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MONOIDS OF ND-FULL HYPERSUBSTITUTIONS

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Abstract

An nd-full hypersubstitution maps any operation symbols to the set of full terms of type τ_n. Nd-full hypersubstitutions can be extended to mappings which map sets of full terms to sets of full terms. The aims of this paper are to show that the extension of an nd-full hypersubstitution is an endomorphism of some clone and that the set of all nd-full hypersubstitutions forms a monoid.

Keywords: superposition of sets of terms, clone, hypersubstitution.

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Dedicated to Professor Klaus Denecke on the occasion of his 75th birthday

1. Introduction

Now we consider algebras of n-ary type, that is, all operation symbols have the same fixed arity n. Let $\tau_n := (n_i)_{i\in I}$ be a fixed type where $n_i = n$ for all $i \in I$ with operation symbols $(f_i)_{i\in I}$ indexed by some set I.

Definition [2]**.** Let H_n be the set of all permutations $s : \{1, \dots, n\} \to \{1, \dots, n\}$ and let f_i be an operation symbol of type τ_n. Full terms of type τ_n are defined in the following way:

(1) $f_i(x_{s(1)}, \dots, x_{s(n)})$ is a full term of type τ_n.

(2) If $t_1, \dots, t_n$ are full terms of type τ_n, then $f_i(t_1, \dots, t_n)$ is a full term of type τ_n.

The set of all full terms of type τ_n is denoted by $W^F_{\tau_n}(X_n)$.

Example 1. Let $\tau_2 := (2,2)$ and let $s : \{1,2\} \to \{1,2\}$ and $r : \{1,2\} \to \{1,2\}$ which are defined by $s(1) = 2, s(2) = 1$ and $r(1) = 1, r(2) = 2$. Then $g(x_{s(1)}, x_{s(2)}), f(x_{r(1)}, x_{r(2)})$ and $f(g(x_{s(1)}, x_{s(2)}), f(x_{r(1)}, x_{r(2)}))$ are full terms of type τ_2.

Definition [3]**.** Let $W^F_{\tau_n}(X_n)$ be a set of full terms of type τ_n. Then the superposition operations

$$S^n : \left(W^F_{\tau_n}(X_n)\right)^{n+1} \to W^F_{\tau_n}(X_n),$$

are defined in the following way. For $t, t_q \in W^F_{\tau_n}(X_n), 1 \leq q \leq n, n \in \mathbb{N}$;

(1) if $t = f_i(x_{s(1)}, \ldots, x_{s(n)})$ where $s \in H_n$, then $S^n(f_i(x_{s(1)}, \ldots, x_{s(n)}), t_1, \ldots, t_n) := f_i(t_{s(1)}, \ldots, t_{s(n)})$,

(2) if $t = f_i(s_1, \ldots, s_n)$ and if we assume that $S^n(s_q, t_1, \ldots, t_n)$ are already defined, then $S^n(f_i(s_1, \ldots, s_n), t_1, \ldots, t_n) := f_i(S^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_n, t_1, \ldots, t_n))$.

For a full term t we need the full term t_s arising from t by replacement a variable x_i in t by a variable $x_{s(i)}$ for a mapping $s \in H_n$. This can be defined as follows.

Definition [3]**.** Let t be a full term in $W^F_{\tau_n}(X_n)$ and let $s, r \in H_n$. We define the full term t_s in the following step:

(1) If $t = f_i(x_{r(1)}, \ldots, x_{r(n)})$, then $t_s := f_i(x_{s(r(1))}, \ldots, x_{s(r(n))})$.

(2) If $t = f_i(t_{r(1)}, \ldots, t_{r(1)})$, then $t_s := f_i(t_{s(r(1))}, \ldots, t_{s(r(1))})$.

Example 2. Let $\tau_2 := (2,2)$ and let $s : \{1,2\} \to \{1,2\}$ and $r : \{1,2\} \to \{1,2\}$ which are defined by $s(1) = 2, s(2) = 1$ and $r(1) = 1, r(2) = 2$. Let $t = f(g(x_{s(1)}, x_{s(2)}), f(x_{r(1)}, x_{r(2)}))$. Then

$$\begin{aligned} t_s &= (f(g(x_{s(1)}, x_{s(2)}), f(x_{r(1)}, x_{r(2)})))_s \\ &= (f(g(x_2, x_1), f(x_1, x_2)))_s \\ &= f(g(x_{s(2)}, x_{s(1)}), f(x_{s(1)}, x_{s(2)})) \\ &= f(g(x_1, x_2), f(x_2, x_1)). \end{aligned}$$

Let $\mathcal{P}(W^F_{\tau_n}(X_n))$ be a set of all subsets of $W^F_{\tau_n}(X_n)$. Let $T = \{t \mid t \in W^F_{\tau_n}(X_n)\}$ and let $s \in H_n$. Then we set $T_s := \{t_s \mid t \in W^F_{\tau_n}(X_n)\}$ and $T_s := \emptyset$ if $T = \emptyset$.

2. Superposition operations of sets of full terms

Let us consider the following superposition operation

$$\dot{S}^n : \left(\mathcal{P}(W^F_{\tau_n}(X_n))\right)^{n+1} \to \mathcal{P}\left(W^F_{\tau_n}(X_n)\right),$$

which is defined by $\dot{S}^n(T, T_1, \ldots, T_n) := \{S^n(t, t_1, \ldots, t_n) \mid t \in T, t_q \in T_q, 1 \le q \le n, n \in \mathbb{N}\}$, where $T, T_q \subseteq W^F_{\tau_n}(X_n)$. Such superposition operation does not satisfy the superassociative law, as the following example.

Example 3. Let $\tau_2 = (2,2,2,2,2,2)$, $X_2 = \{x_1, x_2\}$ and $W^F_{\tau_2}(X_2)$ be a set of all full terms of type τ_2. Let $T, T_q, S_q \subseteq W^F_{\tau_2}(X_2), 1 \le q \le 2$ where $T = \{f(x_1,x_2)\}$, $T_1 = \{g_1(x_1,x_2)\}, T_2 = \{g_2(x_2,x_1)\}$, $S_1 = \{h_1(x_2,x_1)\}$ and $S_2 = \{h_2(x_1,x_2), h_3(x_2,x_1)\}$. Then let us consider the following equations:

$$\begin{aligned}
\dot{S}^2(T_1, S_1, S_2) &= \dot{S}^2(\{g_1(x_1,x_2)\}, \{h_1(x_2,x_1)\}, \{h_2(x_1,x_2), h_3(x_2,x_1)\}) \\
&= \{S^2(g_1(x_1,x_2), h_1(x_2,x_1), h_2(x_1,x_2))\} \\
&\quad \cup \{S^2(g_1(x_1,x_2), h_1(x_2,x_1), h_3(x_2,x_1)\} \\
&= \{g_1(h_1(x_2,x_1), h_3(x_2,x_1))\} \cup \{g_1(h_1(x_2,x_1), h_3(x_2,x_1))\} \\
&= \{g_1(h_1(x_2,x_1), h_3(x_2,x_1))\} \text{ and,}
\end{aligned}$$

