Discussiones Mathematicae
General Algebra and Applications 39 (2019) 153–164
doi:10.7151/dmgaa.1310

ON EQUALITY OF CERTAIN DERIVATIONS OF LIE ALGEBRAS

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Abstract

Let L be a Lie algebra. A derivation α of L is a commuting derivation (central derivation), if $\alpha(x) \in C_L(x)$ $(\alpha(x) \in Z(L))$ for each $x \in L$. We denote the set of all commuting derivations (central derivations) by $\mathcal{D}(L)$ $(Der_z(L))$. In this paper, first we show $\mathcal{D}(L)$ is subalgebra from derivation algebra L, also we investigate the conditions on the Lie algebra L where commuting derivation is trivial and finally we introduce the family of nilpotent Lie algebras in which $Der_z(L) = \mathcal{D}(L)$.

Keywords: derivation, central derivation, centralizer, commuting derivation.

2010 Mathematics Subject Classification: Primary: 17B40, 16W25; Secondary: 17B99.

1. Introduction

Let G be a group, and let $Aut(G)$ be the group of all automorphisms of G. Let

$$\mathcal{A}(G) = \{\alpha \in Aut(G) | x\alpha(x) = \alpha(x)x \text{ for all } x \in G\}.$$

Any element of this set is called a commuting automorphism. This definition was first considered for rings (see [1, 5, 10]).

The following question was raised by Herstein in [8]: Under which conditions $\mathcal{A}(G) = \{1\}$? he find if G is a simple nonabelian group, then $\mathcal{A}(G) = \{1\}$. Afterward, Laffey [9] and Pettet [12] separately extended the result of Herstein. Deaconescu, Silberberg, and Walls in [3] showed that $\mathcal{A}(G)$ is not generally a subgroup of $Aut(G)$ for example see page 425. They raised this question: In which family of groups, $\mathcal{A}(G)$ is a subgroup of $Aut(G)$? For more information, we refer to [4, 6, 16].

In this paper, we work on the structure of Lie algebras and their derivations, in the first we introduce derivation of a Lie algebra. Suppose L be a Lie algebra over an arbitrary field F with bracket $[-,-]$, a Linear transformation $\alpha : L \longrightarrow L$ is a derivation when we have $\alpha([x,y]) = [\alpha(x), y] + [x, \alpha(y)]$ for all $x, y \in L$. The set of all derivations L denote by $Der(L)$ that itself forms a Lie algebra over field F. We define

$$\mathcal{D}(L) = \{\alpha \in Der(L) \mid \alpha(x) \in C_L(x) \quad \text{for all } x \in L\},$$

where $C_L(x)$ is the centralizer of x in L, and each element of the above set is called *commuting derivation*. The study of the commuting derivations for this reason is interesting that in spite of this in groups $A(G)$ is not generally a subgroup from $Aut(G)$ but in Lie algebras $\mathcal{D}(L)$ is always a subalgebra of $Der(L)$ also we answer this question under what conditions $\mathcal{D}(L) = \{0\}$.

A derivation α of a Lie algebra L is called a *central derivation* if its image is contained in the center of L. The set of all central derivations is denoted by $Der_z(L)$. It is easy to see that $Der_z(L)$ is an ideal of $Der(L)$. A Lie algebra L is called *Heisenberg* when $L^2 = Z(L)$ and $dimL^2 = 1$. Heisenberg Lie Algebras are from odd dimension. Such algebras denote by $H(k)$ wherein $dimH(k) = 2k+1$ for all $k \in \mathbb{N}$. Now, the question arises as to whether $Der_z(L) = \mathcal{D}(L)$, in following we give the example that always this is not going to happen.

