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AN INJECTIVE PSEUDO-BCI ALGEBRA IS TRIVIAL

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Abstract

Injective pseudo-BCI algebras are studied. There is shown that the only injective pseudo-BCI algebra is the trivial one.

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1. Introduction

In 1966, Imai and Iséki [11, 12] defined two classes of algebras called BCK-algebras and BCI-algebras as algebras connected with some logics. They have connections with BCK/BCI-logic being the BCK/BCI-system in combinatory logic. Next, in 2001, Georgescu and Iorgulescu [10] introduced the notion of pseudo-BCK algebras as an extension of BCK-algebras, and in 2008, Dudek and Jun [2] defined pseudo-BCI algebras as generalization of BCI-algebras as well as pseudo-BCK algebras.

Pseudo-BCI algebras are algebraic models of some extension of a non-commutative version of the BCI-logic. These algebras have also connections with other algebras of logic such as, for instant pseudo-MV algebras [8] and pseudo-BL algebras [9]. So results obtained for pseudo-BCI algebras are, in some sense, fundamental for other algebras of logic.

In [5] the author investigates the category **psBCI** of pseudo-BCI algebras and homomorphisms between them. He shows that the category **psBCI** has zero objects, zero morphisms, products, equalizers, coequalizers, pullbacks and limits, and that it is concrete, complete, is not balanced and is not abelian. Moreover, considering the category $\mathbf{psBCI_p}$ of p-semisimple pseudo-BCI algebras and homomorphisms between them, the author shows in [5] that the category $\mathbf{psBCI_p}$

is isomorphic with the category **Grp** of groups and group homomorphisms and that it is a full and reflective subcategory of the category **psBCI**.

This paper is a continuation of [5]. We investigate the categorical notion of injectivity of pseudo-BCI algebras. Here we show that the trivial pseudo-BCI algebra is the only injective object in the category **psBCI**.

2. Preliminaries

A *pseudo-BCI algebra* is a structure $(X; \leq, \rightarrow, \rightsquigarrow, 1)$, where $\leq$ is binary relation on X, $\rightarrow$ and $\rightsquigarrow$ are binary operations on X and 1 is an element of X such that for all $x, y, z \in X$, we have

(a1) $x \rightarrow y \leq (y \rightarrow z) \rightsquigarrow (x \rightarrow z), \quad x \rightsquigarrow y \leq (y \rightsquigarrow z) \rightarrow (x \rightsquigarrow z)$,

(a2) $x \leq (x \rightarrow y) \rightsquigarrow y, \quad x \leq (x \rightsquigarrow y) \rightarrow y$,

(a3) $x \leq x$,

(a4) if $x \leq y$ and $y \leq x$, then $x = y$,

(a5) $x \leq y$ iff $x \rightarrow y = 1$ iff $x \rightsquigarrow y = 1$.

It is obvious that any pseudo-BCI algebra $(X; \leq, \rightarrow, \rightsquigarrow, 1)$ can be regarded as a universal algebra $(X; \rightarrow, \rightsquigarrow, 1)$ of type $(2, 2, 0)$. Note that every pseudo-BCI algebra satisfying $x \rightarrow y = x \rightsquigarrow y$ for all $x, y \in X$ is a BCI-algebra.

Every pseudo-BCI algebra satisfying $x \leq 1$ for all $x \in X$ is a pseudo-BCK algebra. A pseudo-BCI algebra which is not a pseudo-BCK algebra will be called *proper*.

Troughout this paper we will often use X to denote a pseudo-BCI algebra. Any pseudo-BCI algebra X satisfies the following, for all $x, y, z \in X$,

(b1) if $1 \leq x$, then $x = 1$,

(b2) if $x \leq y$, then $y \rightarrow z \leq x \rightarrow z$ and $y \rightsquigarrow z \leq x \rightsquigarrow z$,

(b3) if $x \leq y$ and $y \leq z$, then $x \leq z$,

(b4) $x \rightarrow (y \rightsquigarrow z) = y \rightsquigarrow (x \rightarrow z)$,

(b5) $x \leq y \rightarrow z$ iff $y \leq x \rightsquigarrow z$,

(b6) $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y), \quad x \rightsquigarrow y \leq (z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow y)$,

(b7) if $x \leq y$, then $z \rightarrow x \leq z \rightarrow y$ and $z \rightsquigarrow x \leq z \rightsquigarrow y$,

(b8) $1 \rightarrow x = 1 \rightsquigarrow x = x$,

(b9) $((x \rightarrow y) \rightsquigarrow y) \rightarrow y = x \rightarrow y, \quad ((x \rightsquigarrow y) \rightarrow y) \rightsquigarrow y = x \rightsquigarrow y$, (b10) $x \to y \leq (y \to x) \rightsquigarrow 1, \quad x \rightsquigarrow y \leq (y \rightsquigarrow x) \to 1,$

(b11) $(x \to y) \to 1 = (x \to 1) \rightsquigarrow (y \rightsquigarrow 1), \quad (x \rightsquigarrow y) \rightsquigarrow 1 = (x \rightsquigarrow 1) \to (y \to 1),$

(b12) $x \to 1 = x \rightsquigarrow 1.$

If $(X; \leq, \to, \rightsquigarrow, 1)$ is a pseudo-BCI algebra, then, by (a3), (a4), (b3) and (b1), $(X; \leq)$ is a poset with 1 as a maximal element. Note that a pseudo-BCI algebra has also other maximal elements.