$$\begin{aligned}
\dot{S}^2(T_2, S_1, S_2) &= \dot{S}^2(\{g_2(x_2,x_1)\}, \{h_1(x_2,x_1)\}, \{h_2(x_1,x_2), h_3(x_2,x_1)\}) \\
&= \{S^2(g_2(x_2,x_1), h_1(x_2,x_1), h_2(x_1,x_2))\} \\
&\quad \cup \{S^2(g_2(x_2,x_1), h_1(x_2,x_1), h_3(x_2,x_1)\} \\
&= \{g_2(h_2(x_1,x_2), h_1(x_2,x_1))\} \cup \{g_2(h_3(x_2,x_1), h_1(x_2,x_1))\} \\
&= \{g_2(h_2(x_1,x_2), h_1(x_2,x_1)), g_2(h_3(x_2,x_1), h_1(x_2,x_1))\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\dot{S}^2(T, \dot{S}^2(T_1, S_1, S_2), \dot{S}^2(T_2, S_1, S_2)) \\
&= \{S^2(f(x_1,x_2), g_1(h_1(x_2,x_1), h_3(x_2,x_1)), g_2(h_2(x_1,x_2), h_1(x_2,x_1)))\} \\
&\quad \cup \{S^2(f(x_1,x_2), g_1(h_1(x_2,x_1), h_1(x_2,x_1)), g_2(h_3(x_2,x_1), h_1(x_2,x_1)))\} \\
&= \{f(g_1(h_1(x_2,x_1), h_3(x_2,x_1)), g_2(h_2(x_1,x_2), h_1(x_2,x_1)))\} \\
&\quad \cup \{f(g_1(h_1(x_2,x_1), h_1(x_2,x_1)), g_2(h_3(x_2,x_1), h_1(x_2,x_1)))\} \\
&= \{f(g_1(h_1(x_2,x_1), h_3(x_2,x_1)), g_2(h_2(x_1,x_2), h_2(x_1,x_2))), \\
&\quad f(g_1(h_1(x_2,x_1), h_1(x_2,x_1)), g_2(h_3(x_2,x_1), h_1(x_2,x_1)))\}.
\end{aligned}$$

Let us consider the other equations:

$$\begin{aligned}
\dot{S}^2(T, T_1, T_2) &= \{S^2(f(x_1,x_2), g_1(x_1,x_2), g_2(x_2,x_1))\} \\
&= \{f(g_1(x_1,x_2), g_2(x_2,x_1))\} \text{ and,}
\end{aligned}$$

$$\begin{aligned}
&\dot{S}^2(\dot{S}^2(T,T_1,T_2),S_1,S_2)\\
&=\dot{S}^2(\{f(g_1(x_1,x_2),g_2(x_2,x_1))\},S_1,S_2)\\
&=\{S^2(f(g_1(x_1,x_2),g_2(x_2,x_1)),h_1(x_2,x_1),h_2(x_1,x_2))\}\\
&\quad\cup\{S^2(f(g_1(x_1,x_2),g_2(x_2,x_1)),h_1(x_2,x_1),h_3(x_2,x_1))\}\\
&=\{f(g_1(h_1(x_2,x_1),h_2(x_1,x_2)),g_2(h_2(x_1,x_2),h_1(x_2,x_1)))\}\\
&\quad\cup\{f(g_1(h_1(x_2,x_1),h_3(x_2,x_1)),g_2(h_3(x_2,x_1),h_1(x_2,x_1)))\}\\
&=\{f(g_1(h_1(x_2,x_1),h_2(x_1,x_2)),g_2(h_2(x_1,x_2),h_1(x_2,x_1))),\\
&\quad f(g_1(h_1(x_2,x_1),h_3(x_2,x_1)),g_2(h_3(x_2,x_1),h_1(x_2,x_1)))\}.
\end{aligned}$$

Therefore we have $\dot{S}^2(T,\dot{S}^2(T_1,S_1,S_2),\dot{S}^2(T_2,S_1,S_2)) \neq \dot{S}^2(\dot{S}^2(T,T_1,T_2), S_1,S_2)$.

Definition. Let $W^F_{\tau_n}(X_n)$ be the set of all n-ary full terms of type τ_n. Then the superposition operations

$$S^n_{nd} : \left(\mathcal{P}(W^F_{\tau_n}(X_n))\right)^{n+1} \to \mathcal{P}\left(W^F_{\tau_n}(X_n)\right),$$

for $T, T_q \subseteq W^F_{\tau_n}(X_n), 1 \le q \le n, n \in \mathbb{N}$ such that T, T_q are non-empty sets, the $S^n_{nd}(T,T_1,\ldots,T_n)$ are defined in the following way:

(1) If $T = \{f_i(x_{s(1)},\ldots,x_{s(n)})\}$ where $s \in H_n$, then
$S^n_{nd}(\{f_i(x_{s(1)},\ldots,x_{s(n)})\},T_1,\ldots,T_n) := \{f_i(t_{s(1)},\ldots,t_{s(n)}) \mid t_{s(q)} \in T_{s(q)}\}$.

(2) If $T = \{f_i(t_1,\ldots,t_n)\}$ where $t_1,\ldots,t_n \in W^F_{\tau_n}(X_n)$, then
$S^n_{nd}(\{f_i(t_1,\ldots,t_n)\},T_1,\ldots,T_n) := \{f_i(r_1,\ldots,r_n) \mid r_q \in S^n_{nd}(\{t_q\},T_1,\ldots,T_n)\}$.

(3) If T is an arbitrary subset of $W^F_{\tau_n}(X_n)$, then
$S^n_{nd}(T,T_1,\ldots,T_n) := \bigcup_{t\in T} S^n_{nd}(\{t\},T_1,\ldots,T_n)$.

If at least one of the sets $T,T_1,\ldots,T_n$ is an empty set, then $S^n_{nd}(T,T_1,\ldots,T_n) := \emptyset$.

Example 4. Let $\tau_2 := (2,2)$ and let $s : \{1,2\} \to \{1,2\}$ and $r : \{1,2\} \to \{1,2\}$ which are defined by $s(1) = 2, s(2) = 1$ and $r(1) = 1, r(2) = 2$. Let $T = \{g(x_{s(1)}, x_{s(2)}), f(x_{r(1)},x_{r(2)})\}$, $T_1 = \{f(x_{r(1)},x_{r(2)})\}$ and $T_2 = \{g(x_{s(1)},x_{s(2)})\}$. Then we have

$$\begin{aligned}
S^2_{nd}(\{g(x_{s(1)},x_{s(2)})\},T_1,T_2) &= S^2_{nd}(\{g(x_2,x_1)\},T_1,T_2)\\
&= \{g(v_2,v_1) \mid v_2 \in T_2, v_1 \in T_1\}\\
&= \{g(g(x_{s(1)},x_{s(2)}),f(x_{r(1)},x_{r(2)}))\}\\
&= \{g(g(x_2,x_1),f(x_1,x_2))\} \text{ and,}
\end{aligned}$$

$$\begin{aligned}S^2_{nd}(\{f(x_{r(1)},x_{r(2)})\},T_1,T_2) &= S^2_{nd}(\{f(x_1,x_2)\},T_1,T_2)\\ &= \{f(u_1,u_2) \mid u_1 \in T_1, u_2 \in T_2\}\\ &= \{f(f(x_{r(1)},x_{r(2)}),g(x_{s(1)},x_{s(2)}))\}\\ &= \{f(f(x_1,x_2),g(x_2,x_1))\}.\end{aligned}$$

Therefore we have

$$\begin{aligned}S^2_{nd}(T,T_1,T_2) &= S^2_{nd}(\{g(x_{s(1)},x_{s(2)}),f(x_{r(1)},x_{r(2)})\},T_1,T_2)\\ &= S^2_{nd}(\{g(x_2,x_1),f(x_1,x_2)\},T_1,T_2)\\ &= S^2_{nd}(\{g(x_2,x_1)\},T_1,T_2) \cup S^2_{nd}(\{f(x_1,x_2)\},T_1,T_2)\\ &= \{g(g(x_2,x_1),f(x_1,x_2))\} \cup \{f(f(x_1,x_2),g(x_2,x_1))\}\\ &= \{g(g(x_2,x_1),f(x_1,x_2)),f(f(x_1,x_2),g(x_2,x_1))\}.\end{aligned}$$

Now we give some properties of such superposition.