Example 1.1. Consider the Lie algebra $H(1)$, which has the following representation:

$$H(1) = \langle x_1, x_2, x_3 \mid [x_1, x_2] = x_3 \rangle .$$

For the Lie algebra $H(1)$, the matrix representations each element of $Der_z(H(1))$ and $\mathcal{D}(H(1))$ are as the following forms, respectively:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha_{31} & \alpha_{32} & 0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_{11} & 0 & 0 \\ 0 & \alpha_{11} & 0 \\ \alpha_{31} & \alpha_{32} & 2\alpha_{11} \end{bmatrix}.$$

As we can see $Der_z(H(1)) < \mathcal{D}(H(1))$ and the equality does not hold in general.

For each Lie algerba L, $Der_z(L) \subseteq \mathcal{D}(L) \subseteq Der(L)$. Therefore another interesting question that arises is that in what kind of Lie algebras equal are happening? (i.e., $Der_z(L) = \mathcal{D}(L)$). Moreover, $Der_z(L) = \mathcal{D}(L) = Der(L)$ whenever L is an abelian Lie algebra. To see properties of ideal $Der_z(L)$ of $Der(L)$, we refer to [13, 14, 15].

Let L be a Lie algebra, and consider the set

$$R_2(L) = \{x \in L \mid [[x, y], y] = 0 \text{ for all } y \in L\}$$

of right two-Engel elements of L.

Throughout the paper, we will use the following notion. Given a Lie algebra L, the upper and the lower central series of L are denoted by $Z_i(L)$ and L^i, respectively. The nilpotency class of Lie algebra is denoted by $cl(L)$. If L is an n-dimensional nilpotent Lie algebra of class m, we will write $cocl(L) = n - m$. A Lie algebra L of dimension n is filiform if $\dim L^i = n - i$ $(2 \leq i \leq n)$. For these algebras, we have $cocl(L) = 1$. De Graaf [7], Cicalo and *et al.* [2] classify nilpotent Lie algebras from dimension at most six. They display such algebras in the form of $L_{n,k}$ wherein n is dimension of L and k is its index among the nilpotent Lie algebras with dimension n.

In this paper, we prove $R_2(L)$ and $\mathcal{D}(L)$ have the Lie algebra structure; besides, we show that both of them are strictly related together. If $Der_z(L) = \{0\}$, maybe $\mathcal{D}(L) \neq \{0\}$. In the second section we study some properties of $R_2(L)$ and $\mathcal{D}(L)$, and then we find the relation between them. Afterward, we expose the conditions under which $Der_z(L) = \{0\}$ implies $\mathcal{D}(L) = \{0\}$. In section 3, we find a family of nilpotent Lie algebras L such that $Der_z(L) = \mathcal{D}(L)$.

2. Some Properties of $R_2(L)$ and $\mathcal{D}(L)$

In this section, we state some fundamental facts which will use in what follows. For instance, we show that $\mathcal{D}(L)$ is a Lie subalgebra of $Der(L)$. After that, we assert some properties of $R_2(L)$ and $\mathcal{D}(L)$.

Let L be a Lie algebra. Given $x \in L$, ad_x, i.e., $ad_x(-) = [x, -]$ is an inner derivation of L induced by x, and the set of all inner derivation of L is denoted by $ad(L)$, which is an ideal of $Der(L)$. Evidently, $x \in R_2(L)$ if and only if $ad_x \in \mathcal{D}(L)$. Hence from (ii) next lemma it directly follows that $R_2(L)$ is a Lie subalgebra of L.

Lemma 2.1. *Let L be a Lie algebra.*

(i) *If $\alpha \in \mathcal{D}(L)$ and $x, y \in L$, then $[\alpha(x), y] = [x, \alpha(y)]$.*

(ii) *$\mathcal{D}(L)$ is a Lie subalgebra from $Der(L)$.*

(iii) *Under each derivation of L, $R_2(L)$ is invariant.*

Proof. (i) According to the definition of $\mathcal{D}(L)$, we have $[\alpha(x-y), x-y] = 0$. Thus $[\alpha(x), y] = [x, \alpha(y)]$.

