,r)}(V) := clone_{K^*(n,r)}(\tau_n) / Id^{K^*(n,r)}V.$$

The quotient algebra obtained belongs to V_{Menger} . Note that we have a natural homomorphism

$$nat_{Id^{K^*(n,r)}V} : clone_{K^*(n,r)}(\tau_n) \rightarrow clone_{K^*(n,r)}(\tau_n)(V)$$

such that

$$nat_{Id^{K^*(n,r)}V}(t) = [t]_{Id^{K^*(n,r)}V}.$$

Finally, we prove the following theorem.

Theorem 9. *Let V be a variety of type τ_n . If $s \approx t \in Id^{K^*(n,r)}V$ is an identity in $clone_{K^*(n,r)}(V)$, then $s \approx t$ is $K^*(n, r)$ -full closed identity of V .*

Proof. Assume that $s \approx t \in Id^{K^*(n,r)}V$ is an identity in $clone_{K^*(n,r)}(V)$. Let $\sigma \in Hyp^{K^*(n,r)}(\tau_n)$; then $\hat{\sigma} : clone_{K^*(n,r)}(\tau_n) \rightarrow clone_{K^*(n,r)}(\tau_n)$ is an endomorphism by Theorem 5. Thus

$$nat_{Id^{K^*(n,r)}V} \circ \hat{\sigma} : clone_{K^*(n,r)}(\tau_n) \rightarrow clone_{K^*(n,r)}(V)$$

is a homomorphism. By assumption,

$$\text{nat}_{Id^{K^*(n,r)}V} \circ \widehat{\sigma}(s) = \text{nat}_{Id^{K^*(n,r)}V} \circ \widehat{\sigma}(t).$$

That is

$$\text{nat}_{Id^{K^*(n,r)}V}(\widehat{\sigma}[s]) = \text{nat}_{Id^{K^*(n,r)}V}(\widehat{\sigma}[t]).$$

Thus

$$[\widehat{\sigma}[s]]_{Id^{K^*(n,r)}V} = [\widehat{\sigma}[t]]_{Id^{K^*(n,r)}V},$$

and hence

$$\widehat{\sigma}[s] \approx \widehat{\sigma}[t] \in Id^{K^*(n,r)}V.$$

Hence $s \approx t$ is a $K^*(n, r)$ -full closed identity of V . ■

REFERENCES

- [1] K. Denecke, *Menger algebras and clones of terms*, East-West J. Math. **5** (2003) 179–193.
- [2] K. Denecke and P. Jampaclon, *Clones of full terms*, Algebra and Discrete Math. **4** (2004) 1–11.
- [3] K. Denecke and L. Freiberg, *The algebra of strongly full terms*, Novi Sad J. Math. **34** (2004) 87–98.
- [4] B.M. Schein and V.S. Trochimenko, *Algebras of multiplace functions*, Semigroup Forum **17** (1979) 1–64.
doi:10.1007/BF02194309
- [5] K. Denecke and S.L. Wismath, *Hyperidentities and Clones* (Gordon and Breach Science Publisher, 2000).
doi:10.1201/9781482287516
- [6] J.M. Howie and R.B. McFadden, *Idempotent rank in finite full transformation semi-groups*, Proc. Roy. Soc. Edinb. Sect. A **114** (1990) 161–167.
doi:10.1017/S0308210500024355
- [7] S. Puapong and S. Leeratanavalee, *The algebra of generalized full terms*, Int. J. Open Problems Compt. Math. **4** (2011) 54–65.
doi:10.1155/2013/396464
- [8] P. Sarawut, *Some algebraic properties of generalized clone automorphisms*, Acta Universitatis Apulensis **41** (2015) 165–175.
doi:10.17114/j.aaa.2015.41.13

Received 11 June 2019

Revised 18 July 2019

Accepted 18 July 2019