Proposition 5. *Let* $T, T_q \subseteq W^F_{\tau_n}(X_n), 1 \le q \le n, n \in \mathbb{N}$ *and* $s \in H_n$. *Then we have*

$$S^n_{nd}(T_s,T_1,\ldots,T_n) = S^n_{nd}(T,T_{s(1)},\ldots,T_{s(n)}).$$

Proof. If T is empty, then the claim is clearly true. If T is non-empty, then we consider in the following steps.

(1) If T is a singleton, then

Case 1. $T = \{f_i(x_{r(1)},\ldots,x_{r(n)})\}$ where $r \in H_n$, we have

$$\begin{aligned}S^n_{nd}(T_s,T_1,\ldots,T_n) &= S^n_{nd}((\{f_i(x_{r(1)},\ldots,x_{r(n)})\})_s,T_1,\ldots,T_n)\\ &= S^n_{nd}(\{f_i(x_{s(r(1))},\ldots,x_{s(r(n))})\},T_1,\ldots,T_n)\\ &= \{f_i(t_{s(r(1))},\ldots,t_{s(r(n))}) \mid t_{s(r(q))} \in T_{s(r(q))}\}\\ &= S^n_{nd}(\{f_i(x_{r(1)},\ldots,x_{r(n)})\},T_{s(1)},\ldots,T_{s(n)})\\ &= S^n_{nd}(T,T_{s(1)},\ldots,T_{s(n)}),\end{aligned}$$

Case 2. $T = \{f_i(u_1,\ldots,u_n)\}$ where $u_1,\ldots,u_n \in W^F_{\tau_n}(X_n)$, and we assume that the equations

$$S^n_{nd}(\{u_q\}_s,T_1,\ldots,T_n) = S^n_{nd}(\{u_q\},T_{s(1)},\ldots,T_{s(n)}),$$

are satisfied, we have

$$\begin{aligned}S^n_{nd}(T_s,T_1,\ldots,T_n) &= S^n_{nd}((\{f_i(u_1,\ldots,u_n)\})_s,T_1,\ldots,T_n)\\ &= S^n_{nd}(\{f_i(u_{s(1)},\ldots,u_{s(n)})\},T_1,\ldots,T_n)\\ &= \{f_i(v_{s(1)},\ldots,v_{s(n)}) \mid v_{s(q)} \in S^n_{nd}(\{u_{s(q)}\},T_1,\ldots,T_n)\}\\ &= \{f_i(v_{s(1)},\ldots,v_{s(n)}) \mid v_{s(q)} \in S^n_{nd}(\{u_q\}_s,T_1,\ldots,T_n)\}\\ &= \{f_i(v_{s(1)},\ldots,v_{s(n)}) \mid v_{s(q)} \in S^n_{nd}(\{u_q\},T_{s(1)},\ldots,T_{s(n)})\}\\ &= S^n_{nd}(\{f_i(u_1,\ldots,u_n)\},T_{s(1)},\ldots,T_{s(n)})\\ &= S^n_{nd}(T,T_{s(1)},\ldots,T_{s(n)}).\end{aligned}$$

(2) If T is an arbitrary subset of $W_{\tau_n}^F(X_n)$, then

$$\begin{aligned} S_{nd}^n(T_s, T_1, \ldots, T_n) &= \bigcup_{t\in T} S_{nd}^n(\{t_s\}, T_1, \ldots, T_n) = \bigcup_{t\in T} S_{nd}^n(\{t\}_s, T_1, \ldots, T_n) \\ &= \bigcup_{t\in T} S_{nd}^n\left(\{t\}, T_{s(1)}, \ldots, T_{s(n)}\right) = S_{nd}^n\left(T, T_{s(1)}, \ldots, T_{s(n)}\right). \end{aligned}$$

■

Proposition 6. *Let $T, T_q \subseteq W_{\tau_n}^F(X_n), 1 \le q \le n, n \in \mathbb{N}$ and $s \in H_n$. Then we have*

$$S_{nd}^n(T_s, T_1, \ldots, T_n) = (S_{nd}^n(T, T_1, \ldots, T_n))_s.$$

Proof. If T is empty, then the claim is clearly true. If T is non-empty, then we consider in the following steps.

(1) If T is a singleton, then

Case 1. $T = \{f_i(x_{r(1)}, \ldots, x_{r(n)})\}$ where $r \in H_n$, we have

$$\begin{aligned} S_{nd}^n(T_s, T_1, \ldots, T_n) &= S_{nd}^n((\{f_i(x_{r(1)}, \ldots, x_{r(n)})\})_s, T_1, \ldots, T_n) \\ &= S_{nd}^n(\{f_i(x_{s(r(1))}, \ldots, x_{s(r(n))})\}, T_1, \ldots, T_n) \\ &= \{f_i(t_{s(r(1))}, \ldots, t_{s(r(n))}) \mid t_{s(r(q))} \in T_{s(r(q))}\} \\ &= \{(f_i(t_{r(1)}, \ldots, t_{r(n)}))_s \mid t_{s(r(q))} \in T_{s(r(q))}\} \\ &= (\{f_i(t_{r(1)}, \ldots, t_{r(n)}) \mid t_{r(q)} \in T_{r(q)}\})_s \\ &= (S_{nd}^n(\{f_i(x_{r(1)}, \ldots, x_{r(n)})\}, T_1, \ldots, T_n))_s \\ &= (S_{nd}^n(T, T_1, \ldots, T_n))_s. \end{aligned}$$