(ii) If the characteristic of field is 2 then (ii) follows directly from (i), let $\alpha \in \mathcal{D}(L)$ and $x, y \in L$, if $[x, y] = 0$ (and characteristic of field is not 2), then we have

$$0 = \alpha([x, y]) = [\alpha(x), y] + [x, \alpha(y)] = 2[\alpha(x), y]$$

where last equality occur by (i), now if $\alpha, \beta \in \mathcal{D}(L)$ then we get $[\alpha(\beta(x)), x] = 0$, since $[\beta(x), x] = 0$. Similarly $[\beta(\alpha(x)), x] = 0$, and so

$$[[\alpha, \beta](x), x] = [\alpha(\beta(x)), x] - [\beta(\alpha(x)), x] = 0$$

(iii) Assume that $x \in R_2(L)$, $y \in L$, and $\alpha \in Der(L)$. Using (i) we infer that,

$$\begin{aligned}
0 = \alpha(0) &= \alpha([[x, y], y]) \\
&= [\alpha([x, y]), y] + [[x, y], \alpha(y)] \\
&= [[\alpha(x), y], y] + [[x, \alpha(y)], y] + [[x, y], \alpha(y)] \\
&= [[\alpha(x), y], y] + [ad_x(\alpha(y)), y] + [ad_x(y), \alpha(y)] \\
&= [[\alpha(x), y], y] + [\alpha(y), ad_x(y)] - [\alpha(y), ad_x(y)] \\
&= [[\alpha(x), y], y],
\end{aligned}$$

which means $\alpha(x) \in R_2(L)$, and the proof is completed. ■

Lemma 2.2. *Let L be a Lie algebra over a field F, and let $\alpha \in \mathcal{D}(L)$. Then*

(i) *$\alpha(L)$ is a Lie subalgebra of L.*

(ii) *$\alpha(L) \leqslant R_2(L)$.*

(iii) *$[\alpha(x), [y, z]] = [\alpha(y), [x, z]]$ for each $x, y, z \in L$.*

(iv) *If F has characteristic not 2, then $\alpha(L) \leqslant R_2(L) \bigcap C_L(L^2)$.*

Proof. (i) Let $x, y \in L$, then

$$\begin{aligned}
\alpha^2([x, y]) &= 2\left(\left[\alpha^2(x), y\right] + \left[x, \alpha^2(y)\right]\right) \\
&= 2([\alpha(x), \alpha(y)] + [\alpha(x), \alpha(y)]) \\
&= 4[\alpha(x), \alpha(y)].
\end{aligned}$$

Whence

$$[\alpha(x), \alpha(y)] = \alpha^2(1/4[x, y]).$$

Thus $[\alpha(x), \alpha(y)] \in \alpha^2(L)$. The fact that $\alpha^2(L) \leqslant \alpha(L)$ finishes the proof.

(ii) By Lemma 2.1(i), for each $x, y \in L$, we have

$$\begin{aligned}[][\alpha(x), y], y] &= [[x, \alpha(y)], y] \\ &= -[[y, x], \alpha(y)] - [[\alpha(y), y], x] \\ &= [[x, y], \alpha(y)] \\ &= [\alpha([x, y]), y] \\ &= [[\alpha(x), y], y] + [[x, \alpha(y)], y].\end{aligned}$$

Consequently,

$$[[\alpha(x), y], y] = [[x, \alpha(y)], y] = 0.$$

(iii) We have

$$\begin{aligned}[][x, y], \alpha(z)] &= ad_{[x,y]}(\alpha(z)) \\ &= [ad_x, ad_y](\alpha(z)) \\ &= ad_x \circ ad_{\alpha(y)}(z) - ad_y \circ ad_{\alpha(x)}(z).\end{aligned} \tag{2.1}$$

On the other hand, by the Jacobi identity, we can write

$$\begin{aligned}[]\alpha([x, y]), z] &= [[\alpha(x), y], z] + [[x, \alpha(y)], z] \\ &= ad_{[\alpha(x), y]}(z) + ad_{[x, \alpha(y)]}(z) \\ &= ad_{\alpha(x)} \circ ad_y(z) - ad_y \circ ad_{\alpha(x)}(z) \\ &\quad + ad_x \circ ad_{\alpha(y)}(z) - ad_{\alpha(y)} \circ ad_x(z).\end{aligned} \tag{2.2}$$