For any $x \in X$, by (b12), we can define:

$$x^{-} = x \to 1 = x \rightsquigarrow 1.$$

Then, for any $x, y \in X$, we easily have:

(a1') $x \to y \leq y^{-} \rightsquigarrow x^{-}, \quad x \rightsquigarrow y \leq y^{-} \to x^{-},$

(a2') $x \leq (x^{-})^{-},$

(b2') if $x \leq y$, then $y^{-} \leq x^{-}$,

(b5') $x \leq y^{-}$ iff $y \leq x^{-}$,

(b9') $((x^{-})^{-})^{-} = x^{-},$

(b10') $x \to y \leq (y \to x)^{-}, \quad x \rightsquigarrow y \leq (y \rightsquigarrow x)^{-},$

(b11') $(x \to y)^{-} = x^{-} \rightsquigarrow y^{-}, \quad (x \rightsquigarrow y)^{-} = x^{-} \to y^{-}.$

Proposition 2.1 [6]. *The structure $(X; \leq, \to, \rightsquigarrow, 1)$ is a pseudo-BCI algebra if and only if the algebra $(X; \to, \rightsquigarrow, 1)$ of type $(2, 2, 0)$ satisfies the following:*

(i) $(x \to y) \rightsquigarrow [(y \to z) \rightsquigarrow (x \to z)] = 1,$

(ii) $(x \rightsquigarrow y) \to [(y \rightsquigarrow z) \to (x \rightsquigarrow z)] = 1,$

(iii) $1 \to x = x,$

(iv) $1 \rightsquigarrow x = x,$

(v) $x \to y = 1$ *and* $y \to x = 1$, *then* $x = y$.

Example 2.2. Let $X = \{a, b, c, d, 1\}$ and define the binary operations $\to$ and $\rightsquigarrow$ on X by the following tables:

$\to$	a	b	c	d	1
a	1	b	c	c	1
b	a	1	c	d	1
c	c	c	1	b	c
d	c	c	1	1	c
1	a	b	c	d	1

$\rightsquigarrow$	a	b	c	d	1
a	1	b	c	d	1
b	a	1	c	c	1
c	c	c	1	a	c
d	c	c	1	1	c
1	a	b	c	d	1

Then $(X;\to,\rightsquigarrow,1)$ is a (proper) pseudo-BCI algebra. Observe that it is not a pseudo-BCK algebra because $d \nleq 1$.

Example 2.3 [6]**.** Let $Y = \{a,b,c,d,e,f,1\}$ and define the binary operations $\to$ and $\rightsquigarrow$ on Y by the following tables:

$\to$	a	b	c	d	e	f	1
a	1	d	e	b	c	a	a
b	c	1	a	e	d	b	b
c	e	a	1	c	b	d	d
d	b	e	d	1	a	c	c
e	d	c	b	a	1	e	e
f	a	b	c	d	e	1	1
1	a	b	c	d	e	f	1

$\rightsquigarrow$	a	b	c	d	e	f	1
a	1	c	b	e	d	a	a
b	d	1	e	a	c	b	b
c	b	e	1	c	a	d	d
d	e	a	d	1	b	c	c
e	c	d	a	b	1	e	e
f	a	b	c	d	e	1	1
1	a	b	c	d	e	f	1

Then $(Y;\to,\rightsquigarrow,1)$ is a (proper) pseudo-BCI algebra. Observe that it is not a pseudo-BCK algebra because $a \nleq 1$.