Case 2. $T = \{f_i(u_1, \ldots, u_n)\}$ where $u_1, \ldots, u_n \in W_{\tau_n}^F(X_n)$, and we assume that the equations

$$S_{nd}^n(\{u_q\}_s, T_1, \ldots, T_n) = (S_{nd}^n(\{u_q\}, T_1, \ldots, T_n))_s,$$

are satisfied, we have

$$\begin{aligned} &S_{nd}^n(T_s, T_1, \ldots, T_n) \\ &= S_{nd}^n((\{f_i(u_1, \ldots, u_n)\})_s, T_1, \ldots, T_n) \\ &= S_{nd}^n(\{f_i(u_{s(1)}, \ldots, u_{s(n)})\}, T_1, \ldots, T_n) \\ &= \{f_i(v_{s(1)}, \ldots, v_{s(n)}) \mid v_{s(q)} \in S_{nd}^n(\{u_{s(q)}\}, T_1, \ldots, T_n)\} \\ &= \{f_i(v_{s(1)}, \ldots, v_{s(n)}) \mid v_{s(q)} \in S_{nd}^n(\{u_q\}_s, T_1, \ldots, T_n)\} \\ &= \{f_i(v_{s(1)}, \ldots, v_{s(n)}) \mid v_{s(q)} \in (S_{nd}^n(\{u_q\}, T_1, \ldots, T_n))_s\} \\ &= \{(f_i(v_1, \ldots, v_n))_s \mid (v_q)_s \in (S_{nd}^n(\{u_q\}, T_1, \ldots, T_n))_s\} \\ &= (\{f_i(v_1, \ldots, v_n) \mid v_q \in S_{nd}^n(\{u_q\}, T_1, \ldots, T_n)\})_s \\ &= (S_{nd}^n(\{f_i(u_1, \ldots, u_n)\}, T_1, \ldots, T_n))_s = (S_{nd}^n(T, T_1, \ldots, T_n))_s. \end{aligned}$$

(2) If T is an arbitrary subset of $W_{\tau_n}^F(X_n)$, then

$$\begin{aligned} S_{nd}^n(T_s, T_1, \dots, T_n) &= \bigcup_{t\in T} S_{nd}^n(\{t_s\}, T_1, \dots, T_n) = \bigcup_{t\in T} S_{nd}^n(\{t\}_s, T_1, \dots, T_n) \\ &= \bigcup_{t\in T} (S_{nd}^n(\{t\}, T_1, \dots, T_n))_s = \Big(\bigcup_{t\in T} S_{nd}^n(\{t\}, T_1, \dots, T_n)\Big)_s \\ &= \left(S_{nd}^n(T, T_1, \dots, T_n)\right)_s. \end{aligned}$$

■

By Proposition 5 and Proposition 6 we have:

Proposition 7. *Let $T, T_q \subseteq W_{\tau_n}^F(X_n), 1 \le q \le n, n \in \mathbb{N}$ and $s \in H_n$. Then we have*

$$S_{nd}^n\left(T, T_{s(1)}, \dots, T_{s(n)}\right) = (S_{nd}^n(T, T_1, \dots, T_n))_s.$$

Next theorem we show that the superposition operation S_{nd}^n is satified the superassociative law.

Theorem 8. *Let $T, T_q, S_q \subseteq W_{\tau_n}^F(X_n), 1 \le q \le n, n \in \mathbb{N}$. Then we have*

$$\begin{aligned} &S_{nd}^n(T, S_{nd}^n(S_1, T_1, \dots, T_n), \dots, S_{nd}^n(S_n, T_1, \dots, T_n)) \\ &= S_{nd}^n(S_{nd}^n(T, S_1, \dots, S_n), T_1, \dots, T_n). \end{aligned}$$

Proof. If T is empty, then the claim is clearly true. If T is non-empty, then we consider in the following steps.

(1) If T is a singleton, then

Case 1. $T = \{f_i(x_{s(1)}, \dots, x_{s(n)})\}$ where $s \in H_n$, we have

$$\begin{aligned} &S_{nd}^n(T, S_{nd}^n(S_1, T_1, \dots, T_n), \dots, S_{nd}^n(S_n, T_1, \dots, T_n)) \\ &= S_{nd}^n(\{f_i(x_{s(1)}, \dots, x_{s(n)})\}, S_{nd}^n(S_1, T_1, \dots, T_n), \dots, S_{nd}^n(S_n, T_1, \dots, T_n)) \\ &= \{f_i(r_{s(1)}, \dots, r_{s(n)}) \mid r_{s(q)} \in S_{nd}^n(S_{s(q)}, T_1, \dots, T_n)\} \\ &= \{f_i(v_{s(1)}, \dots, v_{s(n)}) \mid v_{s(q)} \in S_{nd}^n(S_{s(q)}, T_1, \dots, T_n)\} \\ &= \{f_i(v_{s(1)}, \dots, v_{s(n)}) \mid v_{s(q)} \in S_{nd}^n(\{p_{s(q)} \mid p_{s(q)} \in S_{s(q)}\}, T_1, \dots, T_n)\} \\ &= S_{nd}^n(\{f_i(p_{s(1)}, \dots, p_{s(n)}) \mid p_{s(q)} \in S_{s(q)}\}, T_1, \dots, T_n) \\ &= S_{nd}^n(S_{nd}^n(\{f_i(x_{s(1)}, \dots, x_{s(n)})\}, S_1, \dots, S_n), T_1, \dots, T_n) \\ &= S_{nd}^n(S_{nd}^n(T, S_1, \dots, S_n), T_1, \dots, T_n). \end{aligned}$$

Case 2. $T = \{f_i(u_1, \dots, u_n)\}$ where $u_1, \dots, u_n \in W_{\tau_n}^F(X_n)$, and we assume that the equations

$$\begin{aligned} &S_{nd}^n(\{u_q\}, S_{nd}^n(S_1, T_1, \dots, T_n), \dots, S_{nd}^n(S_n, T_1, \dots, T_n)) \\ &= S_{nd}^n(S_{nd}^n(\{u_q\}, S_1, \dots, S_n), T_1, \dots, T_n) \end{aligned}$$

are satisfied, we have

$$
\begin{aligned}
&S^n_{nd}(T, S^n_{nd}(S_1, T_1, \ldots, T_n), \ldots, S^n_{nd}(S_n, T_1, \ldots, T_n))\\
&= S^n_{nd}(\{f_i(u_1, \ldots, u_n)\}, S^n_{nd}(S_1, T_1, \ldots, T_n), \ldots, S^n_{nd}(S_n, T_1, \ldots, T_n))\\
&= \{f_i(r_1, \ldots, r_n) \mid r_q \in S^n_{nd}(\{u_q\}, S^n_{nd}(S_1, T_1, \ldots, T_n), \ldots, S^n_{nd}(S_n, T_1, \ldots, T_n))\}\\
&= \{f_i(r_1, \ldots, r_n) \mid r_q \in S^n_{nd}(S^n_{nd}(\{u_q\}, S_1, \ldots, S_n), T_1, \ldots, T_n)\}\\
&= \{f_i(r_1, \ldots, r_n) \mid r_q \in S^n_{nd}(\{v_q \mid v_q \in S^n_{nd}(\{u_q\}, S_1, \ldots, S_n)\}, T_1, \ldots, T_n)\}\\
&= S^n_{nd}(\{f_i(v_1, \ldots, v_n) \mid v_q \in S^n_{nd}(\{u_q\}, S_1, \ldots, S_n)\}, T_1, \ldots, T_n)\\
&= S^n_{nd}(S^n_{nd}(\{f_i(u_1, \ldots, u_n)\}, S_1, \ldots, S_n), T_1, \ldots, T_n)\\
&= S^n_{nd}(S^n_{nd}(T, S_1, \ldots, S_n), T_1, \ldots, T_n).
\end{aligned}
$$

(2) If T is an arbitrary subset of $W^F_{\tau_n}(X_n)$, then

$$
\begin{aligned}
&S^n_{nd}(T, S^n_{nd}(S_1, T_1, \ldots, T_n), \ldots, S^n_{nd}(S_n, T_1, \ldots, T_n))\\
&= \bigcup_{t \in T} S^n_{nd}(\{t\}, S^n_{nd}(S_1, T_1, \ldots, T_n), \ldots, S^n_{nd}(S_n, T_1, \ldots, T_n))\\
&= \bigcup_{t \in T} S^n_{nd}(S^n_{nd}(\{t\}, S_1, \ldots, S_n), T_1, \ldots, T_n)\\
&= S^n_{nd}(\bigcup_{t \in T} S^n_{nd}(\{t\}, S_1, \ldots, S_n), T_1, \ldots, T_n)\\
&= S^n_{nd}(S^n_{nd}(T, S_1, \ldots, S_n), T_1, \ldots, T_n).
\end{aligned}
$$

■

Using this superposition operation we can form algebra $(\mathcal{P}(W^F_{\tau_n}(X_n)); S^n_{nd})$ of type $(n+1)$. This algebra is called *nd-clone*$_F\tau_n$.