Lemma 2.1(i) implies that (2.1) and (2.2) are equal; so

$$ad_{\alpha(x)} \circ ad_y(z) = ad_{\alpha(y)} \circ ad_x(z),$$

which means that

$$[\alpha(x), [y, z]] = [\alpha(y), [x, z]].$$

(iv) By the part (iii), we have, for each $x, y, z \in L$,

$$\begin{aligned}[]\alpha(x), [y, z]] &= [\alpha(y), [x, z]] \\ \Rightarrow [x, \alpha([y, z])] &= [y, \alpha([x, z])] \\ \Rightarrow 2[x, ([\alpha(y), z])] &= 2[[z, \alpha(x)], y] \\ \Rightarrow [x, ([y, \alpha(z)])] &= [[\alpha(z), x], y] = -[y, [\alpha(z), x]].\end{aligned}$$

The Jacobi identity implies that

$$[x, ([y, \alpha(z)])] = -[\alpha(z), [x, y]] - [y, [\alpha(z), x]],$$

so $[\alpha(z), [x, y]] = 0$. ∎

In the following lemma we give some properties of the elements of $\mathcal{D}(L)$.

Lemma 2.3. *Let L be a Lie algebra over the field F, and let $\alpha, \beta \in \mathcal{D}(L)$.*
(i) *$[\alpha, \beta]$ kills all the elements in L^2.*
(ii) *If the characteristic of F is not equal to two, then $[\alpha, \beta] \in Der_z(L)$.*

Proof. (i) It is enough to prove that $\alpha\beta([x, y]) = \beta\alpha([x, y])$ for each $x, y \in L$. One can see that

$$\begin{aligned}\alpha(\beta([x, y])) &= \alpha(2[x, \beta(y)]) \\ &= 4[\alpha(x), \beta(y)] \\ &= \beta(2[\alpha(x), y]) \\ &= \beta\alpha([x, y]).\end{aligned}$$

(ii) Assume that $x, y \in L$. Since $\mathcal{D}(L)$ is a Lie subalgebra of $Der(L)$, by Lemma 2.1(i), we have

$$[[\alpha, \beta](x), y] = [x, [\alpha, \beta](y)].$$

So

$$[\alpha\beta(x), y] - [\beta\alpha(x), y] = [x, \alpha\beta(y)] - [x, \beta\alpha(y)]$$

$$2[\beta(x), \alpha(y)] = 2[\alpha(x), \beta(y)]$$

$$2[[\alpha, \beta](x), y] = 0.$$

The result follows from the fact that $char(F) \neq 2$. ■

Theorem A. *Let L be a Lie algebra. Then $\mathcal{D}(L)$ is an ideal of $Der(L)$.*

Proof. Assume that $\alpha \in \mathcal{D}(L)$, $\beta \in Der(L)$, and $x \in L$. Therefore,

$$\begin{aligned}[[\alpha, \beta](x), x] &= [\alpha\beta(x), x] - [\beta\alpha(x), x] \\ &= [\beta(x), \alpha(x)] + [x, \beta\alpha(x)] \\ &= \beta[x, \alpha(x)] \\ &= \beta(0) \\ &= 0.\end{aligned}$$
■

In the following, we aim to find the conditions under which $Der_z(L) = \{0\}$ implies $\mathcal{D}(L) = \{0\}$.