Example 2.4 [3]**.** Let $Z = (-\infty,0] \times \mathbb{R}^2$ and define the binary operations $\to$ and $\rightsquigarrow$ on Z by

$$
(x_1,y_1,z_1) \to (x_2,y_2,z_2) =
\begin{cases}
\left(0, y_2-y_1, (z_2-z_1)e^{-y_1}\right) & \text{if } x_1 \leq x_2,\\
\left(\frac{2x_2}{\pi}\arctan\left(\ln\left(\frac{x_2}{x_1}\right)\right), y_2-y_1, (z_2-z_1)e^{-y_1}\right) & \text{if } x_2 < x_1,
\end{cases}
$$

$$
(x_1,y_1,z_1) \rightsquigarrow (x_2,y_2,z_2) =
\begin{cases}
\left(0, y_2-y_1, z_2-z_1e^{y_2-y_1}\right) & \text{if } x_1 \leq x_2,\\
\left(x_2e^{-\tan(\frac{\pi x_1}{2x_2})}, y_2-y_1, z_2-z_1e^{y_2-y_1}\right) & \text{if } x_2 < x_1
\end{cases}
$$

for all $(x_1,y_1,z_1),(x_2,y_2,z_2) \in Z$. Then $(Z;\to,\rightsquigarrow,(0,0,0))$ is a proper pseudo-BCI algebra. Notice that Z is not a pseudo-BCK algebra because there exists $(x,y,z) = (0,1,1) \in Z$ such that $(x,y,z) \nleq (0,0,0)$.

Example 2.5. Let W be the set of all bijections $f : \mathbb{N} \to \mathbb{N}$. Define the binary operations $\to$ and $\rightsquigarrow$ on W by

$$
f \to g = gf^{-1},
$$
$$
f \rightsquigarrow g = f^{-1}g
$$

for all $f,g \in W$. Then the algebra $(W;\to,\rightsquigarrow,id_{\mathbb{N}})$ is a proper pseudo-BCI algebra.

For any pseudo-BCI algebra $(X; \to, \rightsquigarrow, 1)$, the set

$$K(X) = \{x \in X : x \leq 1\}$$

is a subalgebra of X (called the pseudo-BCK part of X). Then $(K(X); \to, \rightsquigarrow, 1)$ is a pseudo-BCK algebra. Note that a pseudo-BCI algebra X is a pseudo-BCK algebra if and only if $X = K(X)$.

It is easily seen that for the pseudo-BCI algebras X, Y, Z and W from Examples 2.2, 2.3, 2.4 and 2.5 we have $K(X) = \{a, b, 1\}$, $K(Y) = \{f, 1\}$, $K(Z) = \{(x, 0, 0) : x \leq 0\}$ and $K(W) = \{id_{\mathbb{N}}\}$, respectively.

We will denote by $M(X)$ the set of all maximal elements of X and call it the p-semisimple part of X. Obviously, $1 \in M(X)$. Notice that $M(X) \cap K(X) = \{1\}$. Indeed, if $a \in M(X) \cap K(X)$, then $a \leq 1$ and a is a maximal element of X, which means that $a = 1$. Moreover, observe that 1 is the only maximal element of a pseudo-BCK algebra. Therefore, for a pseudo-BCK algebra X, $M(X) = \{1\}$. In [4] and [3] there is shown that $M(X) = \{x \in X : x = (x^-)^-\}$ and it is a subalgebra of X.

Observe that for the pseudo-BCI algebras X, Y, Z and W from Examples 2.2, 2.3, 2.4 and 2.5 we have $M(X) = \{c, 1\}$, $M(Y) = \{a, b, c, d, e, 1\}$, $M(Z) = \{(0, y, z) : y, z \in \mathbb{R}\}$ and $M(W) = W$, respectively.

Let $(X; \to, \rightsquigarrow, 1)$ be a pseudo-BCI algebra. Then X is *p-semisimple* if it satisfies, for all $x \in X$,

$$\text{if } x \leq 1, \text{ then } x = 1.$$

Note that X is a p-semisimple pseudo-BCI algebra if and only if $K(X) = \{1\}$. Hence, if X is a p-semisimple pseudo-BCK algebra, then $X = \{1\}$. Moreover, as it is proved in [3], $M(X)$ is a p-semisimple pseudo-BCI subalgebra of X and the following are equivalent: (1) X is p-semisimple, (2) $X = M(X)$, (3) $(x \to y) \rightsquigarrow y = x = (x \rightsquigarrow y) \to y$ for any $x, y \in X$.

It is not difficult to see that the pseudo-BCI algebras X, Y and Z from Examples 2.2, 2.3 and 2.4, respectively, are not p-semisimple and the pseudo-BCI algebra W from Example 2.5 is p-semisimple.

3. Injective pseudo-BCI algebras

The reader can find in [1] all notions from the category theory occuring in this section.

An object Q in a category $\mathbf{C}$ is called *injective* if for any morphism $f : X \to Q$ and any monomorphism $g : X \to Y$ there is a morphism $h : Y \to Q$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[dl, dotted, "h"] \\ Q & \end{tikzcd}$$

commutes, that is, $h \circ g = f$.

Let **psBCK** and **psBCI** denote the categories of pseudo-BCK algebras and pseudo-BCI algebras, respectively, and their corresponding homomorphisms. In [5] we have shown the following fact.

Proposition 3.1 [5]**.** *In the category* **psBCI***, the injective morphisms and monomorphisms coincide.*

Remark. In fact the same is true in the category **psBCK** (see [5]).