3. Nd-full hypersubstitutions

Hypersubstitutions for terms over one-sorted algebras were introduced by E. Graczyńska and Schweigert [5]. Our definitions and the properties of superposition operations can be used to define non-deterministic full hypersubstitutions and their extensions. First we introduce the following notation.

Definition. A mapping $\sigma^{nd} : \{f_i \mid i \in I\} \to \mathcal{P}(W^F_{\tau_n}(X_n))$ is called non-deterministic full hypersubstitution or nd-full hypersubstitution, for short. Let *nd-Hyp*$^F(\tau_n)$ be a set of all nd-full hypersubstitutions. Any such nd-full hypersubstitution, σ^{nd} uniquely determine a mapping

$$\hat{\sigma}^{nd} : \mathcal{P}(W^F_{\tau_n}(X_n)) \to \mathcal{P}(W^F_{\tau_n}(X_n)),$$

is defined in the following way:

(1) $\hat{\sigma}^{nd}[\emptyset] := \emptyset$.

(2) $\hat{\sigma}^{nd}[\{f_i(x_{s(1)},\ldots,x_{s(n)})\}] := (\sigma^{nd}(f_i))_s$ for every $s \in H_n$.

(3) $\hat{\sigma}^{nd}[\{f_i(t_1,\ldots,t_n)\}] := S^n_{nd}(\sigma^{nd}(f_i), \hat{\sigma}^{nd}[\{t_1\}],\ldots,\hat{\sigma}^{nd}[\{t_n\}])$ and we assume that $\hat{\sigma}^{nd}[\{t_1\}],\ldots,\hat{\sigma}^{nd}[\{t_n\}]$ are already defined.

(4) $\hat{\sigma}^{nd}[T] := \bigcup_{t\in T} \hat{\sigma}^{nd}[\{t\}]$ where T is an arbitrary subset of $W^F_{\tau_n}(X_n)$.

Example 9. Let $\tau_2 := (2,2)$ and let $s : \{1,2\} \to \{1,2\}$ and $r : \{1,2\} \to \{1,2\}$ which are defined by $s(1) = 2, s(2) = 1$ and $r(1) = 1, r(2) = 2$. Let $T = \{g(f(x_{r(1)}, x_{r(2)}), g(x_{s(1)}, x_{s(2)})), f(x_{r(1)}, x_{r(2)})\}$, and let $\sigma^{nd} : \{g, f\} \to \mathcal{P}(W^F_{\tau_2}(X_2))$ be defined by $\sigma^{nd}(g) := \{f(x_{r(1)}, x_{r(2)})\}$, $\sigma^{nd}(f) := \{g(x_{s(1)}, x_{s(2)})\}$. Then we have

$$\begin{aligned}\hat{\sigma}^{nd}(T) &= \hat{\sigma}^{nd}(\{g(f(x_{r(1)}, x_{r(2)}), g(x_{s(1)}, x_{s(2)})), f(x_{r(1)}, x_{r(2)})\})\\ &= \hat{\sigma}^{nd}(\{g(f(x_{r(1)}, x_{r(2)}), g(x_{s(1)}, x_{s(2)}))\} \cup \hat{\sigma}^{nd}(\{f(x_{r(1)}, x_{r(2)})\}).\end{aligned}$$

Let us consider the following equations:

$$\begin{aligned}&\hat{\sigma}^{nd}(\{g(f(x_{r(1)}, x_{r(2)}), g(x_{s(1)}, x_{s(2)}))\}\\ &= S^2_{nd}(\sigma^{nd}(g), \hat{\sigma}^{nd}(\{f(x_{r(1)}, x_{r(2)})\}), \hat{\sigma}^{nd}(\{g(x_{s(1)}, x_{s(2)})\}))\\ &= S^2_{nd}(\sigma^{nd}(g), (\sigma^{nd}(f))_r, (\sigma^{nd}(g))_s)\\ &= S^2_{nd}(\sigma^{nd}(g), (\{g(x_{s(1)}, x_{s(2)})\})_r, (\{f(x_{r(1)}, x_{r(2)})\})_s)\\ &= S^2_{nd}(\sigma^{nd}(g), (\{g(x_2, x_1)\})_r, (\{f(x_1, x_2)\})_s)\\ &= S^2_{nd}(\sigma^{nd}(g), \{g(x_{r(2)}, x_{r(1)})\}, \{f(x_{s(1)}, x_{s(2)})\})\\ &= S^2_{nd}(\{f(x_{r(1)}, x_{r(2)})\}, \{g(x_2, x_1)\}, \{f(x_2, x_1)\})\\ &= S^2_{nd}(\{f(x_1, x_2)\}, \{g(x_2, x_1)\}, \{f(x_2, x_1)\})\\ &= \{f(r_1, r_2) \mid r_1 \in \{g(x_2, x_1)\}, r_2 \in \{f(x_2, x_1)\}\}\\ &= \{f(g(x_2, x_1), f(x_2, x_1))\} \text{ and}\end{aligned}$$

$$\begin{aligned}\hat{\sigma}^{nd}(\{f(x_{r(1)}, x_{r(2)})\} &= (\sigma^{nd}(f))_r = (\{g(x_{s(1)}, x_{s(2)})\})_r\\ &= (\{g(x_2, x_1)\})_r = \{g(x_{r(2)}, x_{r(1)})\} = \{g(x_2, x_1)\}.\end{aligned}$$

Therefore,

$$\begin{aligned}\hat{\sigma}^{nd}(T) &= \{f(g(x_2, x_1), f(x_2, x_1))\} \cup \{g(x_2, x_1)\}\\ &= \{f(g(x_2, x_1), f(x_2, x_1)), g(x_2, x_1)\}.\end{aligned}$$

Lemma 10. *Let T be a subset of $W^F_{\tau_n}(X_n)$ and $s \in H_n$. Then we have*

$$\hat{\sigma}^{nd}[T_s] = (\hat{\sigma}^{nd}[T])_s.$$

Proof. If T is empty, then the claim is clearly true. If T is non-empty, then we consider in the following steps.