Theorem B. *Let L be a Lie algebra over the field F with no nonzero central derivations. Then $\mathcal{D}(L) = \{0\}$ if and only if $R_2(L) = Z(L)$. Moreover, if L is a perfect Lie algebra $(L = L^2)$ and $char(F) \neq 2$, then $R_2(L) = Z(L)$.*

Proof. If $\mathcal{D}(L)=\{0\}$, then $R_2(L)=Z(L)$ since $ad(L)\bigcap\mathcal{D}(L)\cong\frac{R_2(L)}{Z(L)}$. Now, if $R_2(L)=Z(L)$, then, by Lemma 2.2(ii) $\alpha(L)\leqslant Z(L)$ then, $\mathcal{D}(L)\subseteq Der_z(L)$, and so $\mathcal{D}(L)=\{0\}$. In addition, if $L=L^2$ and since central derivations vanish the elements in L^2, then $Der_z(L)=\{0\}$. On the other hand, by Lemma 2.2(iv) $C_L(L^2)=C_L(L)=Z(L)$ then $\alpha(L)\leqslant Z(L)$. Since $Der_z(L)=\{0\}$, $\mathcal{D}(L)\subseteq Der_z(L)$. Thus $\mathcal{D}(L)=\{0\}$, and therefore $R_2(L)=Z(L)$. ∎

As direct consequences of above theorem, we have the following corollaries.

Corollary 2.1. *Let L be a Lie algebra such that $Z(L)=\{0\}$ and $\mathcal{D}(L)\neq\{0\}$. Then $R_2(L)\neq\{0\}$.*

Corollary 2.2. *Let L be a Lie algebra such that $R_2(L)=\{0\}$. Then $R_2(Der(L))=\{0\}$.*

Proof. First we claim that if L is a Lie algebra such that $Z(L)=\{0\}$, then $R_2(Der(L))\subseteq\mathcal{D}(L)$. Actually, if $\alpha\in R_2(Der(L))$, then $ad_{[\alpha(x),x]}=[[\alpha,ad_x],ad_x]=0$ for each $x\in L$. On the other hand, $Z(L)=\{0\}$ implies that $\alpha\in\mathcal{D}(L)$, and so the claim is valid. Now, if $R_2(L)=\{0\}$, then $Z(L)=\{0\}$ and thus $\mathcal{D}(L)=\{0\}$. On account of the previous theorem, hence $R_2(Der(L))=\{0\}$. ∎

Example 2.1. Let L is a ten-dimensional Lie algebra generated by $\{x_1,x_2,x_3,\ldots,x_{10}\}$ and the nonzero brackets between basis elements are $[x_1,x_2]=2x_2,[x_1,x_3]=-2x_3,[x_1,x_4]=3x_4,[x_1,x_5]=x_5,[x_1,x_6]=-x_6,[x_1,x_7]=-3x_7,[x_1,x_8]=x_8,[x_1,x_9]=-x_9,[x_2,x_3]=x_1,[x_2,x_5]=3x_4,[x_2,x_6]=2x_5,[x_2,x_7]=x_6,[x_3,x_4]=x_5,[x_3,x_5]=2x_6,[x_3,x_6]=3x_7$ and $[x_8,x_9]=x_{10}$. Then $L^2=L$ and $Z(L)=R_2(L)=\langle x_{10}\rangle$ and the matrix representation each of element in $Der(L)$ is as following:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3\alpha_{53} & \frac{3}{2}\alpha_{42} & -\frac{3}{2}\alpha_{43} & \alpha_{44} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}\alpha_{52} & \alpha_{53} & 0 & \alpha_{44}-\alpha_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_{62} & 0 & 0 & 0 & \alpha_{44}-2\alpha_{22} & 0 & 0 & 0 & 0 \\ -3\alpha_{62} & -\frac{3}{2}\alpha_{72} & 0 & 0 & 0 & 0 & \alpha_{44}-3\alpha_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_{88} & -\alpha_{89} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha_{98} & \alpha_{99} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_{88}+\alpha_{99} \end{bmatrix}$$

A simple verification shows that $Der_z(L)=\{0\}$, thus $\mathcal{D}(L)=R_2(Der(L))=\{0\}$.