First, we study injective objects in the category **psBCK**. The following fact will be needed in the sequel.

Proposition 3.2. *Let X be a pseudo-BCK algebra and let $\delta \notin X$. Then $X \cup \{\delta\}$ is a bounded pseudo-BCK algebra with δ as the smallest element, where $x \to \delta = x \rightsquigarrow \delta = \delta$, $\delta \to x = \delta \rightsquigarrow x = 1$ and $\delta \to \delta = \delta \rightsquigarrow \delta = 1$ for any $x \in X$.*

Proof. Axioms of a pseudo-BCK algebra can be verified by routine calculation. ∎

In order to prove the next theorem, the notion of a retraction will be useful. A morphism $f : X \to Y$ is called a *retraction* if there exists a morphism $g : Y \to X$ such that $f \circ g = id_Y$.

Theorem 3.3. *An object X is injective in the category* **psBCK** *if and only if $X = \{1\}$.*

Proof. It is obvious that $\{1\}$ is injective in **psBCK**. Conversely, assume that X is injective in **psBCK**. Consider a bounded pseudo-BCK algebra $X \cup \{\delta\}$, where $\delta \notin X$, as in Proposition 3.2. Since the inclusion map $i : X \to X \cup \{\delta\}$ is an injective morphism in **psBCK**, it is a monomorphism. Hence and by the fact that X is injective there is a retraction $r : X \cup \{\delta\} \to X$ such that $r \circ i = id_X$:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "id_X"'] & X \cup \{\delta\} \arrow[dl, dotted, "r"] \\ X & \end{tikzcd}$$

Thus we have $r(x) = x$ for any $x \in X$. Now, let $z = r(\delta) \in X$. We have

$$z = r(\delta) = r(z \to \delta) = r(z) \to r(\delta) = z \to z = 1,$$

that is, $r(\delta) = 1$. Hence, for any $x \in X$, we get

$$1 = r(1) = r(\delta \to x) = r(\delta) \to r(x) = 1 \to x = x.$$

Therefore, $X = \{1\}$. ■

Now, we investigate injective objects in the category **psBCI**.

Theorem 3.4. *If X is injective in the category* **psBCI**, *then it is p-semisimple.*

Proof. First, we prove that if X is injective in **psBCI**, then $K(X)$ is injective in **psBCK**. Let $f : Y \to K(X)$ be a morphism in **psBCK** and $g : Y \to Z$ be a monomorphism in **psBCK**. Then we have a morphism $i \circ f : Y \to X$ in **psBCI**, where $i : K(X) \to X$ is the inclusion. Since g is a monomorphism in **psBCK**, it is an injective morphism, that is, it is a monomorphism in **psBCI**. Since X is injective in **psBCI**, there is a morphism $\overline{h} : Z \to X$ in **psBCI** such that

$$\overline{h} \circ g = i \circ f.$$

But Z is an object in **psBCK** whence $z \leq 1$ for any $z \in Z$. Thus $\overline{h}(z) \leq 1$ for any $z \in Z$, that is, $\overline{h}(z) \in K(X)$ for any $z \in Z$. So, $\overline{h}$ determines a morphism $h : Z \to K(X)$ in **psBCK** such that

$$i \circ h = \overline{h}.$$

The following diagram illustrates the situation:

$$\begin{tikzcd}
Y \arrow[r, "g"] \arrow[d, "f"'] & Z \arrow[dl, dotted, "h"] \arrow[ddl, dotted, "\overline{h}"] \\
K(X) \arrow[d, "i"'] & \\
X &
\end{tikzcd}$$

Hence, we have $i \circ h \circ g = \overline{h} \circ g = i \circ f$. Since i is a monomorphism in **psBCI**, we get

$$h \circ g = f.$$

Thus, $K(X)$ is injective in **psBCK**. Now, by Theorem 3.3, $K(X) = \{1\}$. This means that X is p-semisimple. ■

Denote by $\mathbf{psBCI_p}$ the category of p-semisimple pseudo-BCI algebras and homomorphisms between them.

Corollary 3.5. *If X is injective in* $\mathbf{psBCI}$*, then it is injective in* $\mathbf{psBCI_p}$*.*

Now, we have the following two facts.

Proposition 3.6 [5]**.** *The category* $\mathbf{psBCI_p}$ *is isomorphic with the category* $\mathbf{Grp}$ *of groups and group homomorphisms.*

Proposition 3.7 [7]**.** *The only injective object in the category* $\mathbf{Grp}$ *is the trivial group.*

From Corollary 3.5 and Propositions 3.6 and 3.7 we obtain the following.

Theorem 3.8. *An object X is injective in the category* $\mathbf{psBCI}$ *if and only if* $X = \{1\}$*.*

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