(1) If T is a singleton, then

Case 1. $T = \{f_i(x_{r(1)}, \dots, x_{r(n)})\}$ where $r \in H_n$, we have

$$\begin{aligned}\hat{\sigma}^{nd}[T_s] &= \hat{\sigma}^{nd}[(\{f_i(x_{r(1)}, \dots, x_{r(n)})\})_s] = \hat{\sigma}^{nd}[\{f_i(x_{s(r(1))}, \dots, x_{s(r(n))})\}] \\ &= \hat{\sigma}^{nd}[\{f_i(x_{(s\circ r)(1)}, \dots, x_{(s\circ r)(n)})\}] = (\sigma^{nd}(f_i))_{(s\circ r)} \\ &= ((\sigma^{nd}(f_i))_r)_s = (\hat{\sigma}^{nd}[\{f_i(x_{r(1)}, \dots, x_{r(n)})\}])_s = (\hat{\sigma}^{nd}[T])_s.\end{aligned}$$

Case 2. $T = \{f_i(t_1, \dots, t_n)\}$ where $t_1, \dots, t_n \in W^F_{\tau_n}(X_n)$, and we assume that the equations

$$\hat{\sigma}^{nd}[\{t_q\}_s] = (\hat{\sigma}[\{t_q\}])_s, 1 \leq q \leq n, n \in \mathbb{N},$$

are satisfied, we have

$$\begin{aligned}\hat{\sigma}^{nd}[T_s] &= \hat{\sigma}^{nd}[(\{f_i(t_1, \dots, t_n)\})_s] = \hat{\sigma}^{nd}[\{f_i(t_{s(1)}, \dots, t_{s(n)})\}] \\ &= S^n_{nd}(\sigma^{nd}(f_i), \hat{\sigma}^{nd}[\{t_{s(1)}\}], \dots, \hat{\sigma}^{nd}[\{t_{s(n)}\}]) \\ &= S^n_{nd}(\sigma^{nd}(f_i), \hat{\sigma}^{nd}[(\{t_1\})_s], \dots, \hat{\sigma}^{nd}[(\{t_n\})_s]) \\ &= S^n_{nd}(\sigma^{nd}(f_i), (\hat{\sigma}^{nd}[\{t_1\}])_s, \dots, (\hat{\sigma}^{nd}[\{t_n\}])_s) \\ &= (S^n_{nd}(\sigma^{nd}(f_i), \hat{\sigma}^{nd}[\{t_1\}], \dots, \hat{\sigma}^{nd}[\{t_n\}]))_s \\ &= (\hat{\sigma}^{nd}[\{f_i(t_1, \dots, t_n)\}])_s = (\hat{\sigma}^{nd}[T])_s.\end{aligned}$$

(2) If T is an arbitrary subset of $W^F_{\tau_n}(X_n)$, then

$$\hat{\sigma}^{nd}[T_s] = \bigcup_{t\in T} \hat{\sigma}^{nd}[\{t\}_s] = \bigcup_{t\in T} (\hat{\sigma}^{nd}[\{t\}])_s = \Big(\bigcup_{t\in T} \hat{\sigma}^{nd}[\{t\}]\Big)_s = (\hat{\sigma}^{nd}[T])_s.$$ ■

The next theorem will show that this extension is an endomorphism of the *nd-clone*$_F\tau_n$.

Theorem 11. *A mapping* $\hat{\sigma}^{nd} : \mathcal{P}(W^F_{\tau_n}(X_n)) \to \mathcal{P}(W^F_{\tau_n}(X_n))$ *is an endomorphism of nd-clone*$_F\tau_n$.

Proof. Let T and $T_q, 1 \leq q \leq n, n \in \mathbb{N}$ be subsets of $W^F_{\tau_n}(X_n)$. We have to show that the equation hold:

$$\hat{\sigma}^{nd}[S^n_{nd}(T, T_1, \dots, T_n)] = S^n_{nd}\left(\hat{\sigma}^{nd}[T], \hat{\sigma}^{nd}[T_1], \dots, \hat{\sigma}^{nd}[T_n]\right).$$

If T is empty, the claim is clearly true.

(1) If T is a singleton, then

Case 1. $T = \{f_i(x_{s(1)}, \ldots, x_{s(n)})\}$ where $s \in H_n$, we have

$$\begin{aligned}
&\hat{\sigma}^{nd}[S^n_{nd}(T, T_1, \ldots, T_n)] \\
&= \hat{\sigma}^{nd}[S^n_{nd}(\{f_i(x_{s(1)}, \ldots, x_{s(n)})\}, T_1, \ldots, T_n)] \\
&= \hat{\sigma}^{nd}[\{f_i(r_{s(1)}, \ldots, r_{s(n)}) \mid r_{s(q)} \in T_{s(q)}\}] \\
&= S^n_{nd}(\sigma^{nd}(f_i), \hat{\sigma}^{nd}[\{r_{s(1)} \mid r_{s(1)} \in T_{s(1)}\}], \ldots, \hat{\sigma}^{nd}[\{r_{s(n)} \mid r_{s(n)} \in T_{s(n)}\}]) \\
&= S^n_{nd}(\sigma^{nd}(f_i), \hat{\sigma}^{nd}[T_{s(1)}], \ldots, \hat{\sigma}^{nd}[T_{s(n)}]) \\
&= S^n_{nd}(\sigma^{nd}(f_i), (\hat{\sigma}^{nd}[T_1])_s, \ldots, (\hat{\sigma}^{nd}[T_n])_s) \\
&= S^n_{nd}((\sigma^{nd}(f_i))_s, \hat{\sigma}^{nd}[T_1], \ldots, \hat{\sigma}^{nd}[T_n]) \\
&= S^n_{nd}(\hat{\sigma}^{nd}[\{f_i(x_{s(1)}, \ldots, x_{s(n)})\}], \hat{\sigma}^{nd}[T_1], \ldots, \hat{\sigma}^{nd}[T_n]) \\
&= S^n_{nd}(\hat{\sigma}^{nd}[T], \hat{\sigma}^{nd}[T_1], \ldots, \hat{\sigma}^{nd}[T_n]).
\end{aligned}$$

Case 2. $T = \{f_i(t_1, \ldots, t_n)\}$ where $t_1, \ldots, t_n \in W^F_{\tau_n}(X_n)$, and we assume that the equations

$$\hat{\sigma}^{nd}[S^n_{nd}(\{t_q\}, T_1, \ldots, T_n)] = S^n_{nd}(\hat{\sigma}^{nd}[\{t_q\}], \hat{\sigma}^{nd}[T_1], \ldots, \hat{\sigma}^{nd}[T_n]),$$