3. THE FAMILIY OF NILPOTENT LIE ALGEBRAS IN WHICH $Der_z(L)=\mathcal{D}(L)$

Let $\mathcal{X}_{n,m}$ be the family of n-dimensional nilpotent Lie algebras L such that $cocl(L)=m$ and $\dim L^{n-m}=m$ and $\mathcal{X}=\bigcup_{n\geqslant 3,\,m<n}\mathcal{X}_{n,m}$. So we have two

chains below

$$L \supseteq L^2 \supseteq L^3 \supseteq \cdots \supseteq L^{n-m} \supseteq L^{n-m+1} = \{0\}$$

$$\{0\} \subseteq Z(L) \subseteq Z_2(L) \subseteq Z_3(L) \subseteq \cdots \subseteq Z_{n-m-1}(L) \subseteq Z_{n-m}(L) = L$$

If $L \in \mathcal{X}$, then exist n, m such that $L \in \mathcal{X}_{n,m}$ and $\dim L^i = n - i \, (2 \leqslant i \leqslant n - m)$ and $\dim Z_j(L) = j + m - 1 \, (1 \leqslant j \leqslant n - m - 1)$; so $L^{n-m} = Z(L)$. Put $\mathcal{F} = \bigcup_{n \geqslant 3} \mathcal{X}_{n,1}$ denote all *filiform* Lie algebras, then $\mathcal{F}$ is in $\mathcal{X}$.

In this section, we prove that all Lie algebras in $\mathcal{X}$ satisfy in the condition $Der_z(L) = \mathcal{D}(L)$ except $H(1)$. Let L be an n-dimensional Lie algebra with the ordered basis $\{x_1, \ldots, x_n\}$ and $\alpha \in gl(L)$. If we put $\alpha(x_j) = \sum_{i=1}^{n} \alpha_{ij} x_i$ for all $1 \leq i \leq n$, then α has the following matrix representation:

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{bmatrix}.$$

Proposition 3.1. *Let $L \in \mathcal{X}$, and let I be an ideal of L and suppose that $\dim \frac{L}{I} = r$ where $2 \leqslant r \leqslant n - m$. Then $I = L^r$.*

Proof. Since $\left(\frac{L}{I}\right)^r = 0_{\frac{L}{I}}$, we have $L^r \subseteq I$. On the other hand, $r = \dim \frac{L}{L^r} = \dim \frac{L}{I}$ which means $I = L^r$. ■

Assume that A and B are two Lie algebras, then $T(A, B)$ is the set of all Linear transformations from A to B.

Definition 3.1. Let L be a nilpotent Lie algebra and dimension $n \geqslant 4$. We defin two-steps centralizer C_i for all $2 \leqslant i \leqslant n - 2$ as follows

$$C_i = C_L\left(L^i / L^{i+2}\right) = \left\{x \in L \mid [x, y] \in L^{i+2}, \forall y \in L^i\right\}$$

Proposition 3.2. *Let $L \in \mathcal{X}$. Then $C_2 = C_L\left(L^2/L^4\right)$ is a maximal subalgebra of L.*

Proof. One can see that

$$L \supseteq C_2 \supseteq L^2 \supseteq L^3 \supseteq \cdots \supseteq L^{n-m} \supseteq L^{n-m+1} = \{0\}. \tag{1}$$

Let us consider the following map:

$$\begin{cases} \psi : L \to T\left(L^2/L^3, L^3/L^4\right) \\ x \mapsto \psi_x, \end{cases}$$

where

$$\begin{cases} \psi_x : L^2/L^3 \to L^3/L^4 \\ y + L^3 \mapsto [x, y] + L^4. \end{cases}$$

It is easy to check that ψ is a Lie homomorphism and $\ker\psi = C_2$. So $L/C_2 \cong Im\psi$, consequently $\dim(L/C_2) \leqslant dimT\left(L^2/L^3, L^3/L^4\right) = 1$, since $L \in \mathcal{X}$, we have $\dim(L/C_2) = 1$; so $\dim C_2 = n-1$ and $L^2 < C_2$. ∎