are satisfied, we have

$$\begin{aligned}
&\hat{\sigma}^{nd}[S^n_{nd}(T, T_1, \ldots, T_n)] \\
&= \hat{\sigma}^{nd}[S^n_{nd}(\{f_i(t_1, \ldots, t_n)\}, T_1, \ldots, T_n)] \\
&= \hat{\sigma}^{nd}[\{f_i(r_1, \ldots, r_n) \mid r_q \in S^n_{nd}(\{t_q\}, T_1, \ldots, T_n)\}] \\
&= S^n_{nd}(\sigma^{nd}(f_i), \hat{\sigma}^{nd}[\{r_1 \mid r_1 \in S^n_{nd}(\{t_1\}, T_1, \ldots, T_n)\}], \ldots, \\
&\quad \hat{\sigma}^{nd}[\{r_n \mid r_n \in S^n(\{t_n\}, T_1, \ldots, T_n)\}]) \\
&= S^n_{nd}(\sigma^{nd}(f_i), \hat{\sigma}^{nd}[S^n_{nd}(\{t_1\}, T_1, \ldots, T_n)], \ldots, \hat{\sigma}^{nd}[S^n_{nd}(\{t_n\}, T_1, \ldots, T_n)]) \\
&= S^n_{nd}(\sigma^{nd}(f_i), S^n_{nd}(\hat{\sigma}^{nd}[\{t_1\}], \hat{\sigma}^{nd}[T_1], \ldots, \hat{\sigma}^{nd}[T_n]), \ldots, S^n_{nd}(\hat{\sigma}^{nd}[\{t_n\}], \\
&\quad \hat{\sigma}^{nd}[T_1], \ldots, \hat{\sigma}^{nd}[T_n])) \\
&= S^n_{nd}(S^n_{nd}(\sigma^{nd}(f_i), \hat{\sigma}^{nd}[\{t_1\}], \ldots, \hat{\sigma}^{nd}[\{t_n\}]), \hat{\sigma}^{nd}[T_1], \ldots, \hat{\sigma}^{nd}[T_n]) \\
&= S^n_{nd}(\hat{\sigma}^{nd}[\{f_i(t_1, \ldots, t_n)\}], \hat{\sigma}^{nd}[T_1], \ldots, \hat{\sigma}^{nd}[T_n]) \\
&= S^n_{nd}(\hat{\sigma}^{nd}[T], \hat{\sigma}^{nd}[T_1], \ldots, \hat{\sigma}^{nd}[T_n]).
\end{aligned}$$

(2) If T is an arbitrary subset of $W^F_{\tau_n}(X_n)$, then

$$\begin{aligned}\hat{\sigma}^{nd}\left[S^n_{nd}(T,T_1,\dots,T_n)\right] &= \hat{\sigma}^{nd}\Big[\bigcup_{t\in T} S^n_{nd}(\{t\},T_1,\dots,T_n)\Big]\\ &= \bigcup_{t\in T}\hat{\sigma}^{nd}[S^n_{nd}(\{t\},T_1,\dots,T_n)]\\ &= \bigcup_{t\in T} S^n_{nd}(\hat{\sigma}^{nd}[\{t\}],\hat{\sigma}^{nd}[T_1],\dots,\hat{\sigma}^{nd}[T_n])\\ &= S^n_{nd}\Big(\bigcup_{t\in T}\hat{\sigma}^{nd}[\{t\}],\hat{\sigma}^{nd}[T_1],\dots,\hat{\sigma}^{nd}[T_n]\Big)\\ &= S^n_{nd}\left(\hat{\sigma}^{nd}[T],\hat{\sigma}^{nd}[T_1],\dots,\hat{\sigma}^{nd}[T_n]\right). \qquad \blacksquare\end{aligned}$$

Let $\sigma_1^{nd},\sigma_2^{nd}\in nd\text{-}Hyp^F(\tau_n)$. Since the extension of non-derministic full hypersubstitution maps $\mathcal{P}(W^F_{\tau_n}(X_n))$ to $\mathcal{P}(W^F_{\tau_n}(X_n))$ we can define a product $\sigma_1^{nd}\circ_{nd}\sigma_2^{nd}$ by

$$\sigma_1^{nd}\circ_{nd}\sigma_2^{nd} := \hat{\sigma}_1^{nd}\circ\sigma_2^{nd}.$$

Here $\circ$ is the usual composition of mappings. Since $\hat{\sigma}_1^{nd}\circ\sigma_2^{nd}$ maps $\{f_i\mid i\in I\}$ to $\mathcal{P}(W^F_{\tau_n}(X_n))$, it is a non-derterministic full hypersubstitution.

The following lemma shows that the extension of this product is the product of the extensions of σ_1^{nd} and σ_2^{nd}.

Lemma 12. *Let $\sigma_1^{nd},\sigma_2^{nd}\in nd\text{-}Hyp^F(\tau_n)$. Then we have*

$$(\sigma_1^{nd}\circ_{nd}\sigma_2^{nd})\hat{} = \hat{\sigma}_1^{nd}\circ\hat{\sigma}_2^{nd}.$$

Proof. Let T be a subset of $W^F_{\tau_n}(X_n)$. We have to show that

$$(\sigma_1^{nd}\circ_{nd}\sigma_2^{nd})\hat{}[T] = (\hat{\sigma}_1^{nd}\circ\hat{\sigma}_2^{nd})[T].$$

If T is empty, then the claim is clearly true. If T is non-empty, then we consider in the following steps.

(1) If T is a singleton, then

Case 1. $T=\{f_i(x_{s(1)},\dots,x_{s(n)})\}$ where $s\in H_n$, we have

$$\begin{aligned}(\sigma_1^{nd}\circ_{nd}\sigma_2^{nd})\hat{}[T] &= (\sigma_1^{nd}\circ_{nd}\sigma_2^{nd})\hat{}[\{f_i(x_{s(1)},\dots,x_{s(n)})\}]\\ &= ((\sigma_1^{nd}\circ_{nd}\sigma_2^{nd})(f_i))_s = (\hat{\sigma}_1^{nd}[\sigma_2^{nd}(f_i)])_s\\ &= \hat{\sigma}_1^{nd}[(\sigma_2^{nd}(f_i))_s] = \hat{\sigma}_1^{nd}[\hat{\sigma}_2^{nd}[\{f_i(x_{s(1)},\dots,x_{s(n)})\}]]\\ &= \hat{\sigma}_1^{nd}[\hat{\sigma}_2^{nd}[T]] = (\hat{\sigma}_1^{nd}\circ\hat{\sigma}_2^{nd})[T].\end{aligned}$$

Case 2. $T=\{f_i(t_1,\dots,t_n)\}$ where $t_1,\dots,t_n\in W^F_{\tau_n}(X_n)$, and we assume that the equations

$$(\sigma_1^{nd}\circ_{nd}\sigma_2^{nd})\hat{}[\{t_q\}] = (\hat{\sigma}_1^{nd}\circ\hat{\sigma}_2^{nd})[\{t_q\}].$$

where $1 \leq q \leq n, n \in \mathbb{N}$ are satisfied, we have

$$\begin{aligned}
&(\sigma_1^{nd} \circ_{nd} \sigma_2^{nd})\hat{}[T] \\
&= (\sigma_1^{nd} \circ_{nd} \sigma_2^{nd})\hat{}[\{f_i(t_1, \ldots, t_n)\}] \\
&= S_{nd}^n((\sigma_1^{nd} \circ_{nd} \sigma_2^{nd})(f_i), (\sigma_1^{nd} \circ_{nd} \sigma_2^{nd})\hat{}[\{t_1\}], \ldots, (\sigma_1^{nd} \circ_{nd} \sigma_2^{nd})\hat{}[\{t_n\}]) \\
&= S_{nd}^n(\hat{\sigma}_1^{nd}[\sigma_2^{nd}(f_i)], (\hat{\sigma}_1^{nd} \circ \hat{\sigma}_2^{nd})[\{t_1\}], \ldots, (\hat{\sigma}_1^{nd} \circ \hat{\sigma}_2^{nd})[\{t_n\}]) \\
&= S_{nd}^n(\hat{\sigma}_1^{nd}[\sigma_2^{nd}(f_i)], \hat{\sigma}_1^{nd}[\hat{\sigma}_2^{nd}[\{t_1\}]], \ldots, \hat{\sigma}_1^{nd}[\hat{\sigma}_2^{nd}[\{t_n\}]]) \\
&= \hat{\sigma}_1^{nd}[S_{nd}^n(\sigma_2^{nd}(f_i), \hat{\sigma}_2^{nd}[\{t_1\}], \ldots, \hat{\sigma}_2^{nd}[\{t_n\}])] \\
&= \hat{\sigma}_1^{nd}[\hat{\sigma}_2^{nd}[\{f_i(t_1, \ldots, t_n)\}]] = \hat{\sigma}_1^{nd}[\hat{\sigma}_2^{nd}[T]].
\end{aligned}$$

(2) If T is an arbitrary subset of $W_{\tau_n}^F(X_n)$, then

$$\begin{aligned}
(\sigma_1^{nd} \circ_{nd} \sigma_2^{nd})\hat{}[T] &= \bigcup_{t \in T} (\sigma_1^{nd} \circ_{nd} \sigma_2^{nd})\hat{}[\{t\}] = \bigcup_{t \in T} (\hat{\sigma}_1^{nd} \circ \hat{\sigma}_2^{nd})[\{t\}] \\
&= \bigcup_{t \in T} \hat{\sigma}_1^{nd}[\hat{\sigma}_2^{nd}[\{t\}]] = \hat{\sigma}_1^{nd}\Big[\bigcup_{t \in T} \hat{\sigma}_2^{nd}[\{t\}]\Big] \\
&= \hat{\sigma}_1^{nd}[\hat{\sigma}_2^{nd}[T]] = (\hat{\sigma}_1^{nd} \circ \hat{\sigma}_2^{nd})[T].
\end{aligned}$$

∎

From Lemma 12 we have the binary operation $\circ_{nd}$ is associative.

Lemma 13. *The binary operation $\circ_{nd}$ is associative.*

Proof. Let $\sigma_1^{nd}, \sigma_2^{nd}, \sigma_3^{nd} \in nd\text{-}Hyp^F(\tau_n)$. We have to show that the equation

$$\sigma_1^{nd} \circ_{nd} (\sigma_2^{nd} \circ_{nd} \sigma_3^{nd}) = (\sigma_1^{nd} \circ_{nd} \sigma_2^{nd}) \circ_{nd} \sigma_3^{nd}$$

is satisfied. By Lemma 3.5, we have

$$\begin{aligned}
\sigma_1^{nd} \circ_{nd} (\sigma_2^{nd} \circ_{nd} \sigma_3^{nd}) &= \hat{\sigma}_1^{nd} \circ (\sigma_2^{nd} \circ_{nd} \sigma_3^{nd}) = \hat{\sigma}_1^{nd} \circ (\hat{\sigma}_2^{nd} \circ \sigma_3^{nd}) \\
&= (\hat{\sigma}_1^{nd} \circ \hat{\sigma}_2^{nd}) \circ \sigma_3^{nd} = (\sigma_1^{nd} \circ_{nd} \sigma_2^{nd})\hat{} \circ \sigma_3^{nd} \\
&= (\sigma_1^{nd} \circ_{nd} \sigma_2^{nd}) \circ_{nd} \sigma_3^{nd}.
\end{aligned}$$

∎

Let $\sigma_{id}^{nd} \in nd\text{-}Hyp^F(\tau_n)$. We define $\sigma_{id}^{nd}(f_i) := \{f_i(x_1, \ldots, x_n)\}$ and the next lemma we show that the extension of σ_{id}^{nd} is an identity mapping.

Lemma 14. *Let $T \subseteq W_{\tau_n}^F(X_n)$. Then we have*

$$\hat{\sigma}_{id}^{nd}[T] = T.$$

Proof. If T is empty, then the claim is clearly true. If T is non-empty, then we consider in the following steps.

(1) If T is a singleton, then

Case 1. $T = \{f_i(x_{s(1)}, \dots, x_{s(n)})\}$ where $s \in H_n$, we have

$$\begin{aligned}\hat{\sigma}_{id}^{nd}[T] &= \hat{\sigma}_{id}^{nd}[\{f_i(x_{s(1)}, \dots, x_{s(n)})\}]\\ &= (\sigma_{id}^{nd}(f_i))_s = (\{f_i(x_1, \dots, x_n)\})_s\\ &= \{f_i(x_{s(1)}, \dots, x_{s(n)})\} = T,\end{aligned}$$

Case 2. $T = \{f_i(t_1, \dots, t_n)\}$ where $t_1, \dots, t_n \in W_{\tau_n}^F(X_n)$, and we assume that the equations

$$\hat{\sigma}_{id}^{nd}[\{t_q\}] = \{t_q\}.$$

where $1 \leq q \leq n, n \in \mathbb{N}$ are satisfied, we have

$$\begin{aligned}\hat{\sigma}_{id}^{nd}[T] &= \hat{\sigma}_{id}^{nd}[\{f_i(t_1, \dots, t_n)\}] = S_{nd}^n(\sigma_{id}^{nd}(f_i), \hat{\sigma}_{id}^{nd}[\{t_1\}], \dots, \hat{\sigma}_{id}^{nd}[\{t_n\}])\\ &= S_{nd}^n(\{f_i(x_1, \dots, x_n)\}, \{t_1\}, \dots, \{t_n\})\\ &= \{f_i(r_1, \dots, r_n) \mid r_q \in \{t_q\}\} = \{f_i(t_1, \dots, t_n)\} = T.\end{aligned}$$

(2) If T is an arbitrary subset of $W_{\tau_n}^F(X_n)$, then

$$\hat{\sigma}_{id}^{nd}[T] = \bigcup_{t \in T} \hat{\sigma}_{id}^{nd}[\{t\}] = \bigcup_{t \in T} \{t\} = T.$$

■

Lemma 15. *The σ_{id}^{nd} in nd-$Hyp^F(\tau_n)$ is an identity element in the set nd-$Hyp^F(\tau_n)$ with respect to the associative binary operation $\circ_{nd}$.*

Proof. Let $\sigma^{nd} \in$ *nd-*$Hyp^F(\tau_n)$ and f_i be an operation symbol. We have to show that $(\sigma^{nd} \circ_{nd} \sigma_{id}^{nd})(f_i) = \sigma^{nd}(f_i) = (\sigma_{id}^{nd} \circ_{nd} \sigma^{nd})(f_i)$.

$$\begin{aligned}(\sigma^{nd} \circ_{nd} \sigma_{id}^{nd})(f_i) &= \hat{\sigma}^{nd}[\sigma_{id}^{nd}(f_i)] = \hat{\sigma}^{nd}[\{f_i(x_1, \dots, x_n)\}]\\ &= \sigma^{nd}(f_i) = \hat{\sigma}_{id}^{nd}[\sigma^{nd}(f_i)] = (\sigma_{id}^{nd} \circ_{nd} \sigma^{nd})(f_i).\end{aligned}$$

■

Now we have that:

Theorem 16. *The structure (nd-$Hyp^F(\tau_n); \circ_{nd}, \sigma_{id}^{nd})$ is a monoid.*

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