Theorem 3.1. *Let $L \in \mathcal{X}$ and $x \notin C_2$. Then $C_L(x) = \langle x\rangle + Z(L)$.*

Proof. According to the sequence (1), we put $P_0 = L$, $P_1 = C_2$ and $P_i = L^i\,(2 \leqslant i \leqslant n-m)$. Therefore, $dimP_i = n-i$ and $\dim(P_{i-1}/P_i) = 1(1 \leqslant i \leqslant n-m)$, where $P_{n-m} = Z(L)$. Now suppose $x_0 = x$ hence we have $P_{i-1} = \langle x_{i-1}, x_i, \ldots, x_{n-1}\rangle$, where $x_i = [x_{i-1}, x]$ and $2 \leqslant i \leqslant n-m$. We have $L = \langle x, P_1\rangle$. Assume that $y \in C_L(x)$. So there exist $\alpha, \alpha_1, \ldots, \alpha_{n-1} \in F$ such that $y = \alpha x + \alpha_1 x_1 + \cdots + \alpha_{n-1}x_{n-1}$. Besides

$$\begin{aligned}[x,y] &= \alpha\,[x,x] + \alpha_1\,[x,x_1] + \cdots + \alpha_{n-m-1}\,[x,x_{n-m-1}]\\ &\quad + \alpha_{n-m}\,[x,x_{n-m}] + \cdots + \alpha_{n-1}\,[x,x_{n-1}]\\ &= \alpha_1 x_2 + \alpha_2 x_3 + \cdots + \alpha_{n-m-1}x_{n-m} + \alpha_{n-m}x_{n-m+1} + \cdots + \alpha_{n-1}x_n.\end{aligned}$$

Since $[x,y] = 0$, $x_{n-m}, \ldots, x_{n-1} \in Z(L)$ and $[x, x_{i-1}] \neq 0$ for $2 \leqslant i \leqslant n-m$, then $\alpha_1 = \alpha_2 = \cdots = \alpha_{n-m-1} = 0$, consequently, $y \in \langle x, Z(L)\rangle$. ∎

Corollary 3.1. *Let L be a filiform Lie algebra, and let $x \notin C_2$. Then $C_L(x) = \langle x\rangle + Z(L)$.*

As we mentioned previously, for each Lie algebra, we have $Der_z(L) \leqslant \mathcal{D}(L)$. The following example shows that the equality also occurs.

Example 3.1. Assume that $L_{5,9}$ is a five-dimensional nilpotent Lie algebra with the ordered basis $\{x_1, x_2, \ldots, x_5\}$ and its nonzero brackets are as follows:

$$[x_1,x_2] = x_3,\ \ [x_1,x_3] = x_4 \text{ and } [x_2,x_3] = x_5.$$

We have $L^2 = \langle x_3, x_4, x_5\rangle$, $L^3 = \langle x_4, x_5\rangle$ and $L^4 = \{0\}$, then *coclass* (L)=2 and $dimL^i = 5-i$ for all $2 \leqslant i \leqslant 5-2$, also $Z(L) = \langle x_4, x_5\rangle$, $Z_2(L) = \langle x_3, x_4, x_5\rangle$ and $Z_3(L) = L$, thus $dimZ_j(L) = j+2-1$ for all $1 \leqslant j \leqslant 5-2-1$, therefore $L \in \mathcal{X}$.

By a simple calculation, we can show that the matrix representations of each element in $Der_z(L)$ and $\mathcal{D}(L)$ are equal and have the following form:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ \alpha_{41} & \alpha_{42} & 0 & 0 & 0\\ \alpha_{51} & \alpha_{52} & 0 & 0 & 0\end{bmatrix};$$

so $Der_z(L) = \mathcal{D}(L)$.

Remark 3.1. Only non-abelian nilpotent Lie algebra of dimension n that $n \leqslant 3$ is $H(1)$, where $H(1) \in \mathcal{X}$ but in Example 1.1 is shown that $Der_z(H(1)) \neq \mathcal{D}(H(1))$.

The following theorem, which is the main result of the paper, shows that for each $L \in \mathcal{X}$, $Der_z(L) = \mathcal{D}(L)$.

Theorem C. *Let $L \in \mathcal{X}$ be a Lie algebra of dimension $n \geqslant 4$ over the filed F and $char(F) \neq 2$. Then $Der_z(L) = \mathcal{D}(L)$.*

Proof. First we express that in nilpotent Lie algebras we have $\varphi(L) = L^2$ wherein $\varphi(L)$ is *Frattini* subalgebra of L [11]. Assume that $x \in L \backslash C_2$ and that $x_1 \in C_2 \backslash L^2$. Therefore, $L = \langle x, x_1 \rangle$, and, for each $\alpha \in \mathcal{D}(L)$, we have $\alpha(x) \in C_L(x)$. Hence by Theorem 3.1, there exist a scalar $t \in F$ and $z \in L^{n-m}$ such that $\alpha(x) = tx + z$. We claim that $t = 0$, otherwise, by Lemma 2.2(iv) $\alpha(x) \in C_L(L^2)$, and therefore

$$\begin{aligned} 0 &= [\alpha(x), [x, x_1]] \\ &= [tx + z, [x, x_1]] \\ &= t[x, [x, x_1]], \end{aligned}$$

which means $L^3 = 0$ and $\dim L = 3$. So $L = H(1)$, and therefore we have a contradiction; so $\alpha(x) \in L^{n-m}$. On account of Lemma 2.1(i), we have $[\alpha(x_1), x] = [x_1, \alpha(x)]$, namely, $\alpha(x_1) \in C_L(x)$. Similarly, $\alpha(x_1) \in L^{n-m}$, and therefore $\alpha \in Der_z(L)$. ■

We are emphasizing that in the above theorem the condition $\dim L^{n-m} = m$ is necessary. To illustrate, consider the following example.

Example 3.2. Assume that $L_{5,5}$ is a five-dimensional Lie algebra with the base $\{x_1, \ldots, x_5\}$ and its nonzero commutators are as follows:

$$[x_1, x_2] = x_3, \quad [x_1, x_3] = x_5 \text{ and } [x_2, x_4] = x_5.$$

Evidently, *cocl* $(L) = 2$ and $\dim L^{n-m} = \dim L^3 = 1$. By a simple calculation, we can show that the matrix representation of each element of $\mathcal{D}(L)$ is as follows:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha_{41} & 0 & 0 & 0 \\ \alpha_{41} & 0 & 0 & 0 & 0 \\ \alpha_{51} & \alpha_{52} & -\alpha_{41} & \alpha_{54} & 0 \end{bmatrix},$$

while the matrix representation of $Der_z(L)$ is as follows:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \alpha_{51} & \alpha_{52} & 0 & \alpha_{54} & 0 \end{bmatrix}$$

Corollary 3.2. *Let L be a filiform Lie algebra of dimension $n \geqslant 4$ over the field F with $char(F) \neq 2$. Then $Der_z(L) = \mathcal{D}(L)$.*

The reader should notice that the converse of the above theorem is not true. To this end consider the following example.

Example 3.3. Assume that $L_{6,10}$ is a six-dimensional Lie algebra with the base $\{x_1, \ldots, x_6\}$ and its nonzero brackets are as follows:

$$[x_1, x_2] = x_3, \ \ [x_1, x_3] = x_6 \text{ and } \ [x_4, x_5] = x_6.$$

By a simple calculation, we can show that the matrix representations of each element in $Der_z(L)$ and $\mathcal{D}(L)$ are equal and have the following form:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_{61} & \alpha_{62} & 0 & \alpha_{64} & \alpha_{65} & 0 \end{bmatrix}$$

while $L \notin \mathcal{X}$.

Acknowledgement

The authors would like to thank the referees for valuable suggestions.

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Received 22 December 2018
Revised 24 May 2019
Accepted 1 June